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The closed finite-to-one mappings and their applications

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Abstract. In this paper, we discuss the closed finite-to-one mapping theorems on generalized metric spaces and their applications. It is proved that point-G_δ properties, $\aleph_0$-snf-countability and csf-countability are invariants and inverse invariants under closed finite-to-one mappings. By the relationships between the weak first-countabilities, we obtain the closed finite-to-one mapping theorems of weak quasi-first-countability, quasi-first-countability, snf-countability, gf-countability and sof-countability. Furthermore, these results are applied to the study of symmetric products of topological spaces.

§1 Introduction

In 1961, P.S. Alexandroff [1] put forward the idea of investigating spaces by mappings at the international topological symposium in Prague. The survey paper "Mappings and spaces" written by A.V. Arhangel'skiĭ [2] in 1966 inherited and developed the idea. One of the most basic questions of Alexandroff-Arhangel'skiĭ's idea is what topological properties are preserved by certain mappings [2]?

In general topology, perfect mappings are widely studied and have obtained fruitful results, for example perfect mappings preserve metrizability [7]. However some important topological properties are not preserved under perfect mappings, such as perfect mappings do not preserve g-metrizability [15]. It is known that g-metrizability is preserved under continuous closed and finite-to-one mappings [15]. It shows the importance of finite-to-one mappings. R.F. Gittings [12] and Lin Shou [14] provided special summary reports on open finite-to-one mappings and closed finite-to-one mappings, respectively. These have played an active role in the development of spaces and mappings and their applications. In the late years, Ge Ying [11] proved that closed finite-to-one mappings preserve sn-metrizability; Shen Rongxin [23] proved that closed finite-to-one mappings preserve quasi-first-countability and weak quasi-first-countability; and

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Mou Lei and H. Ohta [22] studied the closed finite-to-one mappings of spaces with sharp bases. Good and Macías [13] recently discussed the symmetric products of generalized metric spaces and pointed out the role of closed finite-to-one mappings. Based on it, Tang Zhongbao, Lin Shou and Lin Fucai [27] constructed two general stability theorems about symmetric products and topological properties by closed finite-to-one mappings. It shows the special role of closed finite-to-one mappings in discussing the mapping properties of spaces and their applications.

There are still some problems whether closed finite-to-one mappings preserve or preserve inversely topological properties to be solved [14]. It is a classical problem whether closed finite-to-one mappings preserve ortho-compact properties [6], and the applications of closed finite-to-one mappings remains to be discovered. In this paper, we mainly study the following generalized metric properties which are preserved and preserved inversely under closed finite-to-one mappings: point-G_δ properties, $\aleph_0$-snf-countability, weak quasi-first-countability, quasi-first-countability, sof-countability, snf-countability, gf-countability, csf-countability and so on. Also the properties of symmetric products of the above properties are discussed.

First we recall some basic concepts used in this paper. Denote by τ_X or τ the topology on a topological space X. All spaces are T_2 unless stated otherwise, all mappings are continuous and onto. Readers may refer to [7, 10] for unstated notation and terminology.

Let X be a space. $P \subset X$ is called a *sequential neighborhood* of x in X if every sequence converging to $x \in X$ is eventually in P, i.e., if a sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to x in X, there exists $m \in \mathbb{N}$ such that $\{x_n : n \geq m\} \subset P$. A subset P of X is called *sequentially open* if P is a sequential neighborhood of each point in P. P is a *sequentially closed* subset of X if $X \setminus P$ is sequentially open.

Definition 1.1 Let $\mathscr{P} = \bigcup_{x\in X} \mathscr{P}_x$ be a family of subsets of a space X satisfying that (a) $\mathscr{P}_x$ is a *network* of x in X for each $x \in X$, i.e., $x \in \bigcap \mathscr{P}_x$ and if $x \in G \in \tau_X$, there exists $P \in \mathscr{P}_x$ such that $P \subset G$; (b) if $U, V \in \mathscr{P}_x$, then $W \subset U \cap V$ for some $W \in \mathscr{P}_x$.

(1) The family $\mathscr{P}$ is called an *sn-network* [16] for X if each element of $\mathscr{P}_x$ is a sequential neighborhood of x in X for each $x \in X$.

(2) The family $\mathscr{P}$ is called an *so-network* [16] for X if each element of $\mathscr{P}_x$ is sequentially open in X for each $x \in X$.

(3) The family $\mathscr{P}$ is called a *weak base* [2] for X if a subset $G \subset X$ is open in X whenever for each $x \in G$ there exists $P \in \mathscr{P}_x$ such that $P \subset G$.

Moreover $\mathscr{P}_x$ is called an *sn*-network (resp. an *so*-network, a weak base) of x. If every $\mathscr{P}_x$ is countable, X is called *snf-countable* (resp. *sof-countable*, *gf-countable*) [2, 17].

Definition 1.2 Let $\mathscr{P} = \{P_x(n, m) : x \in X, n, m \in \mathbb{N}\}$ be a family of subsets of X, where $\{P_x(n, m)\}_{m\in\mathbb{N}}$ is a decreasing network of x in X for any $x \in X$ and $n \in \mathbb{N}$.

(1) A space X is called a *quasi-first-countable space* [25] if, there exists the family $\mathscr{P}$ such that, given $x \in A \subset X$, the set A is a neighborhood of x in X whenever for every $n \in \mathbb{N}$, there is $m \in \mathbb{N}$ such that $P_x(n, m) \subset A$.

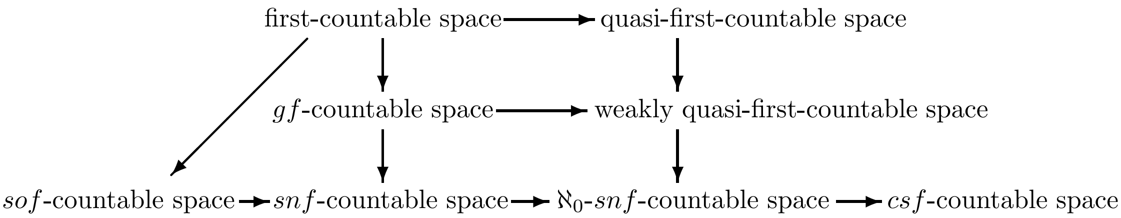
(2) A space X is called a *weakly quasi-first-countable space* [25] if, there exists the family $\mathscr{P}$ such that, given $A \subset X$, the set A is an open set in X whenever if for every $x \in A$ and $n \in \mathbb{N}$,

there is $m \in \mathbb{N}$ such that $P_x(n, m) \subset A$.

