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Existence and Stability of Solutions to Highly Nonlinear Stochastic Differential Delay Equations Driven by G-Brownian Motion

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Abstract. Under linear expectation (or classical probability), the stability for stochastic differential delay equations (SDDEs), where their coefficients are either linear or nonlinear but bounded by linear functions, has been investigated intensively. Recently, the stability of highly nonlinear hybrid stochastic differential equations is studied by some researchers. In this paper, by using Peng's G-expectation theory, we first prove the existence and uniqueness of solutions to SDDEs driven by G-Brownian motion (G-SDDEs) under local Lipschitz and linear growth conditions. Then the second kind of stability and the dependence of the solutions to G-SDDEs are studied. Finally, we explore the stability and boundedness of highly nonlinear G-SDDEs.

§1 Introduction

By using Peng's theory of sublinear expectations, the research of the probability model with ambiguity makes a significant progress. In fact, a sublinear expectation can be represented as the upper expectation of a subset of linear expectations. Moreover, some researchers are focusing on the stochastic calculus of G-Brownian motion (e.g., see, Deng *et.al.* [4], Fei and Fei [6], Fei and Fei [9], Li and Peng [17], Peng [25], and Zhang [32]).

Next, the importance of the study of stochastic differential equations from both the theoretical points of view and their applications is well known. The classical stochastic differential equations with Brownian motion don't take the model uncertainty into consideration. Thus, in

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some complex environments, these equations are too restrictive to describe some phenomena. Recently, with uncertainty, a kind of stochastic differential equations driven by G-Brownian motion is investigated by Bai and Lin [3], Gao [14], Li *et al.* [16], Lin [19], Lin [20], Luo and Wang [21], etc.

We know that the stability of the classical stochastic differential equations is an important topic in the study of stochastic systems (e.g., see, Mao [23], Mao and Yuan [24], reference therein). Recently, Hu *et.al.* [15] initiate the investigation on the stability of hybrid highly nonlinear stochastic delay differential equations driven by Brownian motion. Based on highly nonlinear hybrid SDDEs, the stability of systems is further studied in [7, 8, 11–13, 29–31].

On the one hand, based on the system disturbed by G-Brownian motion providing characterization of the real world with both randomness and ambiguity, it is necessary to investigate the stability of the stochastic differential equations which is similar to a classical stochastic differential equation. One kind of exponential stability for stochastic differential equations driven by G-Brownian motion is discussed by Zhang and Chen [33] where quasi-sure analysis is used. Fei and Fei [10] investigated quasi-sure exponential stability by G-Lyapunov functional method in order to obtain the stability results. The stability of solutions to stochastic differential equations driven by G-Brownian motion is also investigated by Ren *et al.* [26, 27]. The stability of delayed Hopfield neural networks under a sublinear expectation is explored in Li and Yan [18].

On the other hand, in many real systems, such as science, industry, economics and finance etc., we will run into time lag. So it is necessary to explore G-SDDEs. In this paper, we first solve a basic problem in terms of the existence and uniqueness of the solutions to G-SDDEs under the local Lipschitz and the linear growth conditions by Picard iteration method. Next, the second kind of stability of the solution to G-SDDEs is discussed, and the dependence of solution to G-SDDE on initial data has been analysed as well. Finally, we investigate the existence and uniqueness, the asymptotic stability and the boundedness of the solutions to the highly nonlinear G-SDDEs with the local Lipschitz condition.

The arrangement of the paper is presented as follows. In Section 2, we give preliminaries on sublinear expectations and G-Bwownian motions. Furthermore, we characterize the properties of G-Brownian motions and G-martingales. In Section 3, the existence and uniqueness theorem of the solutions to G-SDDEs is proved. Moreover, in Section 4, under the linear growth conditions, we discuss the stability of the solutions to G-SDDEs, and the dependence of the solutions with respect to initial data. Section 5 investigates the existence and the stability of the solution to highly nonlinear G-SDDEs. Finally, the conclusion appears in Section 6.

§2 Preliminaries on Sublinear Expectation

In this section, we first give the notion of sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$, where Ω is a given state set and $\mathcal{H}$ a linear space of real valued functions defined on Ω. The space $\mathcal{H}$ can be considered as the space of random variables. The following concepts stem from Peng [25].

Definition 2.1. *A sublinear expectation $\hat{\mathbb{E}}$ is a functional $\hat{\mathbb{E}}$: $\mathcal{H} \to \mathbb{R}$ satisfying*
(i) Monotonicity: $\hat{\mathbb{E}}[X] \geq \hat{\mathbb{E}}[Y]$ if $X \geq Y$;
(ii) Constant preserving: $\hat{\mathbb{E}}[c] = c$;
(iii) Sub-additivity: For each $X, Y \in \mathcal{H}$, $\hat{\mathbb{E}}[X + Y] \leq \hat{\mathbb{E}}[X] + \hat{\mathbb{E}}[Y]$;
(iv) Positivity homogeneity: $\hat{\mathbb{E}}[\lambda X] = \lambda \hat{\mathbb{E}}[X]$ for $\lambda \geq 0$.

Definition 2.2. *Let $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ be a sublinear expectation space. $(X(t))_{t\geq 0}$ is called a d-dimensional stochastic process if for each $t \geq 0$, $X(t)$ is a d-dimensional random vector in $\mathcal{H}$.*

A d-dimensional process $(B(t))_{t\geq 0}$ on a sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is called a G-Brownian motion if the following properties are satisfies:
(i) $B_0(\omega) = 0$;
(ii) for each $t, s \geq 0$, the increment $B(t+s) - B(t)$ is $N(\{0\} \times s\Sigma)$-distributed and is independent from $(B(t_1), B(t_2), \cdots, B(t_n))$, for each $n \in \mathbb{N}$ and $0 \leq t_1 \leq \cdots \leq t_n \leq t$, where Σ is a bounded, convex and closed subset in the family of $d \times d$ nonnegative definite symmetric matrices. Let $< B > (\cdot)$ be the quadratic variation process of $B(\cdot)$.

We now give the definition of the Itô integral. For the technical simplicity, in the rest of the paper, we introduce Itô integral with respect to one-dimensional G-Brownian motion with $G(\alpha) := \frac{1}{2}\hat{\mathbb{E}}[\alpha B(1)^2] = \frac{1}{2}(\bar{\sigma}^2\alpha^+ - \underline{\sigma}^2\alpha^-)$, where $\hat{\mathbb{E}}[B(1)^2] = \bar{\sigma}^2, \mathcal{E}[B(1)^2] = \underline{\sigma}^2$, $0 < \underline{\sigma} \leq \bar{\sigma} < \infty$, where lower expectation $\mathcal{E}[X] := -\hat{\mathbb{E}}[-X]$ for each $X \in \mathcal{H}$.

Let $p \geq 1$ be fixed. We consider the following type of simple processes: for a given partition $\pi_T = (t_0, \cdots, t_N)$ of $[0, T]$, where T can take ∞, we get

$$\eta(t, \omega) = \sum_{k=0}^{N-1} \xi_k(\omega) I_{[t_k, t_{k+1})}(t),$$

where $\xi_k \in L_G^p(\Omega_{t_k}), k = 0, 1, \cdots, N-1$ are given. The collection of these processes is denoted by $\mathcal{M}_G^{p,0}(0, T)$. We denote by $\mathcal{M}_G^p(0, T)$ the completion of $\mathcal{M}_G^{p,0}(0, T)$ with the norm

$$\|\eta\|_{\mathcal{M}_G^p(0,T)} := \left\{ \hat{\mathbb{E}} \int_0^T |\eta(t)|^p dt \right\}^{1/p} < \infty.$$

More details on the notions of G-expectation $\hat{\mathbb{E}}$ and G-Brownian motion, and the definition of stochastic integral $\int_0^T \eta(t) dB(t)$ on the sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ can be found in Peng [25].

