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On a certain classes of meromorphic functions with positive coefficients

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Abstract. In this paper certain classes of meromorphic functions in punctured unit disk are defined. Some properties including coefficient inequalities, convolution and other results are investigated.

§1 Introduction and preliminaries

Let Σ denote the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n,$$

which are analytic in $D = \{z : 0 < |z| < 1\}$, having a simple pole at the origin. Motivated by M. L. Mogra [1] we define the following class of meromorphic functions and investigate some properties of this class.

A function $f \in \Sigma$ is said to be in the class $\Sigma(A, B, \lambda)$ if it satisfies the condition

$$\frac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda)f(z) + \lambda z f'(z)} = -\frac{1 + A\omega(z)}{1 + B\omega(z)}, \tag{1.1}$$

where $\omega(z)$ is analytic and $|\omega(z)| \le |z|$ in the unit disc U; A and B are real constants satisfying $0 < -A \le B < 1$ and λ is a real constant satisfying $0 \le \lambda \le 1$, $\lambda \ne 1/2$. From (1.1), we have that $f(z) \in \Sigma(A, B, \lambda)$ if and only if

$$\frac{zF'(z)}{F(z)} = -\frac{1 + A\omega(z)}{1 + B\omega(z)}, \tag{1.2}$$

where

$$F(z) = \frac{1}{1-2\lambda}\{(1-\lambda)f(z) + \lambda z f'(z)\} = \frac{1}{z} + \cdots . \tag{1.3}$$

Let $C(A, B, \lambda)$ be the class of functions $f \in \Sigma$ such that $-zf'(z) \in \Sigma(A, B, \lambda)$. Also let Σ_p be the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n, \ a_n \ge 0, \tag{1.4}$$

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which are analytic and in D. We define $\Sigma_p(A,B,\lambda) = \Sigma_p \cap \Sigma(A,B,\lambda)$ and $C_p(A,B,\lambda) = \Sigma_p \cap C_p(A,B,\lambda)$. The convolution or Hadamard product of two meromorphic functions $f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n$ and $g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} b_n z^n$ with $a_n, b_n \geq 0$ is defined by

$$f(z) * g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n b_n z^n. \tag{1.5}$$

The main aim of the present paper is to establish certain result concerning the convolution of meromorphic functions analogous to Padmanabhan and Ganesan [3]. Also M.L Mogra etal [2] have studied some convolution properties of a special class of meromorphic univalent functions which is close to our class and we extend their results in some directions. On the other hand we extend some corresponding results in A Schild and H. Silverman [4] for meromorphic functions with positive coefficients in our class.

In the sequel for real constants A, B and λ satisfying $0 < -A \leq B < 1$, $0 \leq \lambda \leq 1$, $\lambda \neq 1/2$, we define

$$U_{n,\lambda}(A,B) = \frac{(1+\lambda(n-1))(n(B+1)+A+1)}{|1-2\lambda|(B-A)}. \tag{1.6}$$

§2 Main results

Theorem 2.1. *If an univalent function $f(z)$ is in $\Sigma_p(A,B,\lambda)$ with $0 < -A \leq 1/3$, $-A \leq B \leq (1+A)/2$, then $G(z) = z^2F(z)$ is starlike univalent in $|z| < 1$, where $F(z)$ is given in (1.3). Moreover,*

$$\frac{zG'(z)}{G(z)} \prec \frac{1+(2B-A)z}{1+Bz}, \tag{2.1}$$

where $\prec$ denotes the subordination.

Proof. If $G(z) = z^2F(z)$, then

$$\frac{zG'(z)}{G(z)} = \frac{zF'(z)}{F(z)} + 2.$$

Applying (1.2), we obtain

$$\frac{zG'(z)}{G(z)} = \frac{1+(2B-A)\omega(z)}{1+B\omega(z)}.$$

where $\omega(z)$ is analytic and $|\omega(z)| \leq |z|$ in the unit disc U. This gives (2.1) because under the assumptions, we have $2B - A \leq 1$. Moreover, in this case we have

$$\mathfrak{Re}\left\{\frac{1+(2B-A)\omega(z)}{1+B\omega(z)}\right\} > 0 \quad |z| < 1,$$

then $G(z) = z^2F(z)$ is starlike univalent in $|z| < 1$. □

Theorem 2.2. *A function $f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n, a_n \geq 0$ is in $\Sigma_p(A,B,\lambda)$ if and only if*