(3) A space X is called an $\aleph_0$-*snf-countable space* [20] if, there exists the family $\mathscr{P}$ such that, given $A \subset X$, the set A is a sequentially open subset of X whenever if for every $x \in A$ and $n \in \mathbb{N}$, there is $m \in \mathbb{N}$ such that $P_x(n, m) \subset A$.

Definition 1.3 A space X is called a *csf-countable space* [17] if for every $x \in X$, there is a countable family $\mathscr{P}_x$ of subsets of X satisfying as follows: (a) $x \in \bigcap \mathscr{P}_x$; (b) if $x \in U \in \tau_X$ and a sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to x in X, then there exists $P \in \mathscr{P}_x$ such that $P \subset U$ and the sequence $\{x_n\}_{n\in\mathbb{N}}$ is eventually in P. The family $\mathscr{P}_x$ is called a countable *cs*-network of x in X.

The basic relationships between the spaces described above are in the following diagram [19, 20], and these spaces are also known as *weakly first-countable spaces*:



§2 Lemmas

In this section, we study some relationships between several weak first-countabilities. Some auxiliary results will be cited or proved.

Lemma 2.1 *The following are equivalent for a space X:*

(1) X is an $\aleph_0$-snf-countable space.

(2) For each $x \in X$, there exists a family $\mathscr{P}_x = \{P_x(n, m) : n, m \in \mathbb{N}\}$ of subsets of X satisfying:

(2.1) $\{P_x(n, m)\}_{m\in\mathbb{N}}$ is a decreasing network of x in X for each $n \in \mathbb{N}$.

(2.2) For each n, $m_n \in \mathbb{N}$, $\bigcup_{n\in\mathbb{N}} P_x(n, m_n)$ is a sequential neighborhood of x.

(3) For each $x \in X$, there exists a family $\mathscr{P}_x = \{P_x(n, m) : n, m \in \mathbb{N}\}$ of subsets of X satisfying:

(3.1) $\{P_x(n, m)\}_{m\in\mathbb{N}}$ is a decreasing network of x in X for each $n \in \mathbb{N}$.

(3.2) If a sequence $\{x_k\}_{k\in\mathbb{N}}$ in X converges to x, there exist $n \in \mathbb{N}$ and a subsequence $\{x_{k_m}\}_{m\in\mathbb{N}}$ of $\{x_k\}_{k\in\mathbb{N}}$ such that each $x_{k_m} \in P_x(n, m)$.

Proof. (1) $\Rightarrow$ (3). Suppose that $\mathscr{P}$ is a family of subsets of X which satisfies Definition 1.2(3). Let $\mathscr{P}_x = \{P_x(n, m) : n, m \in \mathbb{N}\}$ for each $x \in X$. Then we only need to show that (3.2) holds. Assume that a sequence $\{x_k\}_{k\in\mathbb{N}}$ in X converges to x. Since $\mathscr{P}$ is a network of X, without loss of generality, we may assume that $x_k \neq x$ for all $k \in \mathbb{N}$. Put $H = X \setminus \{x_k : k \in \mathbb{N}\}$. For a point $z \in H$ with $z \neq x$ and $n \in \mathbb{N}$, since H is a neighborhood of z, there exists $m \in \mathbb{N}$ such that $P_z(n, m) \subset H$. Since the set H is not a sequentially open subset of X, according to Definition 1.2(3), there exists $n \in \mathbb{N}$ such that $P_x(n,m) \not\subset H$ for any $m \in \mathbb{N}$. Put $T_m = P_x(n,m) \cap \{x_k : k \in \mathbb{N}\}$. Then $T_m \neq \varnothing$. If T_{m_0} is a finite set for some $m_0 \in \mathbb{N}$, then there exists $m_1 > m_0$ such that $P_x(n,m_1) \subset X \setminus T_{m_0}$, thus $T_{m_1} = \varnothing$, which is a contradiction. So each T_m is an infinite set. Hence there is a subsequence $\{x_{k_m}\}_{m\in\mathbb{N}}$ of $\{x_k\}_{k\in\mathbb{N}}$ such that each $x_{k_m} \in P_x(n,m)$.

(3) $\Rightarrow$ (2). Suppose that a family $\mathscr{P}_x = \{P_x(n,m) : n,m \in \mathbb{N}\}$ of subsets of X satisfies condition (3) for each $x \in X$. Put $P = \bigcup_{n\in\mathbb{N}} P_x(n,m_n)$ for each n, $m_n \in \mathbb{N}$. If P is not a sequential neighborhood of x in X, there is a sequence $\{x_k\}_{k\in\mathbb{N}}$ of X converging to x such that $x_k \notin P$ for each $k \in \mathbb{N}$. According to (3.2), there exist $n \in \mathbb{N}$ and a subsequence $\{x_{k_m}\}_{m\in\mathbb{N}}$ of $\{x_k\}_{k\in\mathbb{N}}$ such that each $x_{k_m} \in P_x(n,m)$. Take $m' \in \mathbb{N}$ such that $m' > m_n$. Then $x_{k_{m'}} \in P_x(n,m') \subset P_x(n,m_n) \subset P$, which is a contradiction. Thus $\mathscr{P}_x$ satisfies (2) for each $x \in X$.

(2) $\Rightarrow$ (1). Suppose that a family $\mathscr{P}_x = \{P_x(n,m) : n,m \in \mathbb{N}\}$ of subsets of X satisfies condition (2) for each $x \in X$. Put $\mathscr{P} = \bigcup_{x\in X} \mathscr{P}_x$. If a subset A of X satisfies that for any $x \in A$ and $n \in \mathbb{N}$, there exists $m_n \in \mathbb{N}$ such that $P_x(n,m_n) \subset A$, then $\bigcup_{n\in\mathbb{N}} P_x(n,m_n) \subset A$. According to (2.2), A is a sequential neighborhood of x. Thus A is a sequential neighborhood of each point in A, i.e., A is a sequentially open subset of X. Therefore, X is an $\aleph_0$-snf-countable space. This completes the proof.