For convenience, we give the following Burkholder-Davis-Gundy inequality (see, e.g., Gao [14, Theorems 2.1-2.2]).

Lemma 2.3. *(Burkholder-Davis-Gundy inequality) Let $p \geq 2$ and $\zeta = \{\zeta(s), s \in [0,T]\} \in \mathcal{M}_G^p(0,T)$. Then, for all $t \in [0,T]$ such that*

$$\hat{\mathbb{E}} \sup_{s \leq u \leq t} \Big| \int_s^u \zeta(v) d < B > (v) \Big|^p \leq (t-s)^{p-1} C_1(p,\bar{\sigma}) \hat{\mathbb{E}} \int_s^t |\zeta(v)|^p dv,$$

$$\hat{\mathbb{E}} \sup_{s \leq u \leq t} \Big| \int_s^u \zeta(v) dB(v) \Big|^p \leq C_2(p,\bar{\sigma}) \hat{\mathbb{E}} \Big(\int_s^t |\zeta(v)|^2 dv \Big)^{p/2},$$

where the constants $C_i(p,\bar{\sigma}), i = 1,2$ depend on parameters p and $\bar{\sigma}$.

We provide the following property which stems from Denis et al. [5] or Zhang and Chen [33].

Proposition 2.4. *Let $\hat{\mathbb{E}}$ be G-expectation. Then there exists a weakly compact family of probability measures $\mathcal{P}$ on $(\Omega, \mathcal{B}(\Omega))$ such that for all $X \in \mathcal{H}, \hat{\mathbb{E}}[X] = \max_{P \in \mathcal{P}} E_P[X]$, where $E_P[\cdot]$ is the linear expectation with respect to P.*

From the above proposition, we know that the weakly compact family of probability measures $\mathcal{P}$ characterizes the degree of Knightian uncertainty. Especially, if $\mathcal{P}$ is singleton, i.e. $\{P\}$, then the model has no ambiguity. Moreover, the related calculus reduces to a classical one. We now define G-upper capacity $\mathbb{V}(\cdot)$ and G-lower capacity $\mathcal{V}(\cdot)$ by

$$\mathbb{V}(A) = \sup_{P \in \mathcal{P}} P(A), \ \forall A \in \mathcal{B}(\Omega),$$

$$\mathcal{V}(A) = \inf_{P \in \mathcal{P}} P(A), \ \forall A \in \mathcal{B}(\Omega).$$

Thus a property is called to hold quasi surely (q.s.) if there exists a polar set D with $\mathbb{V}(D) = 0$ such that it holds for each $\omega \in D^c$. We say that a property holds $\mathcal{P}$-q.s. means that it holds P-a.s. for each $P \in \mathcal{P}$. If an event A fulfills $\mathbb{V}(A) = 1$, then we call the event A occurs $\mathbb{V}$-a.s.

§3 Existence and uniqueness of solutions to G-SDDEs

For convenience of expounding problem, throughout this paper, all stochastic processes take values in $\mathbb{R}$. If A is a subset of Ω, denote by I_A its indicator function. Let $(\Omega, \mathcal{H}, \{\Omega_t\}_{t \geq 0}, \hat{\mathbb{E}}, \mathbb{V})$ be a generalized filtered sublinear expectation space, and $(B(t))_{t \geq 0}$ one-dimensional G-Brownian motion defined on the generalized filtered sublinear expectation space.

Let $f, g, h : \mathbb{R} \times \mathbb{R} \times [0,T] \to \mathbb{R}$ be Borel measurable functions. Consider one-dimensional highly nonlinear G-SDDE

$$\begin{aligned} dX(t) =& f(X(t), X(t-\tau), t)dt + g(X(t), X(t-\tau), t)d < B > (t) \\ &+ h(X(t), X(t-\tau), t)dB(t) \end{aligned} \tag{1}$$

on $t \geq 0$ with nonrandom initial data

$$\{X(t) = \xi(t) : -\tau \leq t \leq 0\} = \xi \in C([-\tau, 0]; \mathbb{R}). \tag{2}$$

The existence and uniqueness of solutions to stochastic differential equations driven by G-Brownian motion (G-SDE) has been presently proved under Lipschitz coefficients with the linear growth condition, see, e.g., Peng [25]. To our best knowledge, however, the existence and uniqueness of G-SDDE with local Lipschtz coefficients and the linear growth condition has not been proved yet. We now discuss the existence and uniqueness of solutions to G-SDDE (1). Let us provide these conditions for our aim.

Assumption 3.1. *Assume that for any $m > 0$, there exists a positive constant K_m such that*

$$\begin{aligned} &|f(x, y, t) - f(\bar{x}, \bar{y}, t)| \vee |g(x, y, t) - g(\bar{x}, \bar{y}, t)| \vee |h(x, y, t) - h(\bar{x}, \bar{y}, t)| \\ &\leq K_m(|x - \bar{x}| + |y - \bar{y}|) \end{aligned} \tag{3}$$

for all $x, \bar{x}, y, \bar{y} \in \mathbb{R}$ with $|x| \vee |\bar{x}| \vee |y| \vee |\bar{y}| \leq m$ and all $t \in [0, T]$. Assume moreover that there exists constant $K > 0$ such that

$$|f(x, y, t)| \vee |g(x, y, t)| \vee |h(x, y, t)| \leq K(1 + |x| + |y|) \tag{4}$$

for all $x \in \mathbb{R}, t \in [0, T]$.

Next, in order to get the existence and uniqueness theorem of solutions to G-SDDEs under Assumption 3.1, we first prove a lemma which gives a bound of the solution.

Lemma 3.2. *Let the linear growth condition (4) hold. If $X(\cdot)$ is a solution to G-SDDE (1) with the initial data (2), then we have*

$$\hat{\mathbb{E}}\Big(\sup_{0 \leq t \leq T} |X(t)|^2\Big) \leq A_1(T, \bar{\sigma}, \tau)e^{A_2(T, \bar{\sigma}, \tau)T}, \tag{5}$$

where

$$\begin{aligned} A_1(T, \bar{\sigma}, \tau) =& 12K^2T(T + TC_1(2, \bar{\sigma}) + C_2(2, \bar{\sigma})) \\ &+ 4\|\xi\|^2(1 + 3K^2T\tau + 3K^2T\tau C_1(2, \bar{\sigma}) + 3K^2\tau C_2(2, \bar{\sigma})), \\ A_2(T, \bar{\sigma}, \tau) =& 24K^2(T + TC_1(2, \bar{\sigma}) + C_2(2, \bar{\sigma})), \end{aligned}$$

and $C_1(2, \bar{\sigma})$ and $C_2(2, \bar{\sigma})$ are defined in Lemma 2.3. Specially, $X(\cdot)$ belongs to $\mathcal{M}_G^2(0, T)$.

