Appl. Math. J. Chinese Univ.
2019, 34(2): 173-183

The geometrical properties of parity and time reversal operators in two dimensional spaces

HUANG Min-yi [1] YANG Yu [2] WU Jun-de [1] CHO Min-Hyung [3*]

Abstract. The parity operator $\mathcal{P}$ and time reversal operator $\mathcal{T}$ are two important operators in the quantum theory, in particular, in the $\mathcal{PT}$-symmetric quantum theory. By using the concrete forms of $\mathcal{P}$ and $\mathcal{T}$, we discuss their geometrical properties in two dimensional spaces. It is showed that if $\mathcal{T}$ is given, then all $\mathcal{P}$ links with the quadric surfaces; if $\mathcal{P}$ is given, then all $\mathcal{T}$ links with the quadric curves. Moreover, we give out the generalized unbroken $\mathcal{PT}$-symmetric condition of an operator. The unbroken $\mathcal{PT}$-symmetry of a Hermitian operator is also showed in this way.

§1 Introduction

Quantum theory is one of the most important theories in physics. It is a fundamental axiom in quantum mechanics that the Hamiltonians should be Hermitian, which implies that the values of energy are real numbers. However, non-Hermitian Hamiltonians are also studied in physics. One of the attempts is Bender's $\mathcal{PT}$-symmetric theory [1]. In this theory, Bender and his colleagues attributed the reality of the energies to the $\mathcal{PT}$-symmetric property, where $\mathcal{P}$ is a parity operator and $\mathcal{T}$ is a time reversal operator. Since then, many physicists discussed the properties of $\mathcal{PT}$-symmetric quantum systems [2]. It also has theoretical applications in quantum optics, quantum statistics and quantum field theory [4,5,9,10]. Recently, Bender, Brody and Muller constructed a Hamiltonian operator H with the property that if its eigenfunctions obey a suitable boundary condition, then the associated eigenvalues correspond to the nontrivial zeros of the Riemann zeta function, where H is not Hermitian in the conventional sense, while iH has a broken $\mathcal{PT}$-symmetry. This result may shed light on the new application of $\mathcal{PT}$-symmetric theory in discussing the Riemann hypothesis [3]. It was discovered by Mostafazadeh that the $\mathcal{PT}$-symmetric case can be generalized to a more general pseudo-Hermitian quantum theory, and the generalized $\mathcal{PT}$- symmetry was also discussed [6,8]. Smith studied the time

Received: 2017-08-13. Revised:2018-06-29.
MR Subject Classification: 47B37.
Keywords: $\mathcal{PT}$-symmetry, geometrical property, unbroken condition.
Digital Object Identifier(DOI):https://doi.org/ 10.1007/s11766-017-3568-6.
Supported by Research Fund, Kumoh National Institute of Technology.
*Corresponding author.

reversal operator $\mathcal{T}$ satisfying that $\mathcal{T}^2 = -I$ and the corresponding $\mathcal{PT}$-symmetric quantum theory [10].

In this paper, by using the concrete forms of $\mathcal{P}$ and $\mathcal{T}$ in two dimensional spaces, we discuss their geometry properties. It is showed that if $\mathcal{T}$ is given, then all $\mathcal{P}$ links with the quadric surfaces; if $\mathcal{P}$ is given, then all $\mathcal{T}$ links with the quadric curves. Moreover, we give out the generalized unbroken $\mathcal{PT}$-symmetric condition of an operator H. The unbroken $\mathcal{PT}$-symmetry of a Hermitian operator is also showed in this way.

§2 Preliminaries

In this paper, we only consider finite dimensional complex Hilbert space $\mathbb{C}^n$, whose elements will be denoted by bold fonts. Let $L(\mathbb{C}^n)$ be the complex vector space of all linear operators on $\mathbb{C}^n$, I be the identity operator on $\mathbb{C}^n$, $\overline{z}$ be the complex conjugation of complex number z.

An operator $\mathcal{T}$ on $\mathbb{C}^2$ is said to be anti-linear if $\mathcal{T}(s\mathbf{u}_1+t\mathbf{u}_2) = \overline{s}\mathcal{T}(\mathbf{u}_1)+\overline{t}\mathcal{T}(\mathbf{u}_2)$. It is obvious that the composition of two anti-linear operators is a linear operator and the composition of an anti-linear operator and a linear operator is still anti-linear. Similar to linear operators, anti-linear operators can also correspond to a matrix with slightly different laws of operation [11].

A time reversal operator $\mathcal{T}$ is an anti-linear operator which satisfies $\mathcal{T}^2 = I$ or $\mathcal{T}^2 = -I$. A parity operator $\mathcal{P}$ is a linear operator which satisfies $\mathcal{P}^2 = I$ [6,8,10,12].

