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Wiener Index, Hyper-Wiener Index, Harary Index and Hamiltonicity Properties of graphs

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Abstract. In this paper, in terms of Wiener index, hyper-Wiener index and Harary index, we first give some sufficient conditions for a nearly balance bipartite graph with given minimum degree to be traceable. Secondly, we establish some conditions for a k-connected graph to be Hamilton-connected and traceable for every vertex, respectively.

§1 Introduction

Let G be a simple graph of order n with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$, denoted by $e(G) = |E(G)|$. The distance between two vertices v_i and v_j of G, denoted by $d_G(v_i, v_j)$, is defined as the minimum length of the paths between v_i and v_j in G. Let $N_G(v)$ be the set of vertices which are adjacent to v in G. The degree of v is denoted by $d_G(v) = |N_G(v)|$, the minimum degree of G is denoted by $\delta(G)$. Let $X \subseteq V(G)$, $G - X$ is the graph obtained from G by deleting all vertices in X. G is called *k-connected* (for $k \in \mathbb{N}$) if $|V(G)| > k$ and $G - X$ is connected for every set $X \subseteq V(G)$ with $|X| < k$. Let $G = (X, Y; E)$ be a bipartite graph with two part sets X, Y. If $|X| = |Y|$, $G = (X, Y; E)$ is called a *balanced bipartite graph.* If $|X| = |Y| + 1$, $G = (X, Y; E)$ is called a *nearly balanced bipartite graph.* For two disjoint graphs G_1 and G_2, the union of G_1 and G_2, denoted by $G_1 + G_2$, is defined as $V(G_1 + G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 + G_2) = E(G_1) \cup E(G_2)$; and the join of G_1 and G_2, denoted by $G_1 \vee G_2$, is defined as $V(G_1 \vee G_2) = V(G_1) \cup V(G_2)$, and $E(G_1 \vee G_2) = E(G_1 + G_2) \cup \{xy : x \in V(G_1), y \in V(G_2)\}$. Denote K_n the complete graph on n vertices, O_n the empty graph on n vertices (without edges), $K_{n,m} = O_n \vee O_m$ the complete bipartite graph with two parts having n, m vertices, respectively.

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The *Wiener index* of a connected graph G, denote by $W(G)$, which was introduced by Wiener [15] in 1947, is defined as

$$W(G)=\sum_{v_i,v_j\in V(G)} d_G(v_i,v_j).$$

We denote $D_i=D_G(v_i)=\sum_{v_j\in V(G)} d_G(v_i,v_j)$, then

$$W(G)=\frac{1}{2}\sum_{i=1}^{n} D_i.$$

The hyper-Wiener index, as a generalization of the Wiener index, is traditionally denoted by $WW(G)$. The hyper-Wiener index of acyclic graphs was introduced by Randić [14] in 1993 and extended to all connected graphs by Klein et al. [8]. The *hyper-Wiener index* of a connected graph G is defined as

$$WW(G)=\frac{1}{2}(\sum_{v_i,v_j\in V(G)} d_G(v_i,v_j)+\sum_{v_i,v_j\in V(G)} d_G^2(v_i,v_j)).$$

We denote $DD_i=DD_G(v_i)=\sum_{v_j\in V(G)} d_G^2(v_i,v_j)$, then

$$WW(G)=\frac{1}{4}\sum_{i=1}^{n}(D_i+DD_i).$$

The Harary index of a connected graph is also a useful topological index. It was introduced by Plavšić et al. [11] and by Ivanciuc et al. [7] in 1993, independently. It is defined as

$$H(G)=\sum_{v_i,v_j\in V(G)}\frac{1}{d_G(v_i,v_j)}.$$

We denote $\widetilde{D}_G(v_i)=\sum_{v_j\in V(G)}\frac{1}{d_G(v_i,v_j)}$, then

$$H(G)=\frac{1}{2}\sum_{i=1}^{n}\widetilde{D}_G(v_i).$$

A *Hamiltonian cycle* of a graph G is a cycle of order n contained in G, and a *Hamiltonian path* of G is a path of order n contained in G, where $|V(G)|=n$. The graph G is said to be *Hamiltonian* if it contains a Hamiltonian cycle, and is said to be *traceable* if it contains a Hamiltonian path. If every two vertices of G are connected by a Hamiltonian path, it is said to be *Hamilton-connected*. A graph G is *traceable from a vertex* x if it has a Hamiltonian x-path. A graph is *traceable from every vertex* if it contains a Hamilton path from every vertex. All these concepts belong to Hamiltonian properties of graphs. The problem of deciding whether a graph is Hamiltonian is one of the most difficult problems in graph theory.