Proof. For each integer $m \geq 1$, define the stopping time

$$\nu_m = T \wedge \inf\{t \in [0, T]; |X(t)| \geq m\}.$$

Clearly, $\nu_m \uparrow T$ q.s. Set $X_m(t) = X(t \wedge \nu_m)$ for $t \in [0, T]$. Then $X_m(t)$ satisfies the equation

$$X_m(t) = X_0 + \int_0^t f(X_m(s), X_m(s-\tau), s) I_{[0,\nu_m]}(s) ds$$
$$+ \int_0^t g(X_m(s), X_m(s-\tau), s) I_{[0,\nu_m]}(s) d < B > (s) + \int_0^t h(X_m(s), X_m(s-\tau), s) I_{[0,\nu_m]}(s) dB(s).$$

By the Hölder inequality, the linear growth condition (4) and the Birkholder-Davis-Gundy inequality for G-Brownian motion (see Lemma 2.3), we deduce

$$\begin{aligned}\hat{\mathbb{E}}\Big(\sup_{0\le t\le T} |X_m(t)|^2\Big) &\le 4\|\xi\|^2 + 4K^2T\hat{\mathbb{E}}\int_0^T (1 + |X_m(s)| + |X_m(s-\tau)|)^2 ds \\ &\quad + 4K^2TC_1(2,\bar{\sigma})\hat{\mathbb{E}}\int_0^T (1 + |X_m(s)| + |X_m(s-\tau)|)^2 ds \\ &\quad + 4C_2(2,\bar{\sigma})K^2\hat{\mathbb{E}}\int_0^T (1 + |X_m(s)| + |X_m(s-\tau)|)^2 ds \\ &\le 12K^2T(T + TC_1(2,\bar{\sigma}) + C_2(2,\bar{\sigma})) + 4\|\xi\|^2(1 + 3K^2T\tau + 3K^2T\tau C_1(2,\bar{\sigma}) + 3K^2\tau C_2(2,\bar{\sigma})) \\ &\quad + 24K^2(T + TC_1(2,\bar{\sigma}) + C_2(2,\bar{\sigma}))\hat{\mathbb{E}}\int_0^T |X_m(s)|^2 ds \\ &\le A_1(T,\bar{\sigma},\tau) + A_2(T,\bar{\sigma},\tau)\int_0^T \hat{\mathbb{E}}\Big(\sup_{0\le s\le t} |X_m(t)|^2\Big) dt,\end{aligned}$$

where we use $\int_{-\tau}^0 |X_m(s)|^2 ds \le \|\xi\|^2\tau$. Hence, from the Gronwall inequality, the required inequality follows by letting $m \to \infty$. Thus the proof is complete. $\square$

Theorem 3.3. *Under Assumption 3.1, the G-SDDE (1) with the initial data (2) has a unique solution* $X(\cdot) \in \mathcal{M}_G^2(0, T)$.

Proof. The whole proof is divided into two steps for assertion.

Step 1. Existence. Define $X^0(t) = \xi(t), t \in [-\tau, 0]$ and $X^0(t) = \xi(0)$ for $0 \le t \le T$. For each $n = 1, 2, \cdots$, set $X^n(t) = \xi(t), t \in [-\tau, 0]$ and define, by the Picard iterations,

$$\begin{aligned}X^{n+1}(t) = \xi(0) &+ \int_0^t f(X^n(s), X^n(s-\tau), s) ds + g(X^n(s), X^n(s-\tau), s) d < B > (s) \\ &+ \int_0^t h(X^n(s), X^n(s-\tau), s) dB(s)\end{aligned} \tag{6}$$

for $t \in [0, T]$. Obviously, $X^0(\cdot) \in \mathcal{M}_G^2(0, T)$.

From (4), (6), the Hölder inequality, and Lemma 2.3, by a similar way as the proof of Lemma 3.2 we easily derive

$$\begin{aligned}\hat{\mathbb{E}}|X^{n+1}(t)|^2 &\le 4\|\xi\|^2 + 4\hat{\mathbb{E}}\Big|\int_0^t f(X^n(s), X^n(s-\tau), s) ds\Big|^2 \\ &\quad + 4\hat{\mathbb{E}}\Big|\int_0^t g(X^n(s), X^n(s-\tau), s) d < B > (s)\Big|^2 + 4\hat{\mathbb{E}}\Big|\int_0^t h(X^n(s), X^n(s-\tau), s) dB(s)\Big|^2 \\ &\le \hat{K}(T,\bar{\sigma},\tau)\int_0^t \hat{\mathbb{E}}|X^n(s)|^2 ds\end{aligned}$$

for some constant $\hat{K}(T,\bar{\sigma},\tau)$. Thus, by induction and Lemma 3.2, we know that $X^{n+1}(\cdot) \in \mathcal{M}_G^2(0,T)$. Define now a stopping time $\tau_m := \inf\{t;$ there exists some $i \in \mathbb{N}$ such that $|X^i(t)| \geq m\}$. Then, we claim that for all $n \geq 0$,

$$\hat{\mathbb{E}}\Big[\sup_{0\leq s\leq t\wedge\tau_m} \mid X^{n+1}(s) - X^n(s) \mid^2\Big] \leq \frac{\hat{C}(M_m t)^n}{n!}, \quad t \in [0,T], \tag{7}$$

where $M_m = 12K_m^2(t + C_1 t + C_2)$, and $C_1 = C_1(2,\bar{\sigma}), C_2 = C_2(2,\bar{\sigma}) > 0$ are defined in Lemma 2.3, and $\hat{C}$ is defined below. Thus, we can easily show that

$$\begin{aligned}
&\hat{\mathbb{E}}\Big[\sup_{0\leq s\leq t\wedge\tau_m} \mid X^1(s) - X^0(s) \mid^2\Big] \\
&\leq 3K^2 t \int_0^{t\wedge\tau_m} (1 + |\xi(0)| + |\xi(s-\tau)|)^2 ds \\
&\quad + 3K^2 C_1 t \int_0^{t\wedge\tau_m} (1 + |\xi(0)| + |\xi(s-\tau)|)^2 ds \\
&\quad + 3C_2 K^2 \int_0^{t\wedge\tau_m} (1 + |\xi(0)| + |\xi(s-\tau)|)^2 ds \\
&\leq 9K^2 t^2 (1 + |\xi(0)|^2 + \|\xi\|^2) \\
&\quad + 9K^2 C_1 t^2 (1 + |\xi(0)|^2 + \|\xi\|^2) \\
&\quad + 9K^2 C_2 t (1 + |\xi(0)|^2 + \|\xi\|^2) := \hat{C}.
\end{aligned}$$

So (7) holds for $n = 0$. Next, assume (7) holds for some $n - 1 \geq 0$. Then, we have

$$\begin{aligned}
&\hat{\mathbb{E}}\Big[\sup_{0\leq s\leq t\wedge\tau_m} \mid X^{n+1}(s) - X^n(s) \mid^2\Big] \\
&\leq 3K_m^2(t + C_1 t + C_2)\hat{\mathbb{E}}\int_0^{t\wedge\tau_m} (|X^n(s) - X^{n-1}(s)| + |X^n(s-\tau) - X^{n-1}(s-\tau)|)^2 ds
\end{aligned}$$

$$\begin{aligned}
&\leq 12K_m^2(t + C_1 t + C_2)\hat{\mathbb{E}}\int_0^{t\wedge\tau_m} |X^n(s) - X^{n-1}(s)|^2 ds \\
&\leq M_m \int_0^t \hat{\mathbb{E}}\Big[\sup_{0\leq r\leq s\wedge\tau_m} |X^n(r) - X^{n-1}(r)|^2\Big] ds \\
&\leq M_m \int_0^t \frac{\hat{C}[M_m s]^{n-1}}{(n-1)!} ds = \frac{\hat{C}[M_m t]^n}{(n)!},
\end{aligned} \tag{8}$$

where $M_m := 12K_m^2(t + C_1 t + C_2)$. Thus, from the Chebyshev inequality for sublinear expectation $\hat{\mathbb{E}}$ (see, e.g., Chen *et al.* [2, Proposition 2.1 (2)]), we get

$$\mathbb{V}\Big\{\sup_{0\leq s\leq t\wedge\tau_m} |X^{n+1}(s) - X^n(s)| > \frac{1}{2^n}\Big\} \leq \frac{\hat{C}[4M_m t]^n}{(n)!}.$$

Since $\sum_{n=0}^{\infty} \hat{C}[4M_m t]^n/(n)! < \infty$, from the Borel-Cantelli lemma on upper capacity (see, Chen [1, Lemma 2.7]), we obtain that there exists a positive integer $n_0 = n_0(\omega), \omega \in D_0^c$, where D_0

is a polar set with $\mathbb{V}(D_0) = 0$, such that

$$\sup_{0\le s\le t\wedge\tau_m} |X^{n+1}(s) - X^n(s)| \le \frac{1}{2^n} \quad \text{whenever } n \ge n_0(\omega).$$

Therefore, we have

$$X^0(s) + \sum_{i=0}^{n-1}(X^{i+1}(s) - X^i(s)) = X^n(s) \quad \text{q.s.}$$

are convergent uniformly in $s \in [0, t\wedge\tau_m]$. Denote the limit of $X^n(t)$ by $X(t)$ which is continuous and Ω_t-adapted.