The (1) $\Rightarrow$ (2) in Lemma 2.1 is not obvious. The space in Definition 1.2(3) was called a *sequential network space with a countable fan* by Lin Shou [18]; the space in Lemma 2.1(3) was defined as an $\aleph_0$-*sn weakly first-countable space* by Wang Pei, Li Zhongmin and Liu Shiqin [29]; and the space in Lemma 2.1(2) was defined as an $\aleph_0$-snf-countable space by Lin Shou and Ge Ying [20]. Here, it is proved that these definitions are consistent. In addition, for each $P_x(n,m)$ in Lemma 2.1, the variable n only need be countable, and the variable m need be countable and ordinal.

A space X is said to be a *sequential space* [8] if each sequentially open subset is open in X. A space X is called a *Fréchet space* [8] if, for any subset $A \subset X$ and $x \in \overline{A}$, there is a sequence in A converging to x in X.

Lemma 2.2 [18,20] *(1) A topological space X is a weakly quasi-first-countable space if and only if it is an $\aleph_0$-snf-countable sequential space.*

(2) A topological space X is a quasi-first-countable space if and only if it is an $\aleph_0$-snf-countable Fréchet space.

A space X is called an α_4-*space* [3] if, whenever $x \in X$ and each sequence S_n in X converges to x for any $n \in \mathbb{N}$, then there exists a sequence S in X converging to x such that $\{n \in \mathbb{N} : S \cap S_n \neq \varnothing\}$ is infinite.

Lemma 2.3 [17] *A topological space X is an snf-countable space if and only if it is a csf-countable α_4-space.*

Lemma 2.4 [16] *A topological space X is a gf-countable space if and only if it is an snf-countable sequential space.*

For every topological space (X,τ), a new topology σ_τ on the X can be defined as follows:

$O \in \sigma_\tau$ if and only if O is a sequentially open subset in (X, τ) [9]. The space (X, σ_τ) is called a *sequential coreflection* of (X, τ), which is denoted by σX. It is well-known that σX is a sequential space, X and σX have the same convergent sequences [4,9].

Lemma 2.5 *A topological space X is an sof-countable space if and only if X is an snf-countable space and σX is a Fréchet space.*

Proof. Let X be a space. For every $A \subset X$, let $\mathrm{cl}_{\sigma X}(A)$ be the closure of A in σX.

If X is an sof-countable space. Obviously, X is an snf-countable space. Let $A \subset X$ and $x \in \mathrm{cl}_{\sigma X}(A)$. Suppose that $\mathscr{P}_x = \{P_n\}_{n\in\mathbb{N}}$ is a countable *so*-network of x in X. Since the intersection of any two sequentially open sets of X is still a sequentially open set, without loss of generality, we may assume that $P_{n+1} \subset P_n$ for each $n \in \mathbb{N}$. Since each P_n is an open neighborhood of x in σX, there exists $x_n \in A \cap P_n$. Next we will show that the sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to x in σX. Let U be an arbitrary open neighborhood of x in σX. Then there is $m \in \mathbb{N}$ such that $P_m \subset U$. If not, there is a sequence $\{z_n\}_{n\in\mathbb{N}}$ in X such that $z_n \in P_n \setminus U$ for every $n \in \mathbb{N}$. Since the family $\mathscr{P}_x$ is a decreasing network of x in X, the sequence $\{z_n\}_{n\in\mathbb{N}}$ converges to x in X. However, the set U is a sequential neighborhood of x in X. So the sequence $\{z_n\}_{n\in\mathbb{N}}$ is eventually in U, which is a contradiction. Thus the set $P_m \subset U$ for some $m \in \mathbb{N}$ and $x_n \in P_n \subset P_m \subset U$ whenever $n > m$. It shows that the sequence $\{x_n\}_{n\in\mathbb{N}}$ in A converges to x in σX. Thus, σX is a Fréchet space.

Conversely, assume that X is an snf-countable space and σX is a Fréchet space. For each $x \in X$, let $\mathscr{P}_x = \{P_n\}_{n\in\mathbb{N}}$ be an snf-network of x in X with each $P_{n+1} \subset P_n$. Put $U_n = X \setminus \mathrm{cl}_{\sigma X}(X \setminus P_n)$ for each $n \in \mathbb{N}$. Then U_n is an open set of σX, so U_n is a sequentially open subset of X and $U_n \subset P_n$. If $x \notin U_n$, i.e., $x \in \mathrm{cl}_{\sigma X}(X \setminus P_n)$. Since σX is a Fréchet space, there is a sequence $\{x_k\}_{k\in\mathbb{N}}$ in $X \setminus P_n$ converging to x, which is a contradiction with P_n being a sequential neighborhood of x in X. Thus, $x \in U_n$. Therefore, the family $\{U_n\}_{n\in\mathbb{N}}$ is an *so*-network of x in X. Hence, X is an sof-countable space. This completes the proof.

To compare the relationships between the weakly first-countable spaces described above, the following question is posed:

Question 2.6 How to find a topological property P such that a space X is an $\aleph_0$-snf-countable space if and only if it is a csf-countable space with property P?

Let $f : X \to Y$ be a mapping. f is called a *finite-to-one* (resp. *countable-to-one*) mapping if, $f^{-1}(y)$ is a finite (resp. countable) subset of X for every $y \in Y$.

Lemma 2.7 *Let $f : X \to Y$ be a closed finite-to-one mapping. If a sequence T in X satisfies that $f(T)$ is a convergent sequence in Y, then the sequence T has a convergent subsequence in X.*

Proof. Put $T = \{x_n\}_{n\in\mathbb{N}}$. Assume that $f(T) = \{f(x_n)\}_{n\in\mathbb{N}}$ is a sequence in Y converging to a point y. Put $K = \{y\} \cup \{f(x_n) : n \in \mathbb{N}\}$ and $L = f^{-1}(K)$. Clearly, K is a compact subset of Y and $T \subset L$. Since f is a closed finite-to-one mapping, L is a compact countable subset of X. Since a compact space with a countable network is metrizable [10], L is a compact metrizable subspace. Then the sequence T in L has a convergent subsequence. This completes the proof.