$$\sum_{n=1}^{\infty} U_{n,\lambda}(A,B) a_n \leq 1 \tag{2.2}$$

also f is in $C_p(A,B,\lambda)$ if and only if

$$\sum_{n=1}^{\infty} n U_{n,\lambda}(A,B) a_n \leq 1. \tag{2.3}$$

Proof. Let $f(z)=\frac{1}{z}+\sum_{n=1}^{\infty}a_nz^n, a_n\geq 0$ and (2.2) holds. We show that $f\in\Sigma(A,B,\lambda)$. It is sufficient to show that the function

$$\omega(z)=\frac{\sum_{n=1}^{\infty}(n+1)(1+\lambda(n-1))a_nz^{n+1}}{(1-2\lambda)(B-A)-\sum_{n=1}^{\infty}(A+nB)(1+\lambda(n-1))a_nz^{n+1}}\ (z\in U) \tag{2.4}$$

is analytic, $\omega(0)=0$ and $|\omega(z)|\leq 1$.

We show that ω is analytic, i.e the denominator in (2.4) is not zero. By the assumption (2.2) we have

$$\begin{aligned}0 &\leq |1-2\lambda|(B-A)-\sum_{n=1}^{\infty}(A+1+n(B+1))(1+\lambda(n-1))a_n\\ &< |1-2\lambda|(B-A)-\sum_{n=1}^{\infty}(A+nB)(1+\lambda(n-1))a_n\end{aligned}$$

So

$$\begin{aligned}&|(1-2\lambda)(B-A)-\sum_{n=1}^{\infty}(A+nB)(1+\lambda(n-1))a_nz^{n+1}|\\ \geq\ & |1-2\lambda|(B-A)-|\sum_{n=1}^{\infty}(A+nB)(1+\lambda(n-1))a_nz^{n+1}|\\ \geq\ & |1-2\lambda|(B-A)-\sum_{n=1}^{\infty}(A+nB)(1+\lambda(n-1))a_n|z|^{n+1}\\ \geq\ & |1-2\lambda|(B-A)-\sum_{n=1}^{\infty}(A+nB)(1+\lambda(n-1))a_n>0\quad (z\in U).\end{aligned}$$

This shows that the denominator in (2.4) is not zero.

By (2.2) we have

$$|\omega(z)|\leq\frac{\sum_{n=1}^{\infty}(n+1)(1+\lambda(n-1))a_n}{|1-2\lambda|(B-A)-\sum_{n=1}^{\infty}(A+nB)(1+\lambda(n-1))a_n}\leq 1.$$

Conversely let $f(z)=\frac{1}{z}+\sum_{n=1}^{\infty}a_nz^n\in\sum_p(A,B,\lambda)$. From (2.4) ω satisfies $\omega(0)=0$ and $|\omega(z)|\leq 1$, also ω is analytic in the unit disk U. Since $\mathfrak{Re}\,\omega(z)\leq|\omega(z)|\leq 1(z\in U)$, so for $z=r(0<r<1)$, we have

$$\omega(r)=\mathfrak{Re}\,\omega(r)\leq|\omega(r)|\leq 1,$$

thus

$$\frac{\sum_{n=1}^{\infty}(n+1)(1+\lambda(n-1))a_nr^{n+1}}{|(1-2\lambda)|(B-A)-\sum_{n=1}^{\infty}(A+nB)(1+\lambda(n-1))a_nr^{n+1}}\leq 1.$$

Letting $r\to 1^-$, we get

$$\frac{\sum_{n=1}^{\infty}(n+1)(1+\lambda(n-1))a_n}{|(1-2\lambda)|(B-A)-\sum_{n=1}^{\infty}(A+nB)(1+\lambda(n-1))a_n}\leq 1.$$

Therefore (2.2) now is obtained. For the proof of the second part of the theorem we apply the first part for the function $g(z)=-zf'(z)$. □

Theorem 2.3. *If $f(z)=\frac{1}{z}+\sum_{n=1}^{\infty}a_nz^n$ and $g(z)=\frac{1}{z}+\sum_{n=1}^{\infty}b_nz^n$ are in $\Sigma_p(A,B,\lambda)$, then the Hadamard product $f(z)*g(z)=\frac{1}{z}+\sum_{n=1}^{\infty}a_nb_nz^n$ is in $\Sigma_p(A_1,B_1,\mu)$ with $0<-A_1\leq B_1<1$,*