The Pauli operators will be used frequently in our discussions. Given the basis $\{\mathbf{e}_i\}_{i=1}^2$ of $\mathbb{C}^2$, they are usually defined as follows [7]:

$$\sigma_1(x_1\mathbf{e}_1 + x_2\mathbf{e}_2) = x_2\mathbf{e}_1 + x_1\mathbf{e}_2, \tag{1}$$

$$\sigma_2(x_1\mathbf{e}_1 + x_2\mathbf{e}_2) = -ix_2\mathbf{e}_1 + ix_1\mathbf{e}_2, \tag{2}$$

$$\sigma_3(x_1\mathbf{e}_1 + x_2\mathbf{e}_2) = x_1\mathbf{e}_1 - x_2\mathbf{e}_2. \tag{3}$$

To put it another way, the representation matrices of σ_1, σ_2 and σ_3 are:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Pauli operators have the following useful properties [7]:

$$\sigma_i\sigma_j = -\sigma_j\sigma_i = i\epsilon_{ijk}\sigma_k, \quad i \neq j, \tag{4}$$

$$\sigma_i^2 = I, \tag{5}$$

where $i, j, k \in \{1, 2, 3\}$, ϵ_{ijk} is the Levi-Civita symbol:

$$\epsilon_{ijk} = \begin{cases} \epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1, \\ \epsilon_{132} = \epsilon_{213} = \epsilon_{321} = -1, \\ 0, otherwise. \end{cases}$$

The well known commutation and anti-commutation relations are:

$$\sigma_i\sigma_j - \sigma_j\sigma_i = 2i\epsilon_{ijk}\sigma_k,$$

$$\sigma_i\sigma_j + \sigma_j\sigma_i = 2\delta_{ij}I,$$

where $i, j, k \in \{1, 2, 3\}$ and δ_{ij} is the Kronecker symbol.

Denote I by σ_0, then $\{\sigma_0, \sigma_1, \sigma_2, \sigma_3\}$ is a basis of $L(\mathbb{C}^2)$. Moreover, an operator $\sigma = t\sigma_0 + x\sigma_1 + y\sigma_2 + z\sigma_3 \in L(\mathbb{C}^2)$ is Hermitian if and only if the coefficients $\{t, x, y, z\}$ are real numbers.

Given the basis $\{\mathbf{e}_i\}_{i=1}^2$ of $\mathbb{C}^2$ and any vector $x = \sum x_i\mathbf{e}_i$, one can define an important anti-linear operator, namely the conjugation operator $\mathcal{T}_0$, by $\mathcal{T}_0(x) = \sum \overline{x_i}\mathbf{e}_i$.

Similar to $\mathcal{T}_0$, one can define another important anti-linear operator τ_0 by

$$\tau_0(x_1\mathbf{e}_1 + x_2\mathbf{e}_2) = -\overline{x_2}\mathbf{e}_1 + \overline{x_1}\mathbf{e}_2. \tag{6}$$

Furthermore, define $\tau_1 = \tau_0\sigma_1, \tau_2 = \tau_0\sigma_2, \tau_3 = \tau_0\sigma_3$, that is, τ_i is defined to be the composition of τ_0 and σ_i. The anti-linear operators $\{\tau_0, \tau_1, \tau_2, \tau_3\}$ forms a basis of the anti-linear operator space of $\mathbb{C}^2$. This basis has the following properties [11]:

$$\begin{aligned}
&\tau_0^2 = -I,\\
&\tau_0\sigma_i = -\sigma_i\tau_0 = \tau_i,\\
&\tau_i\tau_0 = -\tau_0\tau_i = \sigma_i,\\
&\tau_i\tau_j = \sigma_i\sigma_j = i\epsilon_{ijk}\sigma_k \quad (i \neq j),\\
&\tau_i\tau_j - \tau_j\tau_i = 2i\epsilon_{ijk}\sigma_k,
\end{aligned}$$

where $i, j \in \{1, 2, 3\}$.

All the equations above can be verified by the using the definitions of Pauli operators and τ_0. However, for further use, we show that $\tau_0\sigma_i = -\sigma_i\tau_0 = \tau_i$ in detail. Consider $\tau_2 = \tau_0\sigma_2$. By (2) and (6), we have

$$\begin{aligned}
\tau_0\sigma_2(x_1\mathbf{e}_1 + x_2\mathbf{e}_2) &= i\overline{x_1}\mathbf{e}_1 + i\overline{x_2}\mathbf{e}_2,\\
\sigma_2\tau_0(x_1\mathbf{e}_1 + x_2\mathbf{e}_2) &= -i\overline{x_1}\mathbf{e}_1 - i\overline{x_2}\mathbf{e}_2.
\end{aligned}$$

Thus, $\tau_0\sigma_2 = -\sigma_2\tau_0 = \tau_2$. Along similar lines, one can verify that $\tau_0\sigma_i = -\sigma_i\tau_0 = \tau_i$ is also valid for σ_1 and σ_3.

Moreover, it follows from $\tau_0\sigma_i = -\sigma_i\tau_0 = \tau_i$ that $\sigma_j\tau_i = \sigma_j\tau_0\sigma_i = -\tau_0\sigma_j\sigma_i$. Combining with (4) and (5), one can further obtain the following relations:

$$\sigma_j\tau_i = \tau_i\sigma_j = -i\epsilon_{ijk}\tau_k, \quad i \neq j, \tag{7}$$

$$\tau_i\sigma_i = -\sigma_i\tau_i = \tau_0, \tag{8}$$

where $i, j, k \in \{1, 2, 3\}$.