A mapping $f : X \to Y$ is called a *sequentially quotient mapping* [5] if, whenever $\{y_n\}_{n\in\mathbb{N}}$

is a convergent sequence in Y, there exists a convergent sequence $\{x_i\}_{i\in\mathbb{N}}$ in X such that each $x_i \in f^{-1}(y_{n_i})$ and $\{y_{n_i}\}_{i\in\mathbb{N}}$ is a subsequence of $\{y_n\}_{n\in\mathbb{N}}$. Lemma 2.7 shows that every closed finite-to-one mapping is a sequentially quotient mapping [27].

Lemma 2.8 *Sequential spaces and Fréchet spaces are invariants and inverse invariants under closed finite-to-one mappings.*

Proof. The following three results are known: (1) Sequential spaces are preserved under quotient mappings [8]; (2) Fréchet spaces are invariants under pseudo-open mappings [8]; (3) Sequential spaces are inverse invariants under closed finite-to-one mappings [30]. It is also known that every closed mapping is a pseudo-open mapping, and each pseudo-open mapping is a quotient mapping [10]. To complete the proof, it suffices to show that Fréchet spaces are inverse invariants under closed finite-to-one mappings. The result was announced in the paper [14] by the second author. Here we give it a complete proof. Let $f : X \to Y$ be a closed finite-to-one mapping with Y being a Fréchet space. Let $A \subset X$ and $x \in \overline{A}$. Put $f^{-1}(f(x)) = \{x_1, x_2, \dots, x_n\}$ with $x_1 = x$ for some $n \in \mathbb{N}$. Since X is a T_2 space, there is an open neighborhood V of x such that $\overline{V} \cap \{x_2, \dots, x_n\} = \varnothing$. Then $x \in V \cap \overline{A} \subset \overline{V \cap A}$. It follows that $f(x) \in f(\overline{V \cap A}) = \overline{f(V \cap A)}$. Since Y is a Fréchet space, there is a sequence $\{a_n\}_{n\in\mathbb{N}}$ in $V \cap A$ such that the sequence $\{f(a_n)\}_{n\in\mathbb{N}}$ converges to $f(x)$. By Lemma 2.7, there is a subsequence $\{a_{n_i}\}_{i\in\mathbb{N}}$ of $\{a_n\}_{n\in\mathbb{N}}$ such that $\{a_{n_i}\}_{i\in\mathbb{N}}$ converges to a point $a \in X$. Clearly, $f(a) = f(x)$. Thus $a \in \overline{V} \cap f^{-1}(f(x)) = \{x\}$, and the sequence $\{a_{n_i}\}_{i\in\mathbb{N}}$ in A converges to x. Hence X is a Fréchet space. This completes the proof.

Corollary 2.9 *Let $f : X \to Y$ be a closed finite-to-one mapping. Then σX is a Fréchet space if and only if σY is a Fréchet space.*

Proof. Define a mapping $g : \sigma X \to \sigma Y$ by $g(x) = f(x)$ for any $x \in X$. By Lemma 2.8, we only need to prove that g is a closed finite-to-one mapping. Obviously, g is a finite-to-one mapping. We will show that g is a closed mapping. Suppose that F is a closed set in σY, i.e., F is a sequentially closed subset of Y. Since f is continuous, it is easy to verify that $f^{-1}(F)$ is a sequentially closed in X. Thus, $g^{-1}(F)$ is closed in σX. It shows that g is continuous. On the other hand, suppose that A is a closed subset of σX. Let $\{y_n\}_{n\in\mathbb{N}}$ be a sequence in $f(A)$ converging to $y \in Y$. Choose a sequence $\{x_n\}_{n\in\mathbb{N}}$ in A such that $y_n = f(x_n)$ for each $n \in \mathbb{N}$. By Lemma 2.7, there is a convergent subsequence $\{x_{n_i}\}_{i\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ in X. Suppose that the sequence $\{x_{n_i}\}_{i\in\mathbb{N}}$ in X converges to x. Since A is sequentially closed in X, the limit $x \in A$. Therefore, $y = f(x) \in f(A)$; so $f(A)$ is sequentially closed in Y, i.e., $g(A)$ is closed in σY. Hence, $g : \sigma X \to \sigma Y$ is a closed mapping. This completes the proof.

Lemma 2.10 *α_4-spaces are invariants and inverse invariants under closed finite-to-one mappings.*

Proof. Let $f : X \to Y$ be a closed finite-to-one mapping. Firstly, suppose that X is an α_4-space. Let $y \in Y$ and a sequence S_n in Y converge to y for any $n \in \mathbb{N}$. By Lemma 2.7, there is a convergent sequence T_n in X such that $f(T_n)$ is a subsequence of S_n for each $n \in \mathbb{N}$. Suppose that T_n converges to t_n, then $f(t_n) = y$. Since $f^{-1}(y)$ is a finite set, there exist $x \in X$ and a subsequence $\{t_{n_i}\}_{i\in\mathbb{N}}$ of $\{t_n\}_{n\in\mathbb{N}}$ such that each $t_{n_i} = x$. Moreover, since X is an α_4-space, there is a sequence T in X converging to x such that $\{i \in \mathbb{N} : T \cap T_{n_i} \neq \varnothing\}$ is an infinite set. Then the sequence $f(T)$ in Y converges to y and $\{n \in \mathbb{N} : f(T) \cap S_n \neq \varnothing\}$ is an infinite set. Therefore, Y is an α_4-space.