Form (7), we also know that $\{X^n(s)\}_{n\ge1}, s \in [0, t\wedge\tau_m]$ is a Cauchy sequence in $\mathcal{M}_G^2(0,T)$. Thus, we also have $X^n(s) \to X(s), s \in [0, t\wedge\tau_m]$, and $X(\cdot) \in \mathcal{M}_G^2(0, t\wedge\tau_m)$. Next we show that $X(\cdot)$ fulfills equation (1). Obviusly, $X^n(\cdot), X(\cdot) \in \mathcal{M}_G^2(0, t\wedge\tau_m)$. Thus, by the Burkholder-Davis-Gundy inequality for G-Brownian motion (see Lemma 2.3), we deduce

$$\begin{aligned}
&\hat{\mathbb{E}}\Big|\int_0^{t\wedge\tau_m}(f(X^n(s), X^n(s-\tau), s) - f(X(s), X(s-\tau)), s)ds\Big|^2\\
&\quad+\hat{\mathbb{E}}\Big|\int_0^{t\wedge\tau_m}(g(X^n(s), X^n(s-\tau), s) - g(X(s), X(s-\tau)), s)d<B>(s)\Big|^2\\
&\quad+\hat{\mathbb{E}}\Big|\int_0^{t\wedge\tau_m}(h(X^n(s), x^n(s-\tau), s) - h(x(s), x(s-\tau), s))dB(s)\Big|^2\\
&\le K_m^2(t + C_1t + C_2)\hat{\mathbb{E}}\int_0^{t\wedge\tau_m}(|X^n(s) - X(s)| + |X^n(s-\tau) - X(s-\tau)|)^2ds\\
&\le 4K_m^2(t + C_1t + C_2)\hat{\mathbb{E}}\int_0^{t\wedge\tau_m}|X^n(s) - X(s)|^2ds \to 0 \quad \text{as } n\to\infty.
\end{aligned}$$

Thus, letting $n \to \infty$, we easily obtain $X(s)$ satisfying equation (1) for $s \in [0, t\wedge\tau_m]$. Then by taking $m \to \infty$ in (6), we obtain $X(t)$ is the solution to (6) for $t \in [0,T]$. Therefore, the existence has also been proved.

Step 2. Uniqueness. Let $X(t)$ and $\bar{X}(t)$ be the two solutions. Define stoping times $\hat{\tau}_m := \inf\{t; |X(t)| \vee |\bar{X}(t)| \ge m\}$. Noting, for each $t \ge 0$,

$$\begin{aligned}
X(t\wedge\hat{\tau}_m) - \bar{X}(t\wedge\hat{\tau}_m) = &\int_0^{t\wedge\hat{\tau}_m}[f(X(s), X(s-\tau), s) - f(\bar{X}(s), \bar{X}(s-\tau), s)]ds\\
&+\int_0^{t\wedge\hat{\tau}_m}[g(X(s), X(s-\tau), s) - g(\bar{X}(s), \bar{X}(s-\tau), s)]d<B>(s)\\
&+\int_0^{t\wedge\hat{\tau}_m}[h(X(s), X(s-\tau), s) - h(\bar{X}(s), \bar{X}(s-\tau), s)]dB(s).
\end{aligned}$$

By Assumption 3.1, the Cauchy-Schwartz inequality and the Burkholder-Davis-Gundy inequality in Lemma 2.3, we can easily show that

$$\hat{\mathbb{E}}\Big[\sup_{0\le s\le t\wedge\hat{\tau}_m} |\, X(s)-\bar{X}(s)\,|^2\Big] \le 12K_m^2(t+C_1t+C_2)\hat{\mathbb{E}}\int_0^{t\wedge\hat{\tau}_m} |X(s)-\bar{X}(s)|^2 ds$$

$$\le 12K_m^2(t+C_1t+C_2)\int_0^{t\wedge\hat{\tau}_m} \hat{\mathbb{E}}\Big[\sup_{0\le r\le s} |\, X(r)-\bar{X}(r)\,|^2\Big] ds.$$

The Gronwall inequality then yields that, fixing $m>0$ arbitrarily,

$$\hat{\mathbb{E}}\Big[\sup_{0\le s\le t\wedge\hat{\tau}_m} |\, X(s)-\bar{X}(s)\,|^2\Big] = 0, \forall t\in[0,T],$$

which shows

$$\hat{\mathbb{E}}\Big[\sup_{0\le s\le t} |\, X(s)-\bar{X}(s)\,|^2\Big] = 0, \forall t\in[0,T]$$

by letting $m\to\infty$. This implies that $X(t)=\bar{X}(t)$ *q.s.* for $t\in[0,T]$. Thus, the uniqueness has been proved. Hence, the proof is complete. □

Next, although the existence and uniqueness of solutions to stochastic differential equations driven by G-Brownian motion (G-SDE) has been presently proved under the non-Lipschitz condition, but the coefficients of G-SDE is often bounded by a linear function, see, e.g., Lin [19]. However, if the coefficients of G-SDDE (1) cannot be bounded by a linear function and are highly nonlinear from the perspective of Hu *et al.* [15] and Fei *et al.* [11, 12], then the global solution does not exist generally. By a similar discussion as in [24], we can prove the equation (1) has a unique maximal solution under the following polynomial growth condition (9) instead of the linear growth condition (4) of Assumption 3.1

$$\begin{aligned} |f(x,y,t)| &\le K(1+|x|^{q_1}+|y|^{q_1}),\\ |g(x,y,t)| &\le K(1+|x|^{q_2}+|y|^{q_2}),\\ |h(x,y,t)| &\le K(1+|x|^{q_3}+|y|^{q_3}), \end{aligned} \tag{9}$$

for all $x\in\mathbb{R}, t\in[0,T]$, and constants $K>0$, $q_i\ge 1, i=1,2,3$. The following theorem shows that the maximal solution exists without the linear growth condition (related notion is referred to Mao and Yuan [24, Definition 7.11]).

Theorem 3.4. *Under Assumption 3.1 with (4) replaced by (9), the G-SDDE (1) with the initial data (2) has a unique maximal solution* $X(\cdot)\in\mathcal{M}_G^2(0,T)$.

Proof. For every $m\ge 1$, define, for $x\in\mathbb{R}$,

$$x^{[m]} = \begin{cases} x & \text{if } |x|\le m,\\ \frac{mx}{|x|} & \text{if } |x|>m. \end{cases}$$

Now define the truncation functions

$$f_m(x,y,t)=f(x^{[m]},y^{[m]},t),\ g_m(x,y,t)=g(x^{[m]},y^{[m]},t),\ h_m(x,y,t)=h_m(x^{[m]},y^{[m]},t).$$

Thus f_m, g_m, h_m satisfy (3) in Assumption 3.1. By Theorem 3.3, there exists a unique solution

$X_m(\cdot) \in \mathcal{M}_G^2(0,T)$ to equation

$$dX_m(t) = f_m(X_m(t), X_m(t-\tau), t)dt + g_m(X_m(t), X_m(t-\tau), t)d < B > (t) + h_m(X_m(t), X_m(t-\tau), t)dB(t) \tag{10}$$

on $t \geq 0$ with nonrandom initial data

$$\{X_m(t) = \xi(t); -\tau \leq t \leq 0\} = \xi \in C([-\tau, 0]; \mathbb{R}).$$