Recently, some topological indices have been applied to the studies of the Hamiltonian properties of graphs. Up to now, there are some references on the Wiener index, hyper-Wiener index and Harary index conditions for a graph to be traceable, hamiltonian, hamilton-connected, traceable from every vertex. We refer readers to see [3, 4, 5, 6, 8, 9, 10, 12, 13, 18]. Especially, Hua and Ning [6] gave conditions for a balanced bipartite graph to be Hamiltonian with Wiener index and Harary index. Cai et al. [4], with hyper-Wiener index, gave conditions for a balanced bipartite graph to be traceable and Hamiltonian, and presented conditions for a k-connected graphs to be Hamiltonian. Li gave conditions for a k-connected graphs to be Hamiltonian in terms of Wiener index and Harary index in [12] and [13] respectively. In this paper, we will investigate the similar problems. In particular, in terms of Wiener index, hyper-Wiener index and Harary index, we will respectively present some conditions for a nearly balanced bipartite graph to be traceable in Section 2, and give the conditions for a k-connected graph to be Hamilton-connected and traceable for every vertex in Section 3 and in Section 4, respectively.

§2 Wiener Index, Hyper-Wiener Index, Harary Index and Traceable of Nearly Balanced Bipartite Graphs

In this paper, when we mention a bipartite graph, we always fix its partite sets, e.g., $O_{n,m}$ and $O_{m,n}$ are considered as different bipartite graphs, unless $m = n$.

Let G_1, G_2 be two bipartite graphs, with the bipartitions X_1, Y_1 and X_2, Y_2, respectively. We use $G_1 \sqcup G_2$ to denote the graph obtained from $G_1 + G_2$ by adding all possible edges between X_1 and Y_2 and possible edges between Y_1 and X_2. We define a classe of graphs as follows:

$$C_n^k = O_{k,n-k} \sqcup K_{n-k-1,k}.$$

Note that $e(C_n^k) = n(n-k-1) + k^2$ and C_n^k is not traceable.

Lemma 2.1 (Yu, Fang and Fan [17]) Let $G \nsubseteq C_n^k$ be a nearly balanced bipartite graph of order $2n-1$, $\delta(G) \geqslant k \geqslant 1$ and $n \geqslant 2k+1$. If

$$e(G) > n(n-k-2) + (k+1)^2,$$

then G is traceable .

Lemma 2.2 If G is a nearly balanced bipartite graph of order $2n-1$, then

$$W(G) \geqslant 5n^2 - 7n + 2 - 2e(G),$$

the equality holds if and only if, for any nonadjacent vertices $u, v \in V(G)$, if u, v are in different partite sets, $d(u,v) = 3$, and if u, v are in the same partite set, $d(u,v) = 2$.

Proof. Let $G = G[X,Y]$, where $X = \{x_1, x_2, \cdots, x_n\}$, $Y = \{y_1, y_2, \cdots, y_{n-1}\}$.

$$\begin{aligned} W(G) &= \sum_{u,v \in V(G)} d_G(u,v) \\ &= \frac{1}{2}\sum_{i=1}^{2n-1} D_i \\ &\geqslant \frac{1}{2}(\sum_{i=1}^{n}(d_G(x_i) + 3(n-1-d_G(x_i)) + 2(n-1))) \\ &\quad + \frac{1}{2}(\sum_{i=1}^{n-1}(d_G(y_i) + 3(n-d_G(y_i)) + 2(n-2))) \\ &= 5n^2 - 7n + 2 - (\sum_{i=1}^{n} d_G(x_i) + \sum_{i=1}^{n-1} d_G(y_i)) \\ &= 5n^2 - 7n + 2 - 2e(G), \end{aligned}$$

the equality holds if and only if all above inequalities become equalities, i.e., for any two nonadjacent vertices $u, v \in V(G)$, if u, v are in different partite sets, $d(u, v) = 3$, and if u, v are in the same partite set, $d(u, v) = 2$. ■

Lemma 2.3 If G is a nearly balanced bipartite graph of order $2n-1$, then

$$WW(G) \geq 9n^2 - 3n - 5e(G),$$

the equality holds if and only if, for any nonadjacent vertices $u, v \in V(G)$, if u, v are in different partite sets, $d(u, v) = 3$, and if u, v are in the same partite set, $d(u, v) = 2$.