Conversely, suppose that Y is an α_4-space. Let $x \in X$ and each T_n in X be a sequence converging to x for any $n \in \mathbb{N}$. Since f is a closed finite-to-one mapping and X is a T_2 space, there is a neighborhood V of x such that $\overline{V} \cap f^{-1}(f(x)) = \{x\}$. Without loss of generality, we may assume that $T_n \subset V$ for each $n \in \mathbb{N}$. Obviously, the sequence $f(T_n)$ in Y converges to $f(x)$. Since Y is an α_4-space, there is a sequence S in Y converging to $f(x)$ such that $\{n \in \mathbb{N} : S \cap f(T_n) \neq \varnothing\}$ is an infinite set. Put $S = \{y_k\}_{k\in\mathbb{N}}$. Without loss of generality, we may assume that there exists $x_k \in T_{n_k}$ such that $y_k = f(x_k)$ and $n_k < n_{k+1}$ for each $k \in \mathbb{N}$. By Lemma 2.7, there is a subsequence $\{x_{k_i}\}_{i\in\mathbb{N}}$ of $\{x_k\}_{k\in\mathbb{N}}$ such that $\{x_{k_i}\}_{i\in\mathbb{N}}$ converges to a point $z \in X$. Then, $z \in \overline{V} \cap f^{-1}(f(x)) = \{x\}$. Put $T = \{x_{k_i}\}_{i\in\mathbb{N}}$. Thus the sequence T converges to x and $\{n \in \mathbb{N} : T \cap T_n \neq \varnothing\}$ is an infinite set. Hence X is an α_4-space. This completes the proof.

§3 The closed finite-to-one mapping theorems

In this section, we mainly prove that the following topological properties are preserved and preserved inversely by closed finite-to-one mappings: point-G_δ properties, $\aleph_0$-snf-countability, weak quasi-first-countability, quasi-first-countability, csf-countability, snf-countability, gf-countability and sof-countability.

A space is called having a *point-G_δ property*, if each singleton in X is a G_δ-set in X.

Theorem 3.1 *Point-G_δ properties are invariants and inverse invariants under closed finite-to-one mappings.*

Proof. Let $f : X \to Y$ be a closed finite-to-one mapping.

(1) Suppose that X has a point-G_δ property. For an arbitrary $y \in Y$, put $f^{-1}(y) = \{x_1, x_2, \cdots, x_n\}$ for some $n \in \mathbb{N}$. Since X is a T_2 space, there is a family $\{U_i\}_{i\le n}$ of disjoint open subsets of X such that $x_i \in U_i$ for each $i \in \{1, ..., n\}$. Since $\{x_i\}$ is a G_δ-set in X, there is a family $\{U_{ik}\}_{k\in\mathbb{N}}$ of open neighborhoods of x_i such that each $U_{ik} \subset U_i$ and $\bigcap_{k\in\mathbb{N}} U_{ik} = \{x_i\}$. Obviously, $\bigcap_{k\in\mathbb{N}} U_{ik} \subset \bigcap_{k\in\mathbb{N}} (\bigcup_{i\le n} U_{ik})$ for each $i \le n$. Thus, $\bigcup_{i\le n} (\bigcap_{k\in\mathbb{N}} U_{ik}) \subset \bigcap_{k\in\mathbb{N}} (\bigcup_{i\le n} U_{ik})$. On the other hand, if $x \in \bigcap_{k\in\mathbb{N}} (\bigcup_{i\le n} U_{ik})$, then $x \in \bigcup_{i\le n} U_i$. Therefore, there is $m \le n$ such that $x \in U_m$. So $x \in U_m \cap (\bigcap_{k\in\mathbb{N}} (\bigcup_{i\le n} U_{ik})) \subset \bigcap_{k\in\mathbb{N}} U_{mk} \subset \bigcup_{i\le n} (\bigcap_{k\in\mathbb{N}} U_{ik})$. Therefore, $\bigcap_{k\in\mathbb{N}} (\bigcup_{i\le n} U_{ik}) = \bigcup_{i\le n} (\bigcap_{k\in\mathbb{N}} U_{ik})$. Obviously, $f^{-1}(y) \subset \bigcup_{i\le n} U_{ik}$ for any $k \in \mathbb{N}$. Since f is a closed mapping, there is an open neighborhood V_k of y such that $f^{-1}(V_k) \subset \bigcup_{i\le n} U_{ik}$. We will show that $\bigcap_{k\in\mathbb{N}} V_k = \{y\}$. If $z \in \bigcap_{k\in\mathbb{N}} V_k$, then

$$f^{-1}(z) \subset \bigcap_{k\in\mathbb{N}} f^{-1}(V_k) \subset \bigcap_{k\in\mathbb{N}} (\bigcup_{i\le n} U_{ik}) = \bigcup_{i\le n} (\bigcap_{k\in\mathbb{N}} U_{ik}) = \bigcup_{i\le n} \{x_i\} = f^{-1}(y).$$

Therefore, $z = y$. Thus, Y has a point-G_δ property.

(2) Suppose that Y has a point-G_δ property. For any $x \in X$, since f is a finite-to-one mapping, and X is a HT_1 space, there is an open set U in X such that $U \cap f^{-1}(f(x)) = \{x\}$. Choose a family $\{O_n\}_{n\in\mathbb{N}}$ of open subsets of Y such that $\{f(x)\} = \bigcap\limits_{n\in\mathbb{N}} O_n$. Then,

$$U \cap (\bigcap_{n\in\mathbb{N}} f^{-1}(O_n)) = U \cap f^{-1}(\bigcap_{n\in\mathbb{N}} O_n) = U \cap f^{-1}(f(x)) = \{x\}.$$

Therefore, X has a point-G_δ property. This completes the proof.

The proof of (2) in Theorem 3.1 does not require that f is a closed mapping and X is only a T_1 space.

Theorem 3.2 *$\aleph_0$-snf-countability is an invariant and an inverse invariant under closed finite-to-one mappings.*

Proof. It pointed out that each countable-to-one and sequentially quotient mapping preserves $\aleph_0$-snf-countability in [20, Theorem 2.5]. By Lemma 2.7, $\aleph_0$-snf-countability is an invariant under closed finite-to-one mappings.