Set the stopping time $\nu_m = T \wedge \inf\{t \in [0,T]; \|X_{m,t}\| \geq m\}$, where $X_{m,t} = \{X_m(t+u); -\tau \leq u \leq 0\}$. Hence we see that

$$X_m(t) = X_{m+1}(t), \quad t \in [-\tau, \nu_m], \tag{11}$$

which shows that ν_m is increasing such that $\nu_\infty = \lim_{m\to\infty} \nu_m$ exists. Further, we define $\{X(t); -\tau \leq t \leq \nu_\infty\}$ by $X(t) = \xi(t)$ on $t \in [-\tau, 0]$ and $X(t) = X_m(t),\ t \in [\nu_{m-1}, \nu_m],\ m \geq 1$, where $\nu_0 = 0$ and we set $X(\nu_\infty) = \infty$ if $\nu_\infty < T$. From (11), we have $X(t \wedge \nu_m) = X_m(t \wedge \nu_m)$. Thus it follows from (10) that

$$X(t \wedge \nu_m) = \xi(0) + \int_0^{t\wedge\nu_m} f(X(s), X(s-\tau, s))ds + \int_0^{t\wedge\nu_m} g(X(s), X(s-\tau), s)d < B > (s) + \int_0^{t\wedge\nu_m} h(X(s), X(s-\tau), s)dB(s)$$

for any $t \in [0,T]$ and $m \geq 1$. Also, it is easy to see that if $\nu_\infty < T$, then we have

$$\lim \sup_{t\to\infty} |X(t)| = \lim \sup_{m\to\infty} X(\nu_m) = \lim \sup_{m\to\infty} |X_m(\nu_m)| = \infty.$$

Thus $\{X(t); 0 \leq t \leq \nu_\infty\}$ is a maximal local solution. By a standard discussion, we easily prove the uniqueness of the solution. The proof thus is complete. □

§4 Second kind of stability for solutions

In this section, let $T < \infty$. We shall investigate the second kind of stability for strong solutions to the G-SDDEs which is stability with respect to the equation coefficients f, g, h. We will show that if approximations f_n, g_n, h_n of the coefficients f, g, h converge to the exact coefficients, then approximate solutions converge to the solution of the equation with exact coefficients as well. Therefore, let $X(\cdot), X^n(\cdot)$ denote strong solutions of the following highly nonlinear G-SDDEs

$$dX(t) = f(X(t), X(t-\tau), t)dt + g(X(t), X(t-\tau), t)d < B > (s) + h(X(t), X(t-\tau), t)dB(t), \quad t \in [0, \infty), \tag{12}$$

$$dX^n(t) = f_n(X^n(t), X^n(t-\tau), t)dt + g_n(X^n(t), X^n(t-\tau), t)d < B > (s) + h_n(X^n(t), X^n(t-\tau), t)dB(t), \quad t \in [0, \infty) \tag{13}$$

with the same nonrandom initial data $\xi = \{\xi(t), t \in [-\tau, 0]\} \in C([-\tau, 0]; \mathbb{R})$.

Theorem 4.1. *Let $f, g, h, f_n, g_n, h_n : \mathbb{R} \times \mathbb{R} \times [0, T] \to \mathbb{R}$ satisfy Assumption 3.1 with the local Lipschitz condition (3) replaced by the global Lipschitz condition, where $K_m = K$ is independent of m. Assume that for every $x, y \in \mathbb{R}$ it holds*

$$\varphi_n(x, y, t) \to \varphi(x, y, t), \tag{14}$$

where $\varphi_n = f_n, g_n, h_n$, respectively, and $\varphi = f, g, h$, respectively.

Then, for the solutions $X, X_n : [-\tau, T] \times \Omega \to \mathbb{R}$ of the equations (12) and (13), we have

$$\hat{\mathbb{E}}\Big[\sup_{t\in[0,T]} |X^n(t) - X(t)|^2\Big] \to 0 \ \text{as } n \to \infty. \tag{15}$$

Proof. By assumptions of theorem, the solutions X to (12) and X_n to (13) exist and are unique due to Theorem 3.3. Define $\tilde{\tau}_m := \inf\{t;\text{there exists some } i \in \mathbb{N} \text{ such that} |X^i(t)| \geq m\}$. Obviously, $\tilde{\tau}_m \to \infty$ as $m \to \infty$. By the Cauchy-Schwartz inequality and Assumption 3.1, we deduce

$$\begin{aligned}
&\hat{\mathbb{E}} \sup_{v\in[0,t\wedge\tilde{\tau}_m]} \Big| \int_0^v (f_n(X^n(s), X^n(s-\tau), s) - f(X(s), X(s-\tau), s))ds\Big|^2 \\
&\leq 2\hat{\mathbb{E}} \sup_{v\in[0,t\wedge\tilde{\tau}_m]} \Big| \int_0^v (f_n(X^n(s), X^n(s-\tau), s) - f_n(X(s), X(s-\tau), s))ds\Big|^2 \\
&\quad + 2\hat{\mathbb{E}} \sup_{v\in[0,t\wedge\tilde{\tau}_m]} \Big| \int_0^v (f_n(X(s), X(s-\tau), s) - f(X(s), X(s-\tau), s))ds\Big|^2 \\
&\leq 4K_m^2 T\hat{\mathbb{E}} \sup_{v\in[0,t\wedge\tilde{\tau}_m]} \int_0^v (|X^n(s) - X(s)|^2 + |X^n(s-\tau) - X(s)|^2)ds \\
&\quad + 2T\hat{\mathbb{E}} \sup_{v\in[0,t\wedge\tilde{\tau}_m]} \int_0^v |f_n(X(s), X(s-\tau), s) - f(X(s), X(s-\tau), s)|^2 ds \\
&\leq 8K_m^2 T\hat{\mathbb{E}} \sup_{v\in[0,t\wedge\tilde{\tau}_m]} \int_0^v |X^n(s) - X(s)|^2 ds \\
&\quad + 2T\hat{\mathbb{E}} \sup_{v\in[0,t\wedge\tilde{\tau}_m]} \int_0^v |f_n(X(s), X(s-\tau), s) - f(X(s), X(s-\tau), s)|^2 ds.
\end{aligned} \tag{16}$$

By using Assumption 3.1, the Cauchy-Schwartz inequality and the Burkholder-Davis-Gundy inequality on the stochastic integral of G-Brownian motion (see, Lemma 2.3), we have

$$
\begin{aligned}
&\hat{\mathbb{E}}\sup_{v\in[0,t\wedge\tilde{\tau}_m]}\Big|\int_0^v (g_n(X^n(s),X^n(s-\tau),s)-g(X(s),X(s-\tau),s))d<B>(s)\Big|^2\\
&\le 2\hat{\mathbb{E}}\sup_{v\in[0,t\wedge\tilde{\tau}_m]}\Big|\int_0^v (g_n(X^n(s),X^n(s-\tau),s)-g_n(X(s),X(s-\tau),s))d<B>(s)\Big|^2\\
&\quad+2\hat{\mathbb{E}}\sup_{v\in[0,t\wedge\tilde{\tau}_m]}\Big|\int_0^v (g_n(X(s),X(s-\tau),s)-g(X(s),X(s-\tau),s))d<B>(s)\Big|^2\\
&\le 4C_1K^2T\hat{\mathbb{E}}\sup_{v\in[0,t\wedge\tilde{\tau}_m]}\int_0^v (|X^n(s)-X(s)|^2+|X^n(s-\tau)-X(s)|^2)ds\\
&\quad+2C_1T\hat{\mathbb{E}}\sup_{v\in[0,t\wedge\tilde{\tau}_m]}\int_0^v |g_n(X(s),X(s-\tau),s)-g(X(s),X(s-\tau),s)|^2ds
\end{aligned}
$$

$$
\begin{aligned}
&\le 8C_1K^2T\hat{\mathbb{E}}\sup_{v\in[0,t\wedge\tilde{\tau}_m]}\int_0^v |X^n(s)-X(s)|^2ds\\
&\quad+2C_1T\hat{\mathbb{E}}\sup_{v\in[0,t\wedge\tilde{\tau}_m]}\int_0^v |g_n(X(s),X(s-\tau),s)-g(X(s),X(s-\tau),s)|^2ds
\end{aligned}
\tag{17}
$$