Proof. Let $G = G[X, Y]$, where $X = \{x_1, x_2, \cdots, x_n\}$, $Y = \{y_1, y_2, \cdots, y_{n-1}\}$.

$$\begin{aligned}
WW(G) &= \frac{1}{2} \sum_{u,v \in V(G)} (d_G(u, v) + d_G^2(u, v)) \\
&= \frac{1}{4} \sum_{i=1}^{2n-1} (D_i + DD_i) \\
&\geqslant \frac{1}{4} \sum_{i=1}^{n} (d_G(x_i) + 3(n-1-d_G(x_i)) + 2(n-1)) \\
&\quad + \frac{1}{4} \sum_{i=1}^{n} (d_G(x_i) + 9(n-1-d_G(x_i)) + 4(n-1)) \\
&\quad + \frac{1}{4} \sum_{i=1}^{n-1} (d_G(y_i) + 3(n-d_G(y_i)) + 2(n-2)) \\
&\quad + \frac{1}{4} \sum_{i=1}^{n-1} (d_G(y_i) + 9(n-d_G(y_i)) + 4(n-2)) \\
&= 9n^2 - 12n + 3 - \frac{5}{2}\left(\sum_{i=1}^{n} d_G(x_i) + \sum_{i=1}^{n-1} d_G(y_i)\right) \\
&= 9n^2 - 12n + 3 - 5e(G),
\end{aligned}$$

the equality holds if and only if all above inequalities become equalities, i.e., for any two nonadjacent vertices $u, v \in V(G)$, if u, v are in different partite sets, $d(u, v) = 3$, and if u, v are in the same partite set, $d(u, v) = 2$. ■

Lemma 2.4 If G is a nearly balanced bipartite graph of order $2n-1$, then

$$H(G) \leqslant \frac{5}{6}n^2 - \frac{4}{3}n + \frac{1}{2} + \frac{2}{3}e(G),$$

the equality holds if and only if, for any two nonadjacent vertices $u, v \in V(G)$, if u, v are in different partite sets, $d(u, v) = 3$, and if u, v are in the same partite set, $d(u, v) = 2$.

Proof. Let $G = G[X, Y]$, where $X = \{x_1, x_2, \cdots, x_n\}$, $Y = \{y_1, y_2, \cdots, y_{n-1}\}$.

$$\begin{aligned}
H(G) = \sum_{u,v\in V(G)} \frac{1}{d_G(u,v)} &= \frac{1}{2}\sum_{i=1}^{n} \widetilde{D}_G(v_i) \\
&\leqslant \frac{1}{2}\sum_{i=1}^{n}(d_G(x_i) + \frac{1}{3}(n-1-d_G(x_i)) + \frac{1}{2}(n-1)) \\
&\quad + \frac{1}{2}\sum_{i=1}^{n-1}(d_G(y_i) + \frac{1}{3}(n-d_G(y_i)) + \frac{1}{2}(n-2)) \\
&= \frac{5}{6}n^2 - \frac{4}{3}n + \frac{1}{2} + \frac{1}{3}(\sum_{i=1}^{n} d_G(x_i) + \sum_{i=1}^{n-1} d_G(y_i)) \\
&= \frac{5}{6}n^2 - \frac{4}{3}n + \frac{1}{2} + \frac{2}{3}e(G),
\end{aligned}$$

the equality holds if and only if all above inequalities become equalities, i.e., for any two nonadjacent vertices $u, v \in V(G)$, if u, v are in different partite sets, $d(u, v) = 3$, and if u, v are in the same partite set, $d(u, v) = 2$. ■

Corollary 2.5 (i) $W(C_n^k) = 3n^2 + 2kn - 5n - 2k^2 + 2$.

(ii) $WW(C_n^k) = 4n^2 + 5kn + 2n - 5k^2$.

(iii) $H(C_n^k) = \frac{3}{2}n^2 - \frac{2}{3}kn - 2n + \frac{2}{3}k^2 + \frac{1}{2}$.