With the help of $\{\sigma_i\}$ and $\{\tau_i\}$, ones can determine the concrete forms of $\mathcal{P}$ and $\mathcal{T}$:

Lemma 2.1. *Let $\mathcal{P}$ be a parity operator and $\mathcal{T}$ be a time reversal operator on $\mathbb{C}^2$. Then*

(i). Either $\mathcal{P} = \pm I$ or $\mathcal{P} = \sum_{i=1}^{3} a_i\sigma_i$, where a_i satisfying $\sum_{i=1}^{3} a_i^2 = 1$. The latter case is referred to as the nontrivial $\mathcal{P}$. A nontrivial $\mathcal{P}$ has the following matrix representation:

$$P = \begin{pmatrix} a_3 & a_1 - ia_2 \\ a_1 + ia_2 & -a_3 \end{pmatrix}. \tag{9}$$

(ii). $\mathcal{T} = \epsilon \sum_{i=0}^{3} c_i \tau_i$, *where* c_i *are real numbers, if* $\mathcal{T}^2 = I$, *then* $\sum_{i=1}^{3} c_i^2 - c_0^2 = 1$; *if* $\mathcal{T}^2 = -I$, *then* $\sum_{i=1}^{3} c_i^2 - c_0^2 = -1$, ϵ *is a unimodular complex number [11].*

Proof. (i). Suppose $\mathcal{P} = \sum_{i=0}^{3} a_i \sigma_i$. According to the properties of Pauli operators, we have $I = \mathcal{P}^2 = (\sum_{i=0}^{3} a_i^2)I + 2a_0(\sum_{i=1}^{3} a_i \sigma_i)$. Note that $\{\sigma_0, \sigma_1, \sigma_2, \sigma_3\}$ is a basis of $L(\mathbb{C}^2)$, we conclude that $\sum_{i=0}^{3} a_i^2 = 1$ and $a_0 a_1 = a_0 a_2 = a_0 a_3 = 0$. If $a_0 \neq 0$, then $a_1 = a_2 = a_3 = 0$, which implies that $\mathcal{P} = \pm I$. If $a_0 = 0$, then the only constraint is $\sum_{i=1}^{3} a_i^2 = 1$ and the matrix takes the form in (9).

(ii). The proof can be found in [11]. □

Example 1. In (9), if we take $a_2 = 0$, a_1, a_3 are real numbers satisfying that $a_1^2 + a_3^2 = 1$, and denote a_1 by $\sin\alpha$, a_3 by $\cos\alpha$, then $\mathcal{P}$ has the matrix representation $\begin{pmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Thus $\mathcal{P}$ is composed of a reflection and a rotation.

Example 2. In (9), if $a_1 = a_2 = 0$, $a_3 = 1$, then $P = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. If $a_2 = a_3 = 0$, $a_1 = 1$, then $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. These two parity operators were widely used in [2].

§3 The existence of $\mathcal{P}$ commuting with $\mathcal{T}$

In physics, it is demanded that $\mathcal{P}$ and $\mathcal{T}$ are commutative, that is, $\mathcal{PT} = \mathcal{TP}$. In finite dimensional spaces case, by using the canonical forms of matrices, one can show that if $\mathcal{T}^2 = I$, then such $\mathcal{P}$ always exists. In two dimensional case, we can prove it by utilizing Pauli operators.

Theorem 3.1. *For each time reversal operator* $\mathcal{T}$, *if* $\mathcal{T}^2 = I$, *then there exists a nontrivial parity operator* $\mathcal{P}$ *such that* $\mathcal{PT} = \mathcal{TP}$. *If* $\mathcal{T}^2 = -I$, *then there is no* $\mathcal{P}$ *commuting with* $\mathcal{T}$ *except* $\mathcal{P} = \pm I$.

Proof. We will use the following well known equation frequently,

$$(\sigma \cdot \mathbf{A})(\sigma \cdot \mathbf{B}) = (\mathbf{A} \cdot \mathbf{B})I + i\sigma \cdot (\mathbf{A} \times \mathbf{B}), \tag{10}$$

where the bold letters $\mathbf{A}$ and $\mathbf{B}$ denote vectors in $\mathbb{C}^3$ and $\sigma = (\sigma_1, \sigma_2, \sigma_3)$. The symbols $\cdot$ and $\times$ represent the dot and cross product of vectors, respectively.

(i). When $\mathcal{T}^2 = I$.

Let $\mathcal{T} = \epsilon \sum_{i=0}^{3} c_i \tau_i$ and $\mathcal{P} = \sum_{i=1}^{3} a_i \sigma_i$, as was given in Lemma 2.1.

According to (7) and (8), $\mathcal{TP} = \mathcal{PT}$ is equivalent to

$$(-c_0\sigma_0 + \sum_{j=1}^{3} c_j\sigma_j)(\sum_{i=1}^{3} \overline{a_i}\sigma_i)\tau_0 = (\sum_{i=1}^{3} a_i\sigma_i)(c_0\sigma_0 - \sum_{j=1}^{3} c_j\sigma_j)\tau_0.$$

Denote $f_i = Re(a_i)$, $b_i = Im(a_i)$, $\mathbf{f} = (f_1, f_2, f_3)$, $\mathbf{b} = (b_1, b_2, b_3)$ and $\mathbf{c} = (c_1, c_2, c_3)$. Utilizing (10) to expand the equation above, we have

$$(\mathbf{f} \cdot \mathbf{c})\sigma_0 - \sigma \cdot [\mathbf{b} \times \mathbf{c} + c_0\mathbf{f}] = 0. \tag{11}$$

It follows that $\mathcal{TP} = \mathcal{PT}$ is equivalent to

$$c_0\mathbf{f} + \mathbf{b} \times \mathbf{c} = 0, \tag{12}$$

$$\mathbf{f} \cdot \mathbf{c} = 0. \tag{13}$$

Similarly, by utilizing (10) and Lemma 2.1, the contraints $\mathcal{P}^2 = I$ and $\mathcal{T}^2 = I$ can be reduced to the equations as follows,

$$\mathbf{f} \cdot \mathbf{b} = 0, \tag{14}$$

$$\|\mathbf{f}\|^2 - \|\mathbf{b}\|^2 = 1, \tag{15}$$

$$\|\mathbf{c}\|^2 - c_0^2 = 1. \tag{16}$$

Thus, the problem of finding a parity operator $\mathcal{P}$ commuting with $\mathcal{T}$ reduces to finding the vectors $\mathbf{f}$ and $\mathbf{b}$ satisfying (12) – (15).