and

$$
\begin{aligned}
&\hat{\mathbb{E}}\sup_{v\in[0,t\wedge\tilde{\tau}_m]}\Big|\int_0^v (h_n(X^n(s),X^n(s-\tau),s)-h(X(s),X(s-\tau),s))dB(s)\Big|^2\\
&\le 2\hat{\mathbb{E}}\sup_{v\in[0,t\wedge\tilde{\tau}_m]}\Big|\int_0^v (h_n(X^n(s),X^n(s-\tau),s)-h_n(X(s),X(s-\tau),s))dB(s)\Big|^2\\
&\quad+2\hat{\mathbb{E}}\sup_{v\in[0,t\wedge\tilde{\tau}_m]}\Big|\int_0^v (h_n(X(s),X(s-\tau),s)-h(X(s),X(s-\tau),s))dB(s)\Big|^2\\
&\le 4C_2K^2\hat{\mathbb{E}}\sup_{v\in[0,t\wedge\tilde{\tau}_m]}\int_0^v (|X^n(s)-X(s)|^2+|X^n(s-\tau)-X(s)|^2)ds\\
&\quad+2C_2\hat{\mathbb{E}}\sup_{v\in[0,t\wedge\tilde{\tau}_m]}\int_0^v |h_n(X(s),X(s-\tau),s)-h(X(s),X(s-\tau),s)|^2ds\\
&\le 8C_2K^2\hat{\mathbb{E}}\sup_{v\in[0,t\wedge\tilde{\tau}_m]}\int_0^v |X^n(s)-X(s)|^2ds\\
&\quad+2C_2\hat{\mathbb{E}}\sup_{v\in[0,t\wedge\tilde{\tau}_m]}\int_0^v |h_n(X(s),X(s-\tau),s)-h(X(s),X(s-\tau),s)|^2ds.
\end{aligned}
\tag{18}
$$

Thus, from (16)-(18), we have, for $t \in [0,T]$,

$$
\begin{aligned}
&\hat{\mathbb{E}}\Big[\sup_{v\in[0,t\wedge\tilde{\tau}_m]}|X^n(v)-X(v)|^2\Big]\\
&\le 3\hat{\mathbb{E}}\sup_{v\in[0,t\wedge\tilde{\tau}_m]}\Big|\int_0^v(f_n(X^n(s),X^n(s-\tau),s)-f(X(s),X(s-\tau),s))ds\Big|^2\\
&\quad+3\hat{\mathbb{E}}\sup_{v\in[0,t\wedge\tilde{\tau}_m]}\Big|\int_0^v(g_n(X^n(s),X^n(s-\tau),s)-g_n(X(s),X(s-\tau),s))d<B>(s)\Big|^2\\
&\quad+3\hat{\mathbb{E}}\sup_{v\in[0,t\wedge\tilde{\tau}_m]}\Big|\int_0^v(h_n(X^n(s),X^n(s-\tau),s)-h_n(X(s),X(s-\tau),s))dB(s)\Big|^2\\
&\le 24K^2(T+C_1T+C_2)\hat{\mathbb{E}}\sup_{v\in[0,t\wedge\tilde{\tau}_m]}\int_0^v|X^n(s)-X(s)|^2ds\\
&\quad+6T\hat{\mathbb{E}}\sup_{v\in[0,t\wedge\tilde{\tau}_m]}\int_0^v|f_n(X(s),X(s-\tau),s)-f(X(s),X(s-\tau),s)|^2ds\\
&\quad+6C_1T\hat{\mathbb{E}}\sup_{v\in[0,t\wedge\tilde{\tau}_m]}\int_0^v|g_n(X(s),X(s-\tau),s)-g(X(s),X(s-\tau),s)|^2ds\\
&\quad+6C_2\hat{\mathbb{E}}\sup_{v\in[0,t\wedge\tilde{\tau}_m]}\int_0^v|h_n(X(s),X(s-\tau),s)-h(X(s),X(s-\tau),s)|^2ds\\
&=: D_1^n+D_2\int_0^t\hat{\mathbb{E}}\Big[\sup_{r\in[0,s\wedge\tilde{\tau}_m]}|X^n(r)-X(r)|^2\Big]ds,
\end{aligned}
$$

where

$$
\begin{aligned}
D_1^n :=&6T\hat{\mathbb{E}}\sup_{v\in[0,t\wedge\tilde{\tau}_m]}\int_0^v|f_n(X(s),X(s-\tau),s)-f(X(s),X(s-\tau),s)|^2ds\\
&+6C_1T\hat{\mathbb{E}}\sup_{v\in[0,t\wedge\tilde{\tau}_m]}\int_0^v|g_n(X(s),X(s-\tau),s)-g(X(s),X(s-\tau),s)|^2ds\\
&+6C_2\hat{\mathbb{E}}\sup_{v\in[0,t\wedge\tilde{\tau}_m]}\int_0^v|h_n(X(s),X(s-\tau),s)-h(X(s),X(s-\tau),s)|^2ds,\\
D_2 :=&24K^2(T+C_1T+C_2).
\end{aligned}
$$

By the Gronwall inequality we obtain

$$\hat{\mathbb{E}}\Big[\sup_{v\in[0,t\wedge\tilde{\tau}_m]}|X^n(v)-X(v)|^2\Big]\le D_1^n e^{D_2T}.$$

Hence, from condition(14), we know that $D_1^n\to 0$ as $n\to\infty$. Thus, we have (15) by letting $m\to\infty$. Therefore, the proof is complete. □

Now we discuss a stability of the solution with respect to initial data. This kind of stability ensures that in the case of replacement of ξ by its approximate data η, the solution of equation with initial data η does not differ much from the solution of equation with initial data ξ. We will show that such the property holds for strong solutions of the G-SDDEs. We denote the strong solutions to the following G-SDDEs by $X(\cdot), Y(\cdot)$

$$\begin{aligned} dX(t) =& f(X(t), X(t-\tau), t)dt + g(X(t), X(t-\tau), t)d < B > (s) \\ &+ h(X(t), X(t-\tau), t)dB(t), \quad t \in [0, \infty), \end{aligned} \tag{19}$$

$$\begin{aligned} dY(t) =& f(Y(t), Y(t-\tau), t)dt + g(Y(t), Y(t-\tau), t)d < B > (s) \\ &+ h(Y(t), Y(t-\tau), t)dB(t), \quad t \in [0, \infty) \end{aligned} \tag{20}$$

with the nonrandom initial data $\xi, \eta \in C([-\tau, 0])$, respectively.

Theorem 4.2. *Let $f, g, h : \mathbb{R} \times \mathbb{R} \times [0, T] \to \mathbb{R}$ satisfy Assumption 3.1. Then, for the solutions $X, Y : [-\tau, T] \times \Omega \to \mathbb{R}$ of the equations (19) and (20) we have*

$$\hat{\mathbb{E}} \sup_{s \in [0, t \wedge \sigma_m]} |X(s) - Y(s)|^2 \leq D(\xi, \eta, m) e^{D(m)t}, \quad \forall t \in [0, T], \tag{21}$$

where $D(\xi, \eta, m) := (4 + 8\tau K_m^2(t + C_1 t + C_2))\|\xi - \eta\|^2, D(m) := 16K_m^2(t + C_1 t + C_2), \sigma_m := \inf\{t; |X(t)| \vee |Y(t)| \geq m\}$. Moreover, if $\xi = \eta$, then $X(t) = Y(t)$ q.s., $\forall t \in [0, T]$.