Proof. We note that $e(C_n^k) = n(n-k-1) + k^2$, and C_n^k satisfies the conditions for the equality holding in Lemmas 2.2, 2.3, and 2.4. Then Lemmas 2.2, 2.3, and 2.4, we get

$$W(C_n^k) = 5n^2 - 7n + 2 - 2e(C_n^k) = 3n^2 + 2kn - 5n - 2k^2 + 2;$$

$$WW(C_n^k) = 9n^2 - 3n - 5e(C_n^k) = 4n^2 + 5kn + 2n - 5k^2;$$

$$H(C_n^k) = \frac{5}{6}n^2 - \frac{4}{3}n + \frac{1}{2} + \frac{2}{3}e(C_n^k) = \frac{3}{2}n^2 - \frac{2}{3}kn - 2n + \frac{2}{3}k^2 + \frac{1}{2}.$$

■

Theorem 2.6 Let G be a nearly balanced bipartite graph of order $2n-1$, $\delta(G) \geqslant k \geqslant 1$ and $n \geqslant 2k+2$. If

$$W(G) \leqslant 3n^2 + 2kn - 5n - 2k^2 + 2,$$

then G is traceable unless $G = C_n^k$.

Proof. Because $W(G) \leqslant 3n^2 + 2kn - 5n - 2k^2 + 2$, then by Lemma 2.2,

$$5n^2 - 7n + 2 - 2e(G) \leqslant W(G) \leq 3n^2 + 2kn - 5n - 2k^2 + 2,$$

we obtain

$$e(G) \geqslant n(n-k-1) + k^2 > n(n-k-2) + (k+1)^2,$$

when $n \geqslant 2k+2$. By Lemma 2.1, G is traceable unless $G \subseteq C_n^k$. If $G \subset C_n^k$, then $W(G) > W(C_n^k)$, This is in contradiction with Corollary 2.5. The proof is completed. ■

Theorem 2.7 Let G be a nearly balanced bipartite graph of order $2n-1$, $\delta(G) \geqslant k \geqslant 1$ and $n \geqslant 2k+2$. If

$$WW(G) \leqslant 4n^2 + 5kn + 2n - 5k^2,$$

then G is traceable unless $G = C_n^k$.

Proof. Since $WW(G) \leqslant 4n^2 + 5kn + 2n - 5k^2$, then by Lemma 2.3,

$$9n^2 - 3n - 5e(G) \leqslant WW(G) \leq 4n^2 + 5kn + 2n - 5k^2,$$

we obtain

$$e(G) \geqslant n(n-k-1) + k^2 > n(n-k-2) + (k+1)^2,$$

when $n \geqslant 2k+2$. By Lemma 2.1, G is traceable unless $G \subseteq C_n^k$. If $G \subset C_n^k$, then $WW(G) > WW(C_n^k)$, a contradiction by Corollary 2.5. This completes the proof. ■

Theorem 2.8 Let G be a nearly balanced bipartite graph of order $2n-1$, $\delta(G) \geqslant k \geqslant 1$ and $n \geqslant 2k+2$. If

$$H(G) \geqslant \frac{3}{2}n^2 - \frac{2}{3}kn - 2n + \frac{2}{3}k^2 + \frac{1}{2},$$

then G is traceable unless $G = C_n^k$.

Proof. Since $H(G) \geqslant \frac{3}{2}n^2 - \frac{2}{3}kn - 2n + \frac{2}{3}k^2 + \frac{1}{2}$, then by Lemma 2.4,

$$\frac{5}{6}n^2 - \frac{4}{3}n + \frac{1}{2} + \frac{2}{3}e(G) \geqslant H(G) \geqslant \frac{3}{2}n^2 - \frac{2}{3}kn - 2n + \frac{2}{3}k^2 + \frac{1}{2},$$

we obtain

$$e(G) \geqslant n(n-k-1) + k^2 > n(n-k-2) + (k+1)^2,$$

when $n \geqslant 2k+2$. By Lemma 2.1, G is traceable unless $G \subseteq C_n^k$. If $G \subset C_n^k$, then $H(G) < H(C_n^k)$, a contradiction by Corollary 2.5. The proof is completed. ■

§3 Wiener Index, Hyper-Wiener Index, Harary Index and Hamilton-Connectivity of Graphs

Lemma 3.1 (Bondy and Murty [2])Let G be a graph of order n with $\delta(G) \geqslant k$, then G must be k-connected.

Lemma 3.2 (Yu, Li and Xing [16]) Let G be a graph of order n, and $\delta(G) \geqslant k \geqslant 2$. If

$$e(G) > \frac{n(n-1) - k(n-k-1)}{2},$$

then G is Hamilton-connected.