If $c_0 = 0$, then we can choose $\mathbf{b} = 0$ and a unit vector $\mathbf{f}$ orthogonal to $\mathbf{c}$. Thus all the conditions (12) – (15) are satisfied.

If $c_0 \neq 0$. Let $\mathbf{b}$ be a vector such that $\mathbf{b}$ is orthogonal to $\mathbf{c}$ and $\|\mathbf{b}\| = |c_0|$. Moreover, take $\mathbf{f} = \frac{1}{c_0}(\mathbf{c} \times \mathbf{b})$. Direct calculations show that such vectors $\mathbf{f}$ and $\mathbf{b}$ satisfy (12) – (15), which completes the proof of the existence of $\mathcal{P}$.

(ii). When $\mathcal{T}^2 = -I$.

The equation (16) is replaced by the following:

$$\|\mathbf{c}\|^2 - c_0^2 = -1. \tag{17}$$

Thus $c_0 \neq 0$. On the other hand, it follows from (12) that

$$\mathbf{f} = \frac{1}{c_0}(\mathbf{c} \times \mathbf{b}). \tag{18}$$

Substituting (17) and (18) into (15), we have $\|\mathbf{f}\|^2 - \|\mathbf{b}\|^2 = 1 < -\frac{1}{c_0^2}\|\mathbf{b}\|^2$, which is a contradiction. Thus, when $\mathcal{T}^2 = -I$, there is no $\mathcal{P}$ commuting with $\mathcal{T}$ except $\mathcal{P} = \pm I$.

□

Remark 3.1. *When the space is $\mathbb{C}^4$, although $\mathcal{T}^2 = -I$, one can find nontrivial $\mathcal{P}$ commuting with $\mathcal{T}$ [10].*

§4 The geometrical properties of $\mathcal{P}$ and $\mathcal{T}$

Theorem 4.1. *Let $\mathcal{T}$ be a time reversal operator satisfying $\mathcal{T}^2 = I$. The set of parity operators $\mathcal{P}$ commuting with $\mathcal{T}$ correspond uniquely to a hyperboloid in $\mathbb{R}^3$.*

Proof. As was mentioned above, the determination of $\mathcal{P}$ is equivalent to finding out $\mathbf{f}$ and $\mathbf{b}$ satisfying (12) − (15). Now consider $\mathbf{m} = \mathbf{f} + \mathbf{b}$. We shall prove that all the $\mathbf{m}$ form a hyperboloid.

To this end, construct a new coordinate system by taking the direction of $\mathbf{c}$ as that of the X' axis. The $Y' - Z'$ plane is perpendicular to $\mathbf{c}$ and contains the origin point of $\mathbb{R}^3$. Assume that $\mathbf{m} = (x', y', z')$ in the new $X'Y'Z'$ coordinate system.

(i). If $c_0 = 0$, then it follows from (12) − (14) that $\mathbf{b}$ is proportional to $\mathbf{c}$ and that $\mathbf{f}$ is orthogonal to both $\mathbf{c}$ and $\mathbf{b}$. Thus, in the new $X'Y'Z'$ coordinate system,

$$\mathbf{b} = (x', 0, 0),$$
$$\mathbf{f} = (0, y', z').$$

On the other hand, equation (15), namely $\|\mathbf{f}\|^2 - \|\mathbf{b}\|^2 = 1$, implies that

$$y'^2 + z'^2 - x'^2 = 1. \tag{19}$$

It is apparent that one pair of $\mathbf{f}$ and $\mathbf{b}$ correspond to one point $\mathbf{m} = (x', y', z')$, and vice versa. In addition, (19) represents a hyperboloid in $\mathbb{R}^3$.

(ii). If $c_0 \neq 0$, then it follows from (16) that $\mathbf{c} = (\sqrt{1 + c_0^2}, 0, 0)$ in the $X'Y'Z'$ coordinate system. In addition, suppose $\mathbf{b} = (x_0, y_0, z_0)$ in the $X'Y'Z'$ coordinate system. By (18), we have $\mathbf{f} = \frac{1}{c_0}(\mathbf{c} \times \mathbf{b}) = \frac{\sqrt{1+c_0^2}}{c_0}(0, -z_0, y_0)$. Substituting $\mathbf{b}$ and $\mathbf{f}$ into (15), we have

$$\frac{1}{c_0^2}(y_0^2 + z_0^2) - x_0^2 = 1. \tag{20}$$

Note that $x_0 = x', y_0 = \frac{\lambda z' + y'}{1+\lambda^2}, z_0 = \frac{z' - \lambda y'}{1+\lambda^2}$, where $\lambda = \frac{\sqrt{1+c_0^2}}{c_0}$. Thus, one pair of $\mathbf{f}$ and $\mathbf{b}$ correspond to one point $\mathbf{m} = (x', y', z')$, and vice versa. Moreover, it follows from (20) that

$$\frac{1}{1 + 2c_0^2}(y'^2 + z'^2) - x'^2 = 1. \tag{21}$$

That is, all the $\mathbf{m}$ form a hyperboloid.