Proof. By Theorem 3.3, the solutions X to (19) and Y to (20) exist and are unique. For $t \geq 0$, by using Assumption 3.1, the Cauchy-Schwartz inequality and the Burkholder-Davis-Gundy inequality on the stochastic integral of G-Brownian motion (see, Lemma 2.3), and from (19)-(20), we have

$$\begin{aligned} &\hat{\mathbb{E}}\Big[\sup_{v \in [0, t \wedge \sigma_m]} |X(v) - Y(v)|^2\Big] \\ &\leq 4\|\xi - \eta\|^2 + 4\hat{\mathbb{E}} \sup_{v \in [0, t \wedge \sigma_m]} \Big| \int_0^v (f(X(s), X(s-\tau), s) - f(Y(s), Y(s-\tau), s))ds\Big|^2 \end{aligned}$$

$$
\begin{aligned}
&+4\hat{\mathbb{E}} \sup_{v\in[0,t\wedge\sigma_m]} \Big| \int_0^v (g(X(s),X(s-\tau),s)-g(X(s),X(s-\tau),s))d<B>(s)\Big|^2 \\
&+4\hat{\mathbb{E}} \sup_{v\in[0,t\wedge\sigma_m]} \Big| \int_0^v (h(X(s),X(s-\tau),s)-h(X(s),X(s-\tau),s))dB(s)\Big|^2 \\
\le\, & 4\|\xi-\eta\|^2 + 8K_m^2 t\hat{\mathbb{E}} \sup_{v\in[0,t\wedge\sigma_m]} \int_0^v (|X(s)-Y(s)|^2+|X(s-\tau)-Y(s-\tau)|^2)ds \\
&+8K_m^2 tC_1\hat{\mathbb{E}} \sup_{v\in[0,t\wedge\sigma_m]} \int_0^v (|X(s)-Y(s)|^2+|X(s-\tau)-Y(s-\tau)|^2)ds \\
&+8K_m^2 C_2\hat{\mathbb{E}} \sup_{v\in[0,t\wedge\sigma_m]} \int_0^v (|X(s)-Y(s)|^2+|X(s-\tau)-Y(s-\tau)|^2)ds \\
\le\, & 4\|\xi-\eta\|^2 + 8K_m^2(t+tC_1+C_2)\int_{-\tau}^0 |\xi(s)-\eta(s)|^2ds \\
&+16K_m^2(t+C_1t+C_2)\int_0^t \hat{\mathbb{E}}\Big[\sup_{v\in[0,s\wedge\sigma_m]} |X(v)-Y(v)|^2\Big]ds \\
\le\, & D(\xi,\eta,m)+D(m)\int_0^t \hat{\mathbb{E}}\Big[\sup_{v\in[0,s\wedge\sigma_m]} |X(v)-Y(v)|^2\Big]ds.
\end{aligned}
$$

Again by the Gronwall inequality, we obtain (21). Especially, if $\xi=\eta$, then $X(t)=Y(t)$ $q.s.$, $\forall t\in[0,\infty)$. Thus the proof is complete. □

§5 Stability and boundedness of solutions

In this section, let $T=\infty$. We shall discuss the existence and uniqueness, the stability and boundedness of solutions to highly nonlinear G-SDDEs.

In condition (9), we have either $q_1>1$, $q_2>1$ or $q_3>1$. It is known that the Lipschitz condition in Assumption 3.1 only guarantees that the G-SDDE (1) with the initial data (2) has a unique maximal solution, which may explode to infinity at a finite time. To avoid such a possible explosion, we need to impose an additional condition by Lyapunov functions. To this end, we need more notations. We denote $C^{2,1}(\mathbb{R}\times\mathbb{R}_+;\mathbb{R}_+)$ as the family of non-negative functions $U(x,t)$ defined on $(x,t)\in\mathbb{R}\times\mathbb{R}_+$ which are continuously twice differentiable in x and once in t. We can now state another assumption.

Assumption 5.1. *Let $H(\cdot)\in C(\mathbb{R}\times\mathbb{R}_+;\mathbb{R}_+)$. Assume that there exists a function $U\in C^{2,1}(\mathbb{R}\times\mathbb{S}\times\mathbb{R}_+;\mathbb{R}_+)$, $q\ge 2(q_1\vee q_2\vee q_3)$, and nonnegative constants c_0,c_1,c_2 such that*

$$c_2<c_1,\quad x^q\le U(x,t)\le H(x,t) \tag{22}$$

and

$$
\begin{aligned}
\mathbb{L}U(x,t) :=& U_t(x,t)+U_x(x,t)f(x,y,t)+G(2g(x,y,t)U_x(x,t)+h^2(x,y,t)U_{xx}(x,y,t)) \\
\le& c_0-c_1H(x,t)+c_2H(y,t-\tau)
\end{aligned} \tag{23}
$$

for all $x \in \mathbb{R}, t \in \mathbb{R}_+$.

Theorem 5.2. *Under Assumption 3.1 with (4) replaced by (9), and Assumption 5.1, the G-SDDE (1) with the initial data (2) has the following assertions:*

(i) *For the G-SDDE (1) with the initial data (2), there exists a unique global solution on* $[-\tau, \infty)$.

(ii) *The solution X(t) obeys*

$$\limsup_{t\to\infty} \hat{\mathbb{E}}|X(t)|^q \le \frac{c_0}{\varepsilon}, \tag{24}$$

$$\sup_{-\tau\le t<\infty} \hat{\mathbb{E}}|X(t)|^q < H(\xi(0),0) + c_2 e^{\varepsilon\tau}\int_{-\tau}^{0} e^{\varepsilon s} H(\xi(s),s)ds + \frac{c_0}{\varepsilon} < \infty \tag{25}$$

and

$$\limsup_{t\to\infty} \frac{1}{t}\mathcal{E}\Big[\int_0^t H(X(s),s)ds\Big] \le \frac{c_0}{c_1 - c_2}, \tag{26}$$

where $\varepsilon > 0$ *is the unique root to the equation*

$$c_1 = \varepsilon + c_2 e^{\varepsilon\tau}. \tag{27}$$

(iii) *If, in addition,* $c_0 = 0$, *then the solution has the moment properties that*

$$\limsup_{t\to\infty} \frac{1}{t}\log(\hat{\mathbb{E}}|X(t)|^q) \le -\varepsilon \tag{28}$$

and

$$\mathcal{E}\Big[\int_0^\infty H(X(t),t)dt\Big] \le \frac{1}{c_1 - c_2}\Big(H(\xi(0),0) + \int_{-\tau}^{0} H(\xi(s),s)ds\Big); \tag{29}$$

while it also has the sample (pathwise) property that

$$\int_0^\infty H(X(t),t)dt < \infty \quad \mathbb{V}\text{-}a.s. \tag{30}$$

Proof. We prove our claims by three steps.

Step 1. Global solution. From Theorem 3.4, we have verified that for the G-SDDE (1) with the coefficients being locally Lipschitz continuous and any given initial data (2), there is a unique maximal local solution $X(t)$ for $\forall t \in [-\tau, \sigma_\infty)$, where σ_∞ is the explosion time. Let $m_0 > 0$ be sufficiently large for $m_0 \ge \|\xi\|$. For each integer $m \ge m_0$, define the stopping time $\check{\tau}_m = \inf\{t \in [0, \sigma_\infty) : |X(t)| \ge m\}$, where, throughout this paper, $\inf \emptyset = \infty$. Clearly, $\check{\tau}_m$ is increasing as $m \to \infty$. Set $\check{\tau}_\infty = \lim_{m\to\infty} \check{\tau}_m$, whence $\check{\tau}_\infty \le \sigma_\infty$ q.s. If we can show that $\check{\tau}_\infty = \infty$ q.s., then $\sigma_\infty = \infty$ q.s., i.e. the global solution exists. Next, we will show that $\check{\tau}_\infty = \infty$ q.s. By the Itô formula (see, e.g., [25]) and condition (23), we can show that, for any $m \ge m_0$ and $t \ge 0$,

$$\begin{aligned}
&\hat{\mathbb{E}}\bar{U}(X(\check{\tau}_m \wedge t), \check{\tau}_m \wedge t) - \bar{U}(X(0),0)\\
&\le \hat{\mathbb{E}}\int_0^{\check{\tau}_m\wedge t} \Big(c_0 - c_1 H(X(s),s) + c_2 H(X(s-\tau), s-\tau)\Big)ds.
\end{aligned}$$