Conversely, let $f : X \to Y$ be a closed finite-to-one mapping and Y be an $\aleph_0$-snf-countable space. Put $\mathscr{P}_y = \{P_y(n,m) : n,m \in \mathbb{N}\}$ in Y satisfying the condition of Lemma 2.1(2) for each $y \in Y$. Since f is a finite-to-one mapping, for each $x \in X$ there are disjoint open sets U_1 and U_2 in X such that $x \in U_1$ and $f^{-1}(f(x)) \setminus \{x\} \subset U_2$. Put $Q_x(n,m) = U_1 \cap f^{-1}(P_{f(x)}(n,m))$ for each $n,m \in \mathbb{N}$. We will show that the family $\mathscr{Q}_x = \{Q_x(n,m) : n,m \in \mathbb{N}\}$ in X satisfies the condition of Lemma 2.1(2), then X is an $\aleph_0$-snf-countable space.

Suppose that $x \in X$ and $n \in \mathbb{N}$. If $x \in U \in \tau_X$, then $f^{-1}(f(x)) \subset (U_1 \cap U) \bigcup U_2$. Since f is a closed mapping, there is a neighborhood O of $f(x)$ in Y such that $f^{-1}(O) \subset (U_1 \cap U) \bigcup U_2$. Since $\{P_{f(x)}(n,m)\}_{m\in\mathbb{N}}$ is a network of $f(x)$ in Y, there exists $m \in \mathbb{N}$ such that $P_{f(x)}(n,m) \subset O$. Therefore, $Q_x(n,m) = U_1 \cap f^{-1}(P_{f(x)}(n,m)) \subset U_1 \cap f^{-1}(O) \subset U$. So $\{Q_x(n,m)\}_{m\in\mathbb{N}}$ is a decreasing network of x in X. On the other hand, given each $n, m_n \in \mathbb{N}$, if $Q = \bigcup_{n\in\mathbb{N}} Q_x(n,m_n)$ is not a sequential neighborhood of x in X, there exists a sequence $\{x_k\}_{k\in\mathbb{N}}$ in X converging to x such that $x_k \notin Q$ for each $k \in \mathbb{N}$. Then the sequence $\{f(x_k)\}_{k\in\mathbb{N}}$ in Y converges to $f(x)$. Since $P = \bigcup_{n\in\mathbb{N}} P_{f(x)}(n,m_n)$ is a sequential neighborhood of $f(x)$ in Y, there is $k_0 \in \mathbb{N}$ such that $x_{k_0} \in U_1$ and $f(x_{k_0}) \in P$. Thus, $x_{k_0} \in U_1 \cap f^{-1}(P) = Q$, which is a contradiction. Therefore, Q is a sequential neighborhood of x. This completes the proof.

By Lemma 2.2, Theorem 3.2 and Lemma 2.8, we have the following two corollaries:

Corollary 3.3 *Weak quasi-first-countability is an invariant and an inverse invariant under closed finite-to-one mappings.*

Corollary 3.4 *Quasi-first-countability is an invariant and an inverse invariant under closed finite-to-one mappings.*

Shen Rongxin [23, Lemma 2.4 and Corollary 2.5] proved that weak quasi-first-countability (resp. quasi-first-countability) is preserved by quotient (resp. pseudo-open) and peripherally countable mappings. Every closed mappings are pseudo-open, and every pseudo-open mappings are quotient [10]. By these results, it can be shown that weak quasi-first-countability, and quasi-first-countability are preserved by closed finite-to-one mappings.

Theorem 3.5 *csf-countability is an invariant and an inverse invariant under closed finite-*

to-one mappings.

Proof. Let $f : X \to Y$ be a closed finite-to-one mapping.

Firstly, suppose that X is a csf-countable space. For an arbitrary $y \in Y$, put $f^{-1}(y) = \{x_1, x_2, \dots, x_n\}$ for some $n \in \mathbb{N}$. Let $\{U_{ij}\}_{j\in\mathbb{N}}$ be a countable cs-network of x_i in X for each $i \leq n$. Put $\mathscr{P}_y = \{f(U_{ij}) : i \leq n \text{ and } j \in \mathbb{N}\}$. Obviously, $y \in \bigcap \mathscr{P}_y$. We will show that the family $\mathscr{P}_y$ satisfies the following condition (∗): if $y \in U \in \tau_Y$ and $\{y_n\}_{n\in\mathbb{N}}$ converges to y in Y, there is $P \in \mathscr{P}_y$ such that $P \subset U$ and P contains a subsequence of the sequence $\{y_n\}_{n\in\mathbb{N}}$. In fact, by Lemma 2.7, f is a sequentially quotient mapping; so there is a convergent sequence $\{z_k\}_{k\in\mathbb{N}}$ in X such that $\{f(z_k)\}_{k\in\mathbb{N}}$ is a subsequence of $\{y_n\}_{n\in\mathbb{N}}$. Suppose that the sequence $\{z_k\}_{k\in\mathbb{N}}$ converges to a point $z \in X$, then $z \in f^{-1}(y)$. Thus, there exists $i \leq n$ such that $z = x_i$. Since $x_i \in f^{-1}(U)$, there is $j \in \mathbb{N}$ such that $U_{ij} \subset f^{-1}(U)$ and the sequence $\{z_k\}_{k\in\mathbb{N}}$ is eventually in U_{ij}. It follows that $f(U_{ij}) \subset U$ and $f(U_{ij})$ contains a subsequence of the sequence $\{y_n\}_{n\in\mathbb{N}}$. Next we will prove that the point y in Y has a countable cs-network. Put $\mathscr{F}_y = \{\bigcup \mathscr{P}'_y : \mathscr{P}'_y \text{ is a finite subset of } \mathscr{P}_y\}$; thus $\mathscr{F}_y$ is countable and $y \in \bigcap \mathscr{F}_y$. If a sequence $\{y_n\}$ in Y converges to $y \in V \in \tau_Y$, put $\{F \in \mathscr{F}_y : F \subset V\} = \{F_i\}_{i\in\mathbb{N}}$. Then there is $k \in \mathbb{N}$ such that the sequence $\{y_n\}_{n\in\mathbb{N}}$ is eventually in $\bigcup_{i\leqslant k} F_i$. If not, there is a subsequence $\{y_{n_k}\}_{k\in\mathbb{N}}$ of $\{y_n\}_{n\in\mathbb{N}}$ such that each $y_{n_k} \in X \setminus \bigcup_{i\leqslant k} F_i$. Since $\mathscr{P}_y$ satisfies the condition (∗), there exist a subsequence $\{y_{n_{k_j}}\}_{j\in\mathbb{N}}$ of $\{y_{n_k}\}$ and $P \in \mathscr{P}_y$ such that each $y_{n_{k_j}} \in P \subset V$. Then $P \in \mathscr{F}_y$. Therefore, there exists $m \in \mathbb{N}$ such that $P = F_m$. Thus, $y_{n_{k_m}} \notin F_m = P$, which is a contradiction. Hence Y is a csf-countable space.