$0 \le \mu \le \mu_0$, *where*

$$\mu_0 = \frac{\alpha - \sqrt{\alpha^2 - 2\beta\gamma}}{\beta}$$

$\alpha = 4U_{2,\lambda}^2(A,B) - 3U_{1,\lambda}^2(A,B) + 1$, $\beta = 12U_{1,\lambda}^2(A,B)$,

$$\gamma = 2U_{2,\lambda}^2(A,B) - 3U_{1,\lambda}^2(A,B) - 1$$

$$-A_1 \le \frac{K(\lambda,\mu_0)}{2 - K(\lambda,\mu_0)}, \frac{K(\lambda,\mu_0) + A_1}{1 - K(\lambda,\mu_0)} \le B_1$$

$$K(\lambda,\mu_0) = \frac{2|1-2\lambda|^2(B-A)^2}{|1-2\lambda|^2(B-A)^2 + (1-2\mu_0)(B+A+2)^2}.$$

The bounds for A_1 and B_1 cannot be improved.

Proof. Suppose $f(z)$ and $g(z)$ are in $\Sigma_p(A,B,\lambda)$. In view of Theorem 2.2, we have

$$\sum_{n=1}^{\infty} U_{n,\lambda}(A,B)a_n \le 1 \tag{2.5}$$

and

$$\sum_{n=1}^{\infty} U_{n,\lambda}(A,B)b_n \le 1. \tag{2.6}$$

We wish to find values of A_1, B_1 and μ for which $f(z) * g(z) \in \Sigma_p(A_1,B_1,\mu)$. Equivalently we want to determine A_1, B_1 and μ satisfying

$$\sum_{n=1}^{\infty} U_{n,\mu}(A_1,B_1)a_n b_n \le 1. \tag{2.7}$$

Using Cauchy Schwarz inequality together with (2.5) and (2.6) we get

$$\sum_{n=1}^{\infty} U_{n,\lambda}(A,B)\sqrt{a_n b_n} \le \left(\sum_{n=1}^{\infty} U_{n,\lambda}(A,B)a_n\right)^{\frac{1}{2}} \left(\sum_{n=1}^{\infty} U_{n,\lambda}(A,B)b_n\right)^{\frac{1}{2}}. \tag{2.8}$$

From (2.5), (2.6) and (2.8), we get

$$\sum_{n=1}^{\infty} U_{n,\lambda}(A,B)\sqrt{a_n b_n} \le 1.$$

So the inequality (2.7) is satisfied if

$$U_{n,\mu}(A_1,B_1)a_n b_n \le U_{n,\lambda}(A,B)\sqrt{a_n b_n}$$

for $n \ge 1$.
That is if

$$U_{n,\mu}(A_1,B_1)\sqrt{a_n b_n} \le U_{n,\lambda}(A,B).$$

Since $U_{n,\lambda}(A,B) \ge 1$ so from (2.8), we have

$$\sqrt{a_n b_n} \le \frac{1}{U_{n,\lambda}(A,B)}.$$

Thus it is enough to find $U_{n,\mu}(A_1,B_1)$ such that

$$U_{n,\mu}(A_1,B_1) \le U_{n,\lambda}^2(A,B). \tag{2.9}$$

The inequality (2.9) is equivalent to

$$\frac{(1+\mu(n-1))(n(B_1+1)+A_1+1)}{|1-2\mu|(B_1-A_1)} \le \left(\frac{(1+\lambda(n-1))(n(B+1)+A+1)}{|1-2\lambda|(B-A)}\right)^2 := u^2.$$

This yields

$$A_1 \leq \frac{u^2|1-2\mu|B_1 + (1+\mu(n-1))(n(B_1+1)+1)}{1+\mu(n-1)+u^2|1-2\mu|}. \tag{2.10}$$

Now (2.10) gives on simplification

$$\frac{B_1 - A_1}{B_1+1} \geq \frac{(1+\mu(n-1))(n+1)}{1+\mu(n-1)+U_{n,\lambda}^2(A,B)|1-2\mu|}. \tag{2.11}$$

It is easy to see that the right hand of (2.11) decreases as n increases and it is maximum for $n = 1$, provided that $0 \leq \mu \leq \mu_0$ and

$$\frac{B_1 - A_1}{B_1+1} \geq \frac{2|1-2\lambda|^2(B-A)^2}{|1-2\lambda|^2(B-A)^2 + (1-2\mu_0)(B+A+2)^2} := K(\lambda, \mu_0), \tag{2.12}$$

where

$$\mu_0 = \frac{\alpha - \sqrt{\alpha^2 - 2\beta\gamma}}{\beta}, \quad \alpha = (4U_{2,\lambda}^2(A,B) - 3U_{1,\lambda}^2(A,B) + 1)$$

and where

$$\beta = 12U_{1,\lambda}^2(A,B), \quad \gamma = 2U_{2,\lambda}^2(A,B) - 3U_{1,\lambda}^2(A,B) - 1.$$