Theorem 3.3 Let G be a graph of order n, and $\delta(G) \geqslant k \geqslant 2$. If

$$W(G) < \frac{n(n-1) + k(n-k-1)}{2},$$

then G is Hamilton-connected.

Proof. Let G be a graph satisfying the conditions in Theorem 3.3. Suppose that G is not

Hamilton-connected. By Lemma 3.2, we get $e(G) \leqslant \frac{n(n-1)-k(n-k-1)}{2}$. Therefore,

$$\begin{aligned} W(G) &= \sum_{u,v\in V(G)} d_G(u,v) \\ &= \frac{1}{2}\sum_{i=1}^{n} D_i \\ &= \frac{1}{2}\sum_{i=1}^{n}(d_G(x_i)+2(n-1-d_G(x_i))) \\ &= n(n-1)-\frac{1}{2}\sum_{i=1}^{n} d_G(x_i) = n(n-1)-e(G) \\ &\geqslant n(n-1)-\frac{n(n-1)}{2}+\frac{1}{2}k(n-k-1) \\ &= \frac{n(n-1)+k(n-k-1)}{2}, \end{aligned}$$

which is a contradiction. The proof is completed. ■

Theorem 3.4 Let G be a graph of order n, and $\delta(G) \geqslant k \geqslant 2$. If

$$WW(G) < \frac{n(n-1)}{2} + k(n-k-1),$$

then G is Hamilton-connected.

Proof. Let G be a graph satisfying the conditions in Theorem 3.4. Suppose that G is not Hamilton-connected. By Lemma 3.2, we get $e(G) \leqslant \frac{n(n-1)-k(n-k-1)}{2}$. Therefore,

$$\begin{aligned} WW(G) &= \frac{1}{2}\sum_{u,v\in V(G)} (d_G(u,v)+d_G^2(u,v)) \\ &= \frac{1}{4}\sum_{i=1}^{n}(D_i+DD_i) \\ &\geqslant \frac{1}{4}\sum_{i=1}^{n}(d_G(x_i)+2(n-1-d_G(x_i))) + \frac{1}{4}\sum_{i=1}^{n}(d_G(x_i)+4(n-1-d_G(x_i))) \\ &= \frac{3}{2}n(n-1)-\sum_{i=1}^{n} d_G(x_i) = \frac{3}{2}n(n-1)-2e(G) \\ &\geqslant \frac{3}{2}n(n-1)-2(\frac{n(n-1)}{2}-\frac{1}{2}k(n-k-1)) \\ &= \frac{n(n-1)}{2}+k(n-k-1), \end{aligned}$$

which is a contradiction. The proof is completed. ■

Theorem 3.5 Let G be a graph of order n, where $\delta(G) \geqslant k \geqslant 2$. If

$$H(G) > \frac{2n(n-1)-k(n-k-1)}{4},$$

then G is Hamilton-connected.

Proof. Let G be a graph satisfying the conditions in Theorem 3.5. Suppose that G is not

Hamilton-connected. By Lemma 3.2, we get $e(G) \leqslant \frac{n(n-1)-k(n-k-1)}{2}$. Therefore,

$$\begin{aligned} H(G) &= \sum_{u,v \in V(G)} \frac{1}{d_G(u,v)} \\ &= \frac{1}{2}\sum_{i=1}^{n} \widetilde{D}_G(v_i) \\ &\leqslant \frac{1}{2}\sum_{i=1}^{n}(d_G(x_i) + \frac{1}{2}(n-1-d_G(x_i))) \\ &= \frac{1}{4}n(n-1) + \frac{1}{4}\sum_{i=1}^{n} d_G(x_i) \\ &= \frac{1}{4}n(n-1) + \frac{1}{2}e(G) \\ &\leqslant \frac{1}{4}n(n-1) + \frac{1}{2}(\frac{n(n-1)}{2} - \frac{1}{2}k(n-k-1)) \\ &= \frac{2n(n-1)-k(n-k-1)}{4}, \end{aligned}$$

which is a contradiction. The proof is completed. ■

§4 Wiener Index, hyper-Wiener Index, Harary Index and Traceability from Every Vertex of Graphs

Lemma 4.1 (Bondy and Murty [1]) Let G be a graph. Then G is traceable from every vertex if and only if $G \vee K_1$ is Hamilton-connected.