Secondly, we will prove the inverse invariant. Suppose that Y is a csf-countable space. For any $x \in X$, since Y is a csf-countable space, there is a countable cs-network $\mathscr{P}_{f(x)}$ of $f(x)$ in Y. Since f is a finite-to-one mapping, there are disjoint open sets U_1 and U_2 in X such that $x \in U_1$ and $f^{-1}(f(x)) \setminus \{x\} \subset U_2$. Put $\mathscr{Q}_x = \{U_1 \cap f^{-1}(P) : P \in \mathscr{P}_{f(x)}\}$. Then $\mathscr{Q}_x$ is a countable cs-network of x in X. In fact, suppose that a sequence $\{x_n\}_{n\in\mathbb{N}}$ in X converges to $x \in V \in \tau_X$. Since $f^{-1}(f(x)) \subset (U_1 \cap V) \bigcup U_2$ and f is a closed mapping, there is a neighborhood O of $f(x)$ in Y such that $f^{-1}(O) \subset (U_1 \cap V) \bigcup U_2$. Since the sequence $\{f(x_n)\}_{n\in\mathbb{N}}$ converges to $f(x) \in O$, there exists $P \in \mathscr{P}_{f(x)}$ such that $P \subset O$ and the sequence $\{f(x_n)\}_{n\in\mathbb{N}}$ is eventually in P. So the sequence $\{x_n\}_{n\in\mathbb{N}}$ is eventually in $U_1 \cap f^{-1}(P)$ and $U_1 \cap f^{-1}(P) \subset U_1 \cap f^{-1}(O) \subset V$. Thus, X is a csf-countable space. This completes the proof.

Corollary 3.6 *snf-countability is an invariant and an inverse invariant under closed finite-to-one mappings.*

Proof. By Lemma 2.3, Theorem 3.5 and Lemma 2.10, snf-countability is an invariant and an inverse invariant under closed finite-to-one mappings. This completes the proof.

Corollary 3.7 *gf-countability is an invariant and an inverse invariant under closed finite-to-one mappings.*

Proof. By Lemma 2.4, Corollary 3.6 and Lemma 2.8, gf-countability is an invariant and an inverse invariant under closed finite-to-one mappings. This completes the proof.

Corollary 3.8 *sof-countability is an invariant and an inverse invariant under closed finite-to-one mappings.*

Proof. By Lemma 2.5, Corollary 3.6 and Lemma 2.9, sof-countability is an invariant and an inverse invariant under closed finite-to-one mappings. This completes the proof.

A mapping $f : X \to Y$ is called a *perfect mapping* [7] if f is a closed mapping and each fiber $f^{-1}(y)$ is compact in X. Obviously, every closed finite-to-one mapping is perfect.

Example 3.9 *None of the following properties is an invariant under perfect mappings: a point-G_δ property, csf-countability, $\aleph_0$-snf-countability, weak quasi-first-countability, quasi-first-countability, snf-countability, gf-countability and sof-countability.*

J.E. Vaughan constructed the perfect image X_M of a first-countable regular space X such that the space X_M is not of a point-G_δ property [28, Example 7.5]. Since a space is first-countable if and only if it is a strongly Fréchet and csf-countable space [20, Theorem 3.6], and every strongly Fréchet space is preserved by a perfect mapping [26, Proposition 3.4], the space X_M is not csf-countable. Hence, a point-G_δ property, csf-countability, $\aleph_0$-snf-countability, weak quasi-first-countability, quasi-first-countability, snf-countability, gf-countability or sof-countability is not an invariant under perfect mappings.

The following question [23, Question 2.10] is answered negatively by Example 3.9: Is a quasi-first-countable space preserved under a perfect mapping?

Example 3.10 *None of the following properties is an inverse invariant under perfect mappings: a point-G_δ property, csf-countability, $\aleph_0$-snf-countability, weak quasi-first-countability, quasi-first-countability, snf-countability, gf-countability and sof-countability.*

It was proved that the one-point compactification $A(\omega_1)$ of the discrete space ω_1 is not csf-countable [24, Lemma 4.1]. It is easy to see the space $A(\omega_1)$ is not of a point-G_δ property. Let $Y = \{0\}$ and define a function $q : A(\omega_1) \to Y$ by $q(x) = 0$ for each $x \in A(\omega_1)$. Let the function q be a quotient mapping. Then q is a perfect mapping, and Y is first-countable. Therefore, a point-G_δ property, csf-countability, $\aleph_0$-snf-countability, weak quasi-first-countability, quasi-first-countability, snf-countability, gf-countability or sof-countability is not an inverse invariant under perfect mappings.

§4 Some applications

In this section, we discuss some applications of closed finite-to-one mapping theorems obtained in Section 3. Recently, Good and Macías [13] studied the symmetric products of generalized metric spaces. They obtained some generalized metric properties P such that for a topological space X and each $n \in \mathbb{N}$, the space X or the product space X^n has the property P if and only if $\mathscr{F}_n(X)$ does. The symmetric product properties are closely related to finite productive properties and closed finite-to-one mapping properties [13, 27].