It is clear that $K(\lambda, \mu_0) < 1$. Fixing A_1 in (2.12), we get $B_1 \geq \frac{K(\lambda,u_0)+A_1}{1-K}$. It is easy to verify that $0 < -A_1 \leq B_1 < 1$. If we take

$$f(z) = g(z) = \frac{1}{z} + |1-2\lambda|\frac{B-A}{B+A+2}z,$$

then

$$U_{n,\mu_0}(A_1, B_1) = \frac{(1-2\mu_0)K(\lambda,\mu_0)}{2-K(\lambda,\mu_0)}.$$

So we get $f(z) * g(z) \in \Sigma_p\left(-\frac{K(\lambda,\mu_0)}{2-K(\lambda,\mu_0)}, \frac{K(\lambda,\mu_0)}{2-K(\lambda,\mu_0)}\right)$ with $K(\lambda, \mu_0)$ as in (2.12). □

Corollary 2.1. *Let $f(z)$ and $g(z)$ be as in Theorem 2.3. Then*

$$h(z) = \frac{1}{z} + \sum_{n=1}^{\infty} u_{n,\lambda}(A,B)\sqrt{a_n b_n} z^n \in \Sigma_p(A, B, \lambda).$$

Proof. The result follows immediately from (2.8) using the Cauchy-Schwarz inequality. For the same functions as in Theorem 2.3, the result is best possible. □

Theorem 2.4. *If $f(z) \in \Sigma_p(A, B, \lambda)$ and $g(z) \in \Sigma_p(A', B', \theta)$ then $f(z) * g(z) \in \Sigma_p(A_1, B_1, \mu)$ with $0 < -A_1 \leq B_1 < 1$, $0 \leq \mu \leq \mu_0$, where*

$$\mu_0 = \frac{\alpha - \sqrt{\alpha^2 - 2\beta\gamma}}{\beta}$$

$$\alpha = 4U_{2,\lambda}(A,B)U_{2,\theta}(A',B') - 3U_{1,\lambda}(A,B)U_{1,\theta}(A',B') + 1$$

$$\beta = 12U_{1,\lambda}(A,B)U_{1,\theta}(A',B')$$

$$\gamma = 2U_{2,\lambda}(A,B)U_{2,\theta}(A',B') - 3U_{1,\lambda}(A,B)U_{1,\theta}(A',B') - 1$$

$$-A_1 \leq \frac{K(\lambda,\theta,\mu_0)}{2-K(\lambda,\theta,\mu_0)}, \frac{K(\lambda,\theta,\mu_0)+A_1}{1-K(\lambda,\theta,\mu_0)} \leq B_1$$

$$K(\lambda,\theta,\mu_0) = \frac{2|1-2\lambda||1-2\theta|(B-A)(B'-A')}{|1-2\lambda||1-2\theta|(B-A)(B'-A') + (1-2\mu_0)(B+A+2)(B'+A'+2)}.$$

The bounds for A_1 and B_1 cannot be improved.

Proof. Proceeding exactly as in Theorem 2.3, we require to show that

$$U_{n,\mu}(A_1, B_1) \leq U_{n,\theta}(A', B')U_{n,\lambda}(A, B)$$

for all $n \geq 1$. This on simplification yields

$$\frac{B_1 - A_1}{B_1 + 1} \geq \frac{(1 + \mu(n-1))(n+1)}{1 + \mu(n-1) + U_{n,\lambda}(A,B)u_{n,\theta}(A',B')|1-2\mu|}. \tag{2.13}$$

The right hand of (2.13) decreases as n increases and it is maximum for $n = 1$ provided that $0 \leq \mu \leq \mu_0$ and

$$\frac{B_1 - A_1}{B_1 + 1} \geq \frac{2|1-2\lambda||1-2\theta|(B-A)(B'-A')}{|1-2\lambda||1-2\theta|(B-A)(B'-A') + (1-2\mu_0)(B+A+2)(B'+A'+2)}, \tag{2.14}$$

where

$$\mu_0 = \frac{\alpha - \sqrt{\alpha^2 - 2\beta\gamma}}{\beta}$$
$$\alpha = 4U_{2,\lambda}(A,B)U_{2,\theta}(A',B') - 3U_{1,\lambda}(A,B)U_{1,\theta}(A',B') + 1$$
$$\beta = 12u_{1,\lambda}(A,B)u_{1,\theta}(A',B')$$
$$\gamma = 2U_{2,\lambda}(A,B)U_{2,\theta}(A',B') - 3U_{1,\lambda}(A,B)U_{1,\theta}(A',B') - 1.$$