□

Theorem 4.2. *Let $\mathcal{P}$ be a nontrivial parity operator and let us consider the time reversal operators of the form $\mathcal{T} = \sum_{i=0}^{3} c_i \tau_i$ commuting with $\mathcal{P}$. All the points $\mathbf{c} = (c_1, c_2, c_3)$ form an ellipse. The length of the semi-major axis is $\|\boldsymbol{f}\|$ and the length of the semi-minor axis is 1.*

Proof. By (13) and (14), we know that both $\mathbf{b}$ and $\mathbf{c}$ are orthogonal to $\mathbf{f}$.

Construct a new $X'Y'Z'$ coordinate system by taking the direction of $\mathbf{f}$ as that of the Z' axis and the direction of $\mathbf{b}$ as that of the X' axis (If $\mathbf{b} = 0$, take any vector orthogonal to $\mathbf{f}$ as the direction vector of the X' axis). Then we have $\mathbf{b} = (x, 0, 0)$, $\mathbf{f} = (0, 0, z)$ and $\mathbf{c} = (c'_1, c'_2, c'_3)$

in the $X'Y'Z'$ coordinate system. Now the conditions (12) − (16) will reduce to

$$xc_3' = 0, \tag{22}$$

$$xc_2' + c_0 z = 0, \tag{23}$$

$$zc_3' = 0, \tag{24}$$

$$z^2 - x^2 = 1, \tag{25}$$

$$\sum_{i=1}^{3}(c_i')^2 - (c_0)^2 = 1,. \tag{26}$$

Note that (25) ensures that $z \neq 0$. Thus, (22) and (24) imply that $c_3' = 0$, $\mathbf{c} = (c_1', c_2', 0)$. In addition, it follows from (23) that $c_0 = -\frac{x}{z}c_2'$. Substituting $c_3' = 0$, $c_0 = -\frac{x}{z}c_2'$ and (25) into (26), we have

$$(c_1')^2 + \frac{(c_2')^2}{(z)^2} = 1. \tag{27}$$

This is an equation of ellipse. Moreover, since $|z| = \|\mathbf{f}\| > 1$, the length of the semi-major axis is $\|\mathbf{f}\|$ and the length of the semi-minor axis is 1.

□

In the following theorem, we only consider the $\mathcal{T}$ with real coefficients.

Theorem 4.3. *Let $\mathcal{T}_1$, $\mathcal{T}_2$ be two time reversal operators, $\mathcal{T}_1 \neq \pm\mathcal{T}_2$. If there exist two nontrivial parity operators $\mathcal{P}_1$ and $\mathcal{P}_2$ such that $\mathcal{P}_i$ commutes with $\mathcal{T}_1$ and $\mathcal{T}_2$ simultaneously, then $\mathcal{P}_1 = \pm\mathcal{P}_2$.*

Proof. Let $\mathcal{T}_1 = \sum_{i=0}^{3} c_i^{(1)}\tau_i$, $\mathcal{T}_2 = \sum_{i=0}^{3} c_i^{(2)}\tau_i$. Denote $\mathbf{c}^{(1)} = (c_1^{(1)}, c_2^{(1)}, c_3^{(1)})$ and $\mathbf{c}^{(2)} = (c_1^{(2)}, c_2^{(2)}, c_3^{(2)})$.

(i). If $c_0^{(1)} \neq 0$ and $c_0^{(2)} = 0$.

Suppose that $\mathcal{P}$ commute with $\mathcal{T}_i$ simultaneously. By (12), we have $\mathbf{c}^{(2)} \times \mathbf{b} = 0$. It follows that $\mathbf{b} = m\mathbf{c}^{(2)}$. On the other hand, (12) implies that $\mathbf{f} = \frac{1}{c_0^{(1)}}(\mathbf{c}^{(1)} \times \mathbf{b})$. Thus, $\mathbf{f} = \frac{m}{c_0^{(1)}}(\mathbf{c}^{(1)} \times \mathbf{c}^{(2)})$. Substituting $\mathbf{f}$ and $\mathbf{b}$ into (15), then we have

$$m^2(\|\frac{1}{c_0^{(1)}}\mathbf{c}^{(1)} \times \mathbf{c}^{(2)}\|^2 - \|\mathbf{c}^{(2)}\|^2) = 1.$$

The equation has at most two real roots, which are opposite to each other. Thus, there exist at most two parity operators $\mathcal{P}$ and $-\mathcal{P}$ commuting with $\mathcal{T}_i$ simultaneously.

(ii). If $c_0^{(1)} = c_0^{(2)} = 0$ and $\mathbf{c}^{(1)} = t\mathbf{c}^{(2)}$, where t is a real number.

It follows from (16) that $\mathcal{T}_1 = \pm\mathcal{T}_2$, which contradicts with the assumption of the theorem.

(iii). If $c_0^{(1)} \neq 0$, $c_0^{(2)} \neq 0$ and $\mathbf{c}^{(1)} = t\mathbf{c}^{(2)}$.