Lemma 4.2 Let G be a graph of order n, and $\delta(G) \geqslant k \geqslant 2$. If

$$e(G) > \frac{n(n-1)-k(n-k)}{2},$$

then G is traceable from every vertex.

Proof. Let $G' = G \vee K_1$, we note that $|V(G')| = n+1$, $\delta(G') \geq k+1 \geq k$, and $e(G') = e(G) + n > \frac{n(n-1)-k(n-k)}{2} + n = \frac{n(n+1)}{2} - \frac{k(n-k)}{2}$. By Lemma 3.2, we get G' is Hamilton-connected. And by Lemma 4.1, then G is traceable from every vertex. ■

Theorem 4.3 Let G be a graph of order n, and $\delta(G) \geqslant k \geqslant 2$. If

$$W(G) < \frac{n(n-1)+k(n-k)}{2},$$

then G is traceable from every vertex.

Proof. Suppose that G is not traceable from every vertex. By Lemma 4.2, we get $e(G) \leqslant$

$\frac{n(n-1)-k(n-k)}{2}$. Therefore,

$$\begin{aligned} W(G) &= \sum_{u,v\in V(G)} d_G(u,v) \\ &= \frac{1}{2}\sum_{i=1}^{n} D_i \\ &= \frac{1}{2}\sum_{i=1}^{n}(d_G(x_i)+2(n-1-d_G(x_i))) \\ &= n(n-1)-\frac{1}{2}\sum_{i=1}^{n} d_G(x_i) = n(n-1)-e(G) \\ &\geqslant n(n-1)-\frac{n(n-1)-k(n-k)}{2} \\ &= \frac{n(n-1)+k(n-k)}{2}, \end{aligned}$$

which is a contradiction. The proof is completed. ■

Theorem 4.4 Let G be a graph of order n, and $\delta(G) \geqslant k \geqslant 2$. If

$$WW(G) < \frac{1}{2}n(n-1)+k(n-k),$$

then G is traceable from every vertex.

Proof. Suppose that G is not traceable from every vertex. By Lemma 4.2, we get $e(G) \leqslant \frac{n(n-1)-k(n-k)}{2}$. Therefore,

$$\begin{aligned} WW(G) &= \frac{1}{2}\sum_{u,v\in V(G)} (d_G(u,v)+d_G^2(u,v)) \\ &= \frac{1}{4}\sum_{i=1}^{n}(D_i+DD_i) \\ &\geqslant \frac{1}{4}\sum_{i=1}^{n}(d_G(x_i)+2(n-1-d_G(x_i)))+\frac{1}{4}\sum_{i=1}^{n}(d_G(x_i)+4(n-1-d_G(x_i))) \\ &= \frac{3}{2}n(n-1)-\sum_{i=1}^{n} d_G(x_i) = \frac{3}{2}n(n-1)-2e(G) \\ &\geqslant \frac{3}{2}n(n-1)-2(\frac{n(n-1)-k(n-k)}{2}) \\ &= \frac{n(n-1)}{2}+k(n-k), \end{aligned}$$

which is a contradiction. The proof is completed. ■

Theorem 4.5 Let G be a graph of order n, where $\delta(G) \geqslant k \geqslant 2$. If

$$H(G) > \frac{2n(n-1)-k(n-k)}{4},$$

then G is traceable from every vertex.

Proof. Suppose that G is not traceable from every vertex. By Lemma 4.2, we get $e(G) \leqslant$

$\frac{n(n-1)-k(n-k)}{2}$. Therefore,

$$\begin{aligned} H(G) &= \sum_{u,v\in V(G)} \frac{1}{d_G(u,v)} \\ &= \frac{1}{2}\sum_{i=1}^{n} \widetilde{D}_G(v_i) \\ &\leqslant \frac{1}{2}\sum_{i=1}^{n}(d_G(x_i)+\frac{1}{2}(n-1-d_G(x_i))) \\ &= \frac{1}{4}n(n-1)+\frac{1}{4}\sum_{i=1}^{n} d_G(x_i) \\ &= \frac{1}{4}n(n-1)+\frac{1}{2}e(G) \\ &\leqslant \frac{1}{4}n(n-1)+\frac{n(n-1)-k(n-k)}{4} \\ &= \frac{2n(n-1)-k(n-k)}{4}, \end{aligned}$$

which is a contradiction. The proof is completed. ■

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