However, we deduce

$$\begin{aligned}&\int_0^{\check{\tau}_m\wedge t} e^{\varepsilon s}H(X(s-\tau),s-\tau)ds\\&\le e^{\varepsilon\tau}\int_{-\tau}^{0} e^{\varepsilon s}H(\xi(s),s)ds+e^{\varepsilon\tau}\int_0^{\check{\tau}_m\wedge t} e^{\varepsilon s}H(X(s),s)ds.\end{aligned}\tag{31}$$

Thus, from (31) with $\varepsilon=0$, we know

$$\hat{\mathbb{E}}|X(\check{\tau}_m\wedge t),t)|^q\le K_1+c_0t+(c_1-c_2)\hat{\mathbb{E}}\Big[-\int_0^{\check{\tau}_m\wedge t}H(X(s),s)ds\Big],\tag{32}$$

where $K_1=H(\xi(0),0)+\int_{-\tau}^{0}H(\xi(s),s)ds$. Noting that

$$c_1>c_2,\ \hat{\mathbb{E}}\Big[-\int_0^{\check{\tau}_m\wedge t}H(X(s),s)ds\Big]\le 0,$$

we get, from (32),

$$\hat{\mathbb{E}}|X(\check{\tau}_m\wedge t),\check{\tau}_m\wedge t)|^q\le K_1+c_0t.$$

Then we have $m^qP(\check{\tau}_m\le t)\le K_1+c_0t$ for each $P\in\mathcal{P}$. Therefore, letting $m\to\infty$ in the inequality above, we have $P(\check{\tau}_\infty\le t)=0$, which shows $P(\check{\tau}_\infty>t)=1$. Due to arbitrariness of $t\ge 0$, we must have $\mathcal{V}(\check{\tau}_\infty=\infty)=\min_{P\in\mathcal{P}}P(\check{\tau}_\infty=\infty)=1$ as required. That is, $\check{\tau}_\infty=\infty$ q.s. as required.

Step 2. Asymptotic boundedness. By the Itô formula (see, e.g., [25]) and Assumption 5.1, we obtain that for $t\ge 0$,

$$\begin{aligned}&\hat{\mathbb{E}}\Big[e^{\varepsilon(\check{\tau}_m\wedge t)}U(X(\check{\tau}_m\wedge t),\check{\tau}_m\wedge t)\Big]-U(X(0),0)\\&=\hat{\mathbb{E}}\Big[\int_0^{\check{\tau}_m\wedge t}e^{\varepsilon s}[U_t(X(s),s)+\varepsilon U(X(s),s)+U_x(X(s),s)f(X(s),X(s-\tau),s)]ds\\&\quad+e^{\varepsilon s}[U_x(X(s),s)g(X(s),X(s-\tau),s)\\&\quad+\frac{1}{2}U_{xx}(X(s),s)h^2(X(s),X(s-\tau),s)]d<B>(s)\Big]\\&\le\hat{\mathbb{E}}\Big[\int_0^{\check{\tau}_m\wedge t}e^{\varepsilon s}[\varepsilon U(X(s),s)+\mathbb{L}U(X(s),X(s-\tau),s)]ds\Big]\end{aligned}\tag{33}$$

$$\begin{aligned}&\le\hat{\mathbb{E}}\Big[\int_0^{\check{\tau}_m\wedge t}e^{\varepsilon s}\Big(c_0-(c_1-\varepsilon)H(X(s),s)+c_2H(X(s-\tau),s-\tau)\Big)ds\Big]\\&\le\frac{c_0}{\varepsilon}e^{\varepsilon t}+\hat{\mathbb{E}}\Big[-(c_1-\varepsilon)\int_0^{\check{\tau}_m\wedge t}e^{\varepsilon s}H(X(s),s)ds\\&\quad+c_2\int_0^{\check{\tau}_m\wedge t}e^{\varepsilon s}H(X(s-\tau),s-\tau)ds\Big].\end{aligned}\tag{34}$$

Thus, we get from (31), (22) and (27) that

$$\hat{\mathbb{E}}\Big[e^{\varepsilon(\check{\tau}_m\wedge t)}|X(\check{\tau}_m\wedge t)|^q\Big]\le\hat{\mathbb{E}}\Big[e^{\varepsilon(\check{\tau}_m\wedge t)}U(X(\check{\tau}_m\wedge t),\check{\tau}_m\wedge t)\Big]\le\tilde{K}+\frac{c_0}{\varepsilon}e^{\varepsilon t},$$

where $\tilde{K}=H(\xi(0),0)+c_2e^{\varepsilon\tau}\int_{-\tau}^{0}e^{\varepsilon s}H(\xi(s),s)ds$. Letting $m\to\infty$ we get that

$$\hat{\mathbb{E}}\Big[e^{\varepsilon t}|X(t)|^q\Big]\le\tilde{K}+\frac{c_0}{\varepsilon}e^{\varepsilon t}.\tag{35}$$

Dividing both sides in (35) by $e^{\varepsilon t}$, we easily obtain the desired assertions (24) and (25).

In order to show (26), from (32) we know that

$$-(c_1-c_2)\hat{\mathbb{E}}\Big[-\int_0^{\tilde{\tau}_m\wedge t} H(X(s),s)ds\Big]\le K_1+c_0t.$$

Thus, letting $m\to\infty$, we get

$$-(c_1-c_2)\hat{\mathbb{E}}\Big[-\int_0^{t} H(X(s),s)ds\Big]\le K_1+c_0t. \tag{36}$$

Dividing both sides in (36) by t and letting $t\to\infty$ we get the assertion (26).

Step 3. Asymptotic stability. Now we consider the case when $c_0=0$. It then follows from (35) that

$$\hat{\mathbb{E}}|X(t)|^q\le \tilde{K}e^{-\varepsilon t},$$

which verifies the required assertion (28). Moreover, from (36), we get

$$(c_1-c_2)\mathcal{E}\Big[\int_0^{t} H(X(s),s)ds\Big]\le K_1.$$

Thus we easily shows (29), which implies (30) by the notion of sublinear lower expectation $\mathcal{E}$. Hence we complete the proof. □

§6 Conclusion

In real systems, we are often faced with two kinds of uncertainties: probability and Knightian ones. By using Peng's sublinear expectation framework, we can characterize the systems with ambiguity. This paper gives a description of the uncertain delay system through G-Brownian motion. We first prove the existence and uniqueness of the solution to G-SDDEs under sublinear expectation with the local Lipschitz and linear growth conditions. Then the second kind of stability and dependence of the solution to G-SDDEs are studied. Finally, we try to give the characterization of stability and boundedness of the solutions to the highly nonlinear G-SDDEs. In this paper, our main contributions are presented as follows: (i) we prove the existence and uniqueness of the solution to G-SDDEs by mathematical technique; (ii) the second kind of stability of the solution to G-SDDEs is also investigated under the linear growth condition; (iii) the stability and boundedness of the solution to the highly nonlinear G-SDDEs are discussed. Our study provides a new perspective which has a close link with the highly nonlinear stochastic delay differential equations under sublinear expectations. In addition, our results on G-SDDEs reduce to the ones of G-SDEs as the time lag is zero. Finally, we note that the guaranteed cost control for nonlinear stochastic systems under the classical probability is studied by Ma *et al.* [22] and Shen *et al.* [28]. We can extend the analysis of stability, the state-estimation and the stochastic control problem for uncertain nonlinear systems under a classical probability to those under the sublinear expectation framework.

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