By (12), we have $\mathbf{f} = \frac{1}{c_0^{(1)}}(\mathbf{c}^{(1)} \times \mathbf{b}) = \frac{1}{c_0^{(2)}}(\mathbf{c}^{(2)} \times \mathbf{b}) = \frac{t}{c_0^{(1)}}(\mathbf{c}^{(2)} \times \mathbf{b})$. It follows that $c_0^{(1)} = tc_0^{(2)}$. Thus, we have $c_i^{(1)} = tc_i^{(2)}$, $(i = 0, 1, 2, 3)$. On the other hand, (16) implies that $t^2 = 1$. Hence $\mathcal{T}_1 = \pm\mathcal{T}_2$, which is a contradiction.

(iv). If $c_0^{(1)} = c_0^{(2)} = 0$ and $\mathbf{c}^{(1)} \neq t\mathbf{c}^{(2)}$.

By (12), we have $\mathbf{c}^{(1)} \times \mathbf{b} = \mathbf{c}^{(2)} \times \mathbf{b} = 0$. However, since $\mathbf{c}^{(1)} \neq t\mathbf{c}^{(2)}$, we have $\mathbf{b} = 0$. Thus (15) implies that $\|\mathbf{f}\| = 1$. Moreover, (13) implies that $\mathbf{f}$ is orthogonal to both $\mathbf{c}^{(1)}$ and $\mathbf{c}^{(2)}$. So

$\mathbf{f}$ can only have two directions, which are opposite to each other. Thus, there exist at most two parity operators $\mathcal{P}$ and $-\mathcal{P}$ commuting with $\mathcal{T}_i$ simultaneously.

(v). If $c_0^{(1)} \neq 0$, $c_0^{(2)} \neq 0$ and $\mathbf{c}^{(1)} \neq t\mathbf{c}^{(2)}$.

Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be two parity operators, which are determined by $(\mathbf{f}^{(1)}, \mathbf{b}^{(1)})$ and $(\mathbf{f}^{(2)}, \mathbf{b}^{(2)})$ respectively. Moreover, suppose that both $\mathcal{P}_1$ and $\mathcal{P}_2$ commute with $\mathcal{T}_i$ simultaneously.

By (12), we have $\mathbf{f}^{(1)} = \frac{1}{c_0^{(1)}}(\mathbf{c}^{(1)} \times \mathbf{b}^{(1)})$ and $\mathbf{f}^{(1)} = \frac{1}{c_0^{(2)}}(\mathbf{c}^{(2)} \times \mathbf{b}^{(1)})$. It follows that

$$\frac{1}{c_0^{(1)}}\mathbf{c}^{(1)} - \frac{1}{c_0^{(2)}}\mathbf{c}^{(2)} = t_1\mathbf{b}^{(1)},$$

where t_1 is a nonzero real number.

Similarly, we have $\mathbf{f}^{(2)} = \frac{1}{c_0^{(1)}}(\mathbf{c}^{(1)} \times \mathbf{b}^{(2)})$ and $\mathbf{f}^{(2)} = \frac{1}{c_0^{(2)}}(\mathbf{c}^{(2)} \times \mathbf{b}^{(2)})$. It follows that

$$\frac{1}{c_0^{(1)}}\mathbf{c}^{(1)} - \frac{1}{c_0^{(2)}}\mathbf{c}^{(2)} = t_2\mathbf{b}^{(2)},$$

So $t_1\mathbf{b}^{(1)} = t_2\mathbf{b}^{(2)}$, which implies that $\mathbf{b}^{(1)} = k\mathbf{b}^{(2)}$. Now $\|\mathbf{f}^{(1)}\|^2 - \|\mathbf{b}^{(1)}\|^2 = k^2(\|\mathbf{f}^{(2)}\|^2 - \|\mathbf{b}^{(2)}\|^2) = 1$, hence $k = \pm 1$. Thus it is apparent that $\mathcal{P}_1 = \pm\mathcal{P}_2$.

Note that (i) – (v) contain all the situations, which completes the proof. □

If we denote $com(\mathcal{T}) = \{\mathcal{P} | \mathcal{P}\mathcal{T} = \mathcal{T}\mathcal{P}, \mathcal{P}^2 = I\}$, then the following corollary can be obtained.

Corollary 4.4. *If* $\mathcal{T}_1 = \sum_{i=0}^{3} c_i^{(1)}\tau_i$, $\mathcal{T}_2 = \sum_{i=0}^{3} c_i^{(2)}\tau_i$ *are two time reversal operators,* $\mathcal{T}_j^2 = I$, $j = 1, 2$. *Then* $com(\mathcal{T}_1) = com(\mathcal{T}_2)$ *if and only if for each* i, $c_i^{(1)} = \epsilon c_i^{(2)}$, *where* ϵ *is a unimodular coefficient.*

§5 $\mathcal{PT}$-symmetric operators and unbroken $\mathcal{PT}$-symmetric condition