Clearly $K(\lambda,\theta,\mu_0) < 1$. Fixing A_1 in (2.14) we get

$$\frac{K(\lambda,\theta,\mu_0) + A_1}{1 - K(\lambda,\theta,\mu_0)} \leq B_1.$$

It is easily seen that the result is best possible for the functions

$$f(z) = \frac{1}{z} + |1-2\lambda|\frac{B-A}{B+A+2}z,$$
$$g(z) = \frac{1}{z} + |1-2\theta|\frac{B-A}{B+A+2}z.$$

□

Corollary 2.2. *If $f(z), g(z), h(z) \in \sum_p(A,B,\lambda)$ then $f(z) * g(z) * h(z) \in \sum_p(A_1,B_1,\mu)$ with $0 \leq \mu \leq \mu_0$ where μ_0 is as in Theorem 2.4, $0 \leq \theta \leq \theta_0$ and*

$$\theta_0 = \frac{\alpha - \sqrt{\alpha^2 - 2\beta\gamma}}{\beta}$$

$\alpha = 4U^2_{2,\lambda}(A,B) - 3U^2_{1,\lambda}(A,B) + 1$, $\beta = 12U^2_{1,\lambda}(A,B)$,

$$\gamma = 2U^2_{2,\lambda}(A,B) - 3U^2_{1,\lambda}(A,B) - 1$$
$$-A_1 \leq \frac{K(\lambda,\theta,\mu_0)}{2 - K(\lambda,\theta,\mu_0)}, \frac{K(\lambda,\theta,\mu_0) + A_1}{1 - K(\lambda,\theta,\mu_0)} \leq B_1$$
$$K(\lambda,\theta,\mu_0) = \frac{2|1-2\lambda||1-2\theta|(B-A)(B'-A')}{|1-2\lambda||1-2\theta|(B-A)(B'-A') + (1-2\mu_0)(B+A+2)(B'+A'+2)}$$
$$-A' \leq \frac{K(\lambda,\theta_0)}{2 - K(\lambda,\theta_0)}, \frac{K(\lambda,\theta_0) + A}{1 - K(\lambda,\theta_0)} \leq B'$$
$$K(\lambda,\theta_0) = \frac{2|1-2\lambda|^2(B-A)^2}{|1-2\lambda|^2(B-A)^2 + (1-2\theta_0)(B+A+2)^2}.$$

Proof. Since $f(z), g(z) \in \sum_p(A,B,\lambda)$ by Theorem 2.4, we have $f(z) * g(z) \in \sum_p(A',B',\theta)$, where $-A' \leq \frac{K(\lambda,\theta_0)}{2-K(\lambda,\theta_0)}, \frac{K(\lambda,\theta_0)+A}{1-K(\lambda,\theta_0)} \leq B'$ with

$$K(\lambda,\theta_0) = \frac{2|1-2\lambda|^2(B-A)^2}{|1-2\lambda|^2(B-A)^2 + (1-2\theta_0)(B+A+2)^2}.$$

Now letting $f(z) * g(z) \in \sum_p(A',B',\theta)$ and $h(z) \in \sum_p(A,B,\lambda)$ the result follows by Theorem 2.4. □

Theorem 2.5. *If $f(z) \in C_p(A,B,\lambda)$ and $g(z) \in C_p(A',B',\theta)$ then $f(z)*g(z) \in C_p(A_1,B_1,\theta)$, where*

$$-A_1 \leq \frac{K(\lambda,\theta,\mu_0)}{2-K(\lambda,\theta,\mu_0)}, \frac{K(\lambda,\theta,\mu_0)+A_1}{1-K(\lambda,\theta,\mu_0)} \leq B_1$$

with $0 \leq \mu \leq \mu_0$ and μ_0 as in Theorem 2.4. The result is best possible.