Definition 4.1 [21] Let (X, τ) be a topological space. 2^X denotes the family of all nonempty and compact sets in X. For each $n \in \mathbb{N}$, put $\mathscr{F}_n(X) = \{A \in 2^X : |A| \leq n\}$. The set 2^X is endowed with the *Vietoris topology*, a base of which consists of all subsets of the following forms:

$$\langle U_1, U_2, \cdots, U_k\rangle = \{A \in 2^X : A \subset \bigcup_{i\leq k} U_i \text{ and } A \cap U_i \neq \varnothing, \text{for each } i \in \{1, \cdots, k\}\},$$

where $k \in \mathbb{N}$ and each U_i is open in X. The set $\mathscr{F}_n(X)$ endowed with the subspace topology of 2^X is called the *n-fold symmetric product* of X for each $n \in \mathbb{N}$.

There are two types of n-fold symmetric product properties. The one is that the n-fold symmetric product $\mathscr{F}_n(X)$ has property P if and only if the space X has property P. It is known that the spaces, such as csf-countable, snf-countable and sof-countable spaces, have this n-fold symmetric product property [27]. The other one is that the n-fold symmetric product $\mathscr{F}_n(X)$ has property P if and only if the product space X^n has property P. It is known that gf-countability has this n-fold symmetric product property [27]. In this section, we will show that the following properties have n-fold symmetric product properties: a point-G_δ property, $\aleph_0$-snf-countability, weak quasi-first-countability and quasi-first-countability.

The following lemma indicates that the relations between closed finite-to-one mappings and n-fold symmetric products.

Lemma 4.2 [13] *For a space X and $n \in \mathbb{N}$. $f_n : X^n \to \mathscr{F}_n(X)$ is a closed finite-to-one mapping, where $f_n : X^n \to \mathscr{F}_n(X)$ is defined by $f_n(x_1, x_2, \ldots, x_n) = \{x_1, x_2, \ldots, x_n\}$.*

Lemma 4.3 *$\aleph_0$-snf-countability is finite productive.*

Proof. Assume that $k \in \mathbb{N}$ and $\{X_i : i \in \{1, 2, \cdots, k\}\}$ is a family of $\aleph_0$-snf-countable spaces. Put $X = \prod_{i\le k} X_i$. For each $x = (x_1, x_2, \cdots, x_k) \in X$ and $i \le k$, since X_i is an $\aleph_0$-snf-countable space, let $\mathscr{P}_{x_i} = \{P_{x_i}(n, m) : n, m \in \mathbb{N}\}$ be a family of subsets in X_i, which satisfies Lemma 2.1(3). Put $P_x(\boldsymbol{n}, m) = \prod_{i\le k} P_{x_i}(n_i, m)$ for each $\boldsymbol{n} = (n_1, n_2, \cdots, n_k) \in \mathbb{N}^k$ and $m \in \mathbb{N}$. We will show that the family $\mathscr{P}_x = \{P_x(\boldsymbol{n}, m) : \boldsymbol{n} \in \mathbb{N}^k, m \in \mathbb{N}\}$ in X satisfies Lemma 2.1(3) for each $x \in X$. Firstly, $\mathbb{N}^k$ is countable. For any $\boldsymbol{n} = (n_1, n_2, \cdots, n_k) \in \mathbb{N}^k$, clearly $\{P_x(\boldsymbol{n}, m)\}_{m\in\mathbb{N}}$ is a decreasing family. Let U be a neighborhood of x in X. Then there is an open set U_i in X_i for each $i \le k$ such that $x \in \prod_{i\le k} U_i \subset U$. Since $\{P_{x_i}(n_i, m)\}_{m\in\mathbb{N}}$ is a network of x_i in X_i for each $i \le k$, there exists $m_i \in \mathbb{N}$ such that $P_{x_i}(n_i, m_i) \subset U_i$. Let $m = \max\{m_i : i \le k\}$. Then $P_x(\boldsymbol{n}, m) \subset \prod_{i\le k} P_{x_i}(n_i, m_i) \subset \prod_{i\le k} U_i \subset U$. It shows that $\{P_x(\boldsymbol{n}, m)\}_{m\in\mathbb{N}}$ is a decreasing network of x in X. On the other hand, assume that a sequence $\{x(j)\}_{j\in\mathbb{N}}$ in X converges to $x \in X$ where $x(j) = (x_1(j), x_2(j), \cdots, x_k(j))$ for each $j \in \mathbb{N}$. Then the sequence $\{x_i(j)\}_{j\in\mathbb{N}}$ in X_i converges to x_i for each $i \le k$. By Lemma 2.1(3.2), there exist $n_i \in \mathbb{N}$ and a subsequence $\{x_i(j_m)\}_{m\in\mathbb{N}}$ of $\{x_i(j)\}_{j\in\mathbb{N}}$ such that each $x_i(j_m) \in P_{x_i}(n_i, m)$. Without loss of generality, we may assume that the sequence $\{j_m\}_{m\in\mathbb{N}}$ and i are irrelevant. Let $\boldsymbol{n} = (n_1, n_2, \cdots, n_k) \in \mathbb{N}^k$. Then the subsequence $\{x(j_m)\}_{m\in\mathbb{N}}$ of $\{x(j)\}_{j\in\mathbb{N}}$ satisfies $x(j_m) \in P_x(\boldsymbol{n}, m)$ for each $m \in \mathbb{N}$. By Lemma 2.1, X is an $\aleph_0$-snf-countable space. This completes the proof.

Theorem 4.4 *For $n \in \mathbb{N}$, a topological space X has a point-G_δ property (resp. $\aleph_0$-snf-countability) if and only if $\mathscr{F}_n(X)$ does.*

Proof. Obviously, the point-G_δ property is finite productive and hereditary. So X has a point-G_δ property if and only if X^n does. By Lemma 4.2 and Theorem 3.1, X^n has a point-G_δ property if and only if the n-fold symmetric product does.

By Lemmas 4.2 and 4.3 and Theorem 3.2, the proof of $\aleph_0$-snf-countability is similar. This completes the proof.

Theorem 4.5 *For a topological space X and $n \in \mathbb{N}$, X^n has weak quasi-first-countability (resp. quasi-first-countability) if and only if $\mathscr{F}_n(X)$ does.*

Proof. By Lemma 4.2 and Corollary 3.3 (resp. Corollary 3.4), X^n has weak quasi-first-countability (resp. quasi-first-countability) if and only if $\mathscr{F}_n(X)$ does. This completes the proof.

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