A linear operator H is said to be $\mathcal{PT}$-symmetric if $H\mathcal{PT} = \mathcal{PT}H$. As is known, in standard quantum mechanics, the Hamiltonians are assumed to be Hermitian such that all the eigenvalues are real and the evolution is unitary. In the $\mathcal{PT}$-symmetric quantum theory, Bender replaced the Hermiticity of the Hamiltonians with $\mathcal{PT}$-symmetry. However, the $\mathcal{PT}$-symmetry of a linear operator does not imply that its eigenvalues must be real. Thus, Bender introduced the unbroken $\mathcal{PT}$-symmetric condition. The Hamiltonian H is said to be unbroken $\mathcal{PT}$-symmetric if there exists a collection of eigenvectors Ψ_i of H such that they span the whole space and $\mathcal{PT}\Psi_i = \Psi_i$. It was shown that for a $\mathcal{PT}$-symmetric Hamiltonian H, its eigenvalues are all real if and only if H is unbroken $\mathcal{PT}$-symmetric [2]. In two dimensional space case, this condition has a much simpler description and an important illustrative example. That is, if $\mathcal{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\mathcal{T} = \mathcal{T}_0$, $H = \begin{pmatrix} re^{i\theta} & s \\ s & re^{-i\theta} \end{pmatrix}$, then H is unbroken $\mathcal{PT}$-symmetric iff $s^2 \geq r^2 \sin^2\theta$ [2].

In the following part, we shall give the unbroken $\mathcal{PT}$-symmetry condition for general $\mathcal{PT}$-symmetric operators. To this end, we need the following proposition.

For a Hamiltonian $H = \sum_{i=0}^{3} h_i\sigma_i$ which is written in terms of Pauli operators, we denote $\mathbf{f}_H = (Re(h_1), Re(h_2), Re(h_3))$ and $\mathbf{b}_H = (Im(h_1), Im(h_2), Im(h_3))$.

Proposition 5.1. *If H is a $\mathcal{PT}$-symmetric operator, then it has four real parameters. Moreover, if $H = \sum_{i=0}^{3} h_i\sigma_i$, then we have*

$$Im(h_0) = 0, \tag{28}$$

$$\boldsymbol{f}_H \cdot \boldsymbol{b}_H = 0. \tag{29}$$

Proof. It is apparent that $\mathcal{PT}$ is also a time reversal operator. Thus we can assume that $\mathcal{PT} = \sum_{j=0}^{3} c_j\tau_j$. Now the condition $\mathcal{PT}H = H\mathcal{PT}$ is equivalent to

$$(\sum_{j=0}^{3} c_j\tau_0\sigma_j)(\sum_{i=0}^{3} h_i\sigma_i) = (\sum_{i=0}^{3} h_i\sigma_i)(\sum_{j=0}^{3} c_j\tau_0\sigma_j).$$

According to (10), this equation can be reduced to

$$c_0(\overline{h_0} - h_0) + \sum_{i=1}^{3} c_i(h_0 - \overline{h_0})\sigma_i + \sum_{i=1}^{3} c_i(h_i + \overline{h_i}) + i\sigma\cdot[\mathbf{c}\times(\overline{\mathbf{h}} - \mathbf{h})] - \sum_{i=1}^{3} c_0(h_i + \overline{h_i})\sigma_i = 0,$$

where $\mathbf{h} = (h_1, h_2, h_3)$ and $\overline{\mathbf{h}} = (\overline{h_1}, \overline{h_2}, \overline{h_3})$.

The equation above is equivalent to

$$Im(h_0) = 0, \tag{30}$$

$$\mathbf{c}\cdot\mathbf{f}_H = 0, \tag{31}$$

$$\mathbf{c}\times\mathbf{b}_H - c_0\mathbf{f}_H = 0. \tag{32}$$

(i). When $c_0 \neq 0$. It follows (32) that $\mathbf{f}_H = \frac{1}{c_0}(\mathbf{c}\times\mathbf{b}_H)$. Thus, the four parameters $Im(h_1)$, $Im(h_2)$, $Im(h_3)$ and $Re(h_0)$ determine H.

Note that (28) is the same as (30). On the other hand, $\mathbf{f}_H = \frac{1}{c_0}(\mathbf{c}\times\mathbf{b}_H)$ implies that $\mathbf{f}_H\cdot\mathbf{b}_H = 0$. Thus, (29) is also valid.

(ii). When $c_0 = 0$. (32) implies that $\mathbf{b}_H = t\mathbf{c}$. Thus, we only need one real parameter t to determine $\mathbf{b}_H$. (31) implies that $\mathbf{f}_H$ should be orthogonal to $\mathbf{c}$. Hence two parameters are needed. With $Re(h_0)$, we have four parameters altogether.

In this case, (29) follows from the fact $\mathbf{b}_H = t\mathbf{c}$ and the equation (31).

□

Theorem 5.2. *If H is a $\mathcal{PT}$-symmetric operator and $\begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}$ is the representation matrix of H, then H is unbroken if and only if $(Re(h_{11}+h_{22}))^2 - 4Re(h_{11}h_{22} - h_{12}h_{21}) \geq 0$.*

Proof. Let $\begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}$ be the matrix of H, λ be an eigenvalue of H, then

$$\lambda^2 - (h_{11}+h_{22})\lambda + h_{11}h_{22} - h_{12}h_{21} = 0. \tag{33}$$

On the other hand, rewrite $H = h_0\sigma_0 + h_1\sigma_1 + h_2\sigma_2 + h_3\sigma_3$. It follows from (28) and (29) that

$$Im(h_{11} + h_{22}) = 2Im(h_0) = 0,$$
$$Im(h_{11}h_{22} - h_{12}h_{21}) = -\mathbf{f}_H \cdot \mathbf{b}_H = 0.$$