Theorem 2.6. *If $f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n, a_n \geq 0$ belongs to $\Sigma_p(A,B,\lambda)$ and $g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} b_n z^n$ with $|b_n| \leq 1, n \geq 1$, then $f(z)*g(z) \in \Sigma(A,B,\lambda)$.*

Proof. Since $f(z) \in \Sigma_p(A,B,\lambda)$, we have

$$\Sigma_{n=1}^{\infty} U_{n,\lambda}(A,B)a_n \leq 1.$$

Furthermore $|b_n| \leq 1, n \geq 1$. Therefore,

$$\Sigma_{n=1}^{\infty} U_{n,\lambda}(A,B)|a_n b_n| = \sum_{n=1}^{\infty} U_{n,\lambda}(A,B)a_n|b_n| \leq 1,$$

this shows that $f(z)*g(z) \in \Sigma(A,B,\lambda)$. □

Corollary 2.3. *If $f(z) \in \sum_p(A,B,\lambda)$ and $g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} b_n z^n; 0 \leq b_n \leq 1$ for $n \geq 1$ then $f(z)*g(z) \in \sum_p(A,B,\lambda)$.*

Theorem 2.7. *If $f(z)$ and $g(z)$ are in $\sum_p(A,B,\lambda)$, then $h(z) = \frac{1}{z} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)z^n \in \sum_p(A_1,B_1,\mu)$, where*

$$0 \leq \mu \leq \mu_0 = \frac{\alpha - \sqrt{\alpha^2 - 2\beta\gamma}}{\beta},$$

$\alpha = 4U_{2,\lambda}^2(A,B) - 3U_{1,\lambda}^2(A,B) + 2$, $\beta = 12U_{1,\lambda}^2(A,B)$,

$$\gamma = 2U_{2,\lambda}^2(A,B) - 3U_{1,\lambda}^2(A,B) - 2,$$

$$-A_1 \leq \frac{K(\lambda,\mu_0)}{2-K(\lambda,\mu_0)}, \frac{K(\lambda,\mu_0)+A_1}{1-K(\lambda,\mu_0)} \leq B_1$$

$$K(\lambda,\mu_0) = \frac{4|1-2\lambda|^2(B-A)^2}{2|1-2\lambda|^2(B-A)^2 + (1-2\mu_0)(B+A+2)^2}.$$

The result is best possible.

Proof. Since $f(z), g(z) \in \sum_p(A,B,\lambda)$, then

$$\sum_{n=1}^{\infty} U_{n,\lambda}(A,B,\lambda)a_n \leq 1$$

and

$$\sum_{n=1}^{\infty} U_{n,\lambda}(A,B,\lambda)b_n \leq 1.$$

Therefore,

$$\sum_{n=1}^{\infty} U_{n,\lambda}^2(A,B,\lambda)a_n^2 \leq 1$$

and

$$\sum_{n=1}^{\infty} U_{n,\lambda}^2(A,B,\lambda)b_n^2 \leq 1.$$

Hence

$$\sum_{n=1}^{\infty} \frac{1}{2} U_{n,\lambda}^2(A,B,\lambda)(a_n^2 + b_n^2) \leq 1. \tag{2.15}$$

We want to find values of A_1, B_1 and μ such that

$$\sum_{n=1}^{\infty} U_{n,\mu}^2(A_1, B_1, \mu)(a_n^2 + b_n^2) \le 1. \tag{2.16}$$

Comparing (2.16) with (2.15) we see that (2.16) is true if

$$2U_{n,\mu}(A_1, B_1, \mu) \le U_{n,\mu}^2(A, B, \lambda)$$

or

$$\frac{B_1 - A_1}{B_1 + 1} \ge \frac{2(1 + \mu(n-1))(n+1)}{2(1 + \mu(n-1)) + U_{n,\lambda}^2(A, B)|1 - 2\mu|} \tag{2.17}$$

for all $n \ge 1$. The right hand side of (2.17) is a decreasing function of n and is maximum for $n = 1$ provided that $0 \le \mu \le \mu_0$ and

$$\frac{B_1 - A_1}{B_1 + 1} \ge \frac{4|1 - 2\lambda|^2(B - A)^2}{2|1 - 2\lambda|(B - A)^2 + (1 - 2\mu_0)(B + A + 2)^2} := K(\lambda, \mu_0). \tag{2.18}$$

Keeping A_1 fixed in (2.18) we get $\frac{K(\lambda,\mu_0)+A_1}{1-K(\lambda,\mu_0)} \le B_1$ and $-A_1 \le \frac{K(\lambda,\mu_0)}{2-K(\lambda,\mu_0)}$ with $K(\lambda, \mu_0)$ given as in (2.18). The functions $f(z) = g(z) = \frac{1}{z} + |1 - 2\lambda|\frac{B-A}{B+A+2}z$ show that our result is best possible. □

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