The two equations above imply that

$$-Im(h_{11} + h_{22})\lambda + Im(h_{11}h_{22} - h_{12}h_{21}) = 0. \tag{34}$$

Substitute (34) into (33). Now the equation (33) reduces to

$$\lambda^2 - Re(h_{11} + h_{22})\lambda + Re(h_{11}h_{22} - h_{12}h_{21}) = 0, \tag{35}$$

According to (35), λ is a real number, that is, H is unbroken $\mathcal{PT}$-symmetric, if and only if

$$(Re(h_{11} + h_{22}))^2 - 4Re(h_{11}h_{22} - h_{12}h_{21}) \geq 0. \tag{36}$$

□

Remark 5.1. *Note that when the equality is valid in (36), H may be non-diagonalisable in general. In this case, the space $\mathbb{C}^2$ is actually spanned an eigenvector ψ_1 satisfying $(H - \lambda_0 I)\psi_1 = 0$ and a generalized eigenvector ψ_2 satisfying $(H-\lambda_0 I)^2\psi_2 = 0$, where $\lambda_0 = \frac{1}{2}Re(h_{11}+h_{22})$ is the eigenvalue.*

Remark 5.2. *Note that Bender's unbroken $\mathcal{PT}$-symmetric condition in [2] is a special case of (36). To see this, let $H = \begin{pmatrix} re^{i\theta} & s \\ s & re^{-i\theta} \end{pmatrix}$, we have*

$$Re(h_{11}) = Re(h_{22}) = r\cos\theta,$$
$$Re(h_{11}h_{22} - h_{12}h_{21}) = r^2 - s^2.$$

Then (36) holds iff $s^2 \geq r^2 \sin^2\theta$.

Remark 5.3. *If H is a Hermitian operator, then it is also unbroken $\mathcal{PT}$-symmetric. Usually, this can be shown by using canonical forms. However, in $\mathbb{C}^2$, it also follows from direct calculation.*

In fact, since $H = \sum_{i=0}^{3} h_i\sigma_i$ is Hermitian, each h_i is a real number. Now we only need to find real coefficients c_0, c_1, c_2 and c_3 such that $\sum_{i=1}^{3} c_i^2 - c_0^2 = 1$ and equations (30) – (32) hold. Take $c_0 = 0$ and c_1, c_2, c_3 are such real numbers that $\mathbf{c}\cdot\mathbf{f}_H = 0$ and $\sum_{i=1}^{3} c_i^2 = 1$. Let $\mathcal{PT} = \sum_{i=0}^{3} c_i\tau_i$. It is apparent that $(\mathcal{PT})^2 = I$ and H is $\mathcal{PT}$-symmetric. Moreover, if we rewrite the Hermitian matrix as $H = \begin{pmatrix} a & b \\ \overline{b} & a \end{pmatrix}$, then $[Re(h_{11} + h_{22})]^2 - 4Re(h_{11}h_{22} - h_{12}h_{21}) = 4a^2 - 4(a^2 - |b|^2) = 4|b|^2 \geqslant 0$ holds, so H is also unbroken.

References

[1] C M Bender, S Boettcher. *Real spectra in non-Hermitian hamiltonians having $\mathcal{PT}$-symmetry*, Phys Rev Lett, 1998, 80(24): 5243.

[2] C M Bender. *Making sense of non-Hermitian hamiltonians*, Rep Prog Phys, 2007, 70(6): 947-1018.

[3] C M Bender, D C Brody, M P Muller. *Hamiltonian for the Zeros of the Riemann Zeta Function*, Phys Rev Lett, 2017,118(13): 130201.

[4] L Chang, X S Jiang, S Y Hua, C Yang, J M Wen, L Jiang, G Y Li, G Z Wang, M Xiao. *Parity-time symmetry and variable optical isolation in active-passive-coupled microresonators*, Nature Photon, 2014,8(7): 524-529.

[5] S Deffner, A Saxena. *Jarzynski equality in $\mathcal{PT}$-symmetric quantum mechanics*, Phys Rev lett, 2015,114(15): 150601.

[6] J W Deng, U Guenther, Q H Wang. *General $\mathcal{PT}$-symmetric matrices*, arXiv preprint arXiv:1212.1861, 2012.

[7] W Greiner. *Quantum mechanics: an introduction*, Springer Science and Business Media, 2011.

[8] A Mostafazadeh. *Pseudo-Hermitian representation of quantum mechanics*, Int J Geom Meth Mod Phys, 2010,7(07): 1191-1306.

[9] C E Rüter, K G Makris, R E Ganainy, D N Christodoulides, M Segev, D Kip. *Observation of parity-time symmetry in optics*, Nature Phys, 2010,6(3): 192.

[10] K J Smith. *Non-Hermitian quantum mechanics*, PhD thesis, Case Western Reserve University, 2010.

[11] A Uhlmann. *Anti-(conjugate) linearity*, Sci China Phys Mech Astron, 2016,59(3): 630301.

[12] E Wigner. *Group theory: and its application to the quantum mechanics of atomic spectra*, Elsevier, 2012.

[1] School of Mathematical Science, Zhejiang University, Hangzhou 310027, People's Republic of China.
Email: 11335001@zju.edu.cn, wjd@zju.edu.cn

[2] Department of Mathematics, National University of Singapore, Singapore 119076, Republic of Singapore.
Email: a0086285@u.nus.edu

[3] Department of Applied Mathematics, Kumoh National Institute of Technology, Kyungbuk, 39177, Korea.
Email: mignon@kumoh.ac.kr