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Finite Determinacy of High Codimension Smooth Function Germs

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Abstract. Mather gave the necessary and sufficient conditions for the finite determinacy smooth function germs with no more than codimension 4. The theorem is very effective on determining low codimension smooth function germs. In this paper, the concept of right equivalent for smooth function germs ring generated by two ideals finitely is defined. The containment relationships of function germs still satisfy finite k-determinacy under sufficiently small disturbance which are discussed in orbit tangent spaces. Furthermore, the methods in judging the right equivalency of Arnold function family with codimension 5 are presented.

§1 Introduction

Finite determinacy of smooth function germs has always been a very active project in the research of singularity theory. Its core idea is taking the jet of function germs, approximating the infinite terms by its finite terms and getting the same topology properties. In recent years, there are a large number of literatures on the study of finite determinacy problem of smooth function germs. For example, Wall showed the necessary and sufficient conditions for the finite determinacy of smooth mapping germs in [14]. Kushner and Leme [5] and Sun and Wilson [12] gave the relationship between the mapping germs of relative stability and the finite relative determinacy. In addition, Liu, Shi and Li provided the definitions and determination methods of finite determinacy and infinite relative determinacy for smooth function germs with certain boundary conditions in [7, 10, 11]. This theory has numerous applications in mathematics and the natural sciences, see [2–4, 6, 13].

The necessary and sufficient conditions, for the finite determinacy of the function germs which are not more than codimension 4, were given by Mather in [9]. The characteristics of

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finite determinacy for smooth function germs are established. His theorem is quite effective on determining low codimension germs. This work is a foundation of the study on the theory of finite determinacy. However, his theorem does not work well on high codimension germs, even such the Arnold function family $N_t(x,y) = xy(x-ty)$, $t \in (1,+\infty)$ (see [1]).

Our work here is a valuable supplement to the previous work mentioned above. We define the concept of right equivalent of two ideals which are finitely generated in the smooth function germ ring (Definition 2.1). And we discuss the containment relationship of function germs which still satisfy finite k-determinacy under sufficiently small disturbance in orbit tangent spaces (Theorem 3.1). Furthermore, we present the judging methods of right equivalency for Arnold function family with codimension 5 (Theorem 3.4 and Example 4.1).

This paper will highlight that not only is our proof the improving and complementary of Mather's finite k-determinacy theorem, but also the idea and methods provided in the proof are more significant. Our works improve the applicability of Mather's finite k-determinacy theorem.

The structure of this paper is as follows. In Section 2, we present some basic notations and preliminaries. In Section 3, the theorems of finite determinacy for smooth function germs with high codimension are established. As an application of the main results, the right equivalency for Arnold function family is presented in Section 4.

All undefined terms and symbols could be found in [8].

§2 The basic concepts and preliminaries

Let E_n be a ring of C^∞ smooth function germs at $0 \in \mathbb{R}^n$, M_n be the only maximal ideal in E_n, M_n^k be the k-th power of M_n, $J(f) = \langle \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n} \rangle_{E_n}$ be the Jacobian ideal of the smooth function germ f. Here $(t,x)=(t,x_1,x_2,\ldots,x_n) \in \mathbb{R} \times \mathbb{R}^n$.

Definition 2.1. Let $I_1 = \langle f_1, f_2, \ldots, f_r \rangle_{E_n}$ and $I_2 = \langle g_1, g_2, \ldots, g_r \rangle_{E_n}$ be finitely generated ideals in E_n. Two ideals I_1 and I_2 are R-equivalent, if there exists an invertible matrix $[u_{ij}]_{r\times r}$ in E_n, such that

$$\begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_r \end{pmatrix} = [u_{ij}]_{r\times r} \begin{pmatrix} g_1 \\ g_2 \\ \vdots \\ g_r \end{pmatrix}.$$

Definition 2.2. Let $f, g \in E_n$. Two function germs f and g are said to be isomorphic(i.e., right equivalence) if there exists a local diffeomorphism germ $\Phi : (\mathbb{R}^n, 0) \to \mathbb{R}^n$ such that $g = f \circ \Phi$.

Definition 2.3. Let $f : (\mathbb{R}^n, 0) \to \mathbb{R}$ be a C^∞ real function germ and k be a positive integer. We say f is k-determined if all the Taylor polynomial germs which have the same order k with f in E_n are right equivalent to f.

In order to prove Theorem 3.4, we will introduce a proposition.

Proposition 2.4. [3] *Let*

$$X = \frac{\partial}{\partial t} + \sum_{i=1}^{n} X_i(x) \frac{\partial}{\partial x_i}$$

be a C^∞ vector field on an open neighborhood of $\mathbb{R} \times \{0\} \subset \mathbb{R} \times \mathbb{R}^n$, $t \in [0,1]$. There exists an open set U containing $[0,1] \times \{0\}$, in which the following system of differential equations has a unique solution.

$$\begin{cases} \frac{d\Phi_1(t,x)}{dt} = X_1(\Phi_1(t,x), \ldots, \Phi_n(t,x)) \\ \frac{d\Phi_2(t,x)}{dt} = X_2(\Phi_1(t,x), \ldots, \Phi_n(t,x)) \\ \qquad \vdots \\ \frac{d\Phi_n(t,x)}{dt} = X_n(\Phi_1(t,x), \ldots, \Phi_n(t,x)), \end{cases}$$

with the initial condition

$$\begin{pmatrix} \Phi_1(0,x) \\ \Phi_2(0,x) \\ \vdots \\ \Phi_n(0,x) \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

Here, $\Phi_t : (U_0, x) \to (U_t, \Phi(t,x))$ is a local diffeomorphism.

§3 Finite determinacy of high codimension smooth function germs

In this section, we present our main results and proofs.

Theorem 3.1. *Let $h(x) \in M_n^k$ be sufficiently small, then $M_n^k \subset M_n \cdot J(f)$ if and only if $M_n^k \subset M_n \cdot J(f + \tau h)$, where $\tau \in [0,1]$.*

Proof. We have

$$\begin{aligned} & M_n \cdot J(f) \\ & = \langle x_1, x_2, \ldots, x_n \rangle_{E_n} \cdot \langle \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n} \rangle_{E_n} \\ & = \langle x_1 \cdot \frac{\partial f}{\partial x_1}, \ldots, x_1 \cdot \frac{\partial f}{\partial x_n}, x_2 \cdot \frac{\partial f}{\partial x_1}, \ldots, x_2 \cdot \frac{\partial f}{\partial x_n}, \ldots, x_n \cdot \frac{\partial f}{\partial x_1}, \ldots, x_n \cdot \frac{\partial f}{\partial x_n} \rangle_{E_n}. \\ & M_n \cdot J(f + \tau h) \\ & = \langle x_1, x_2, \ldots, x_n \rangle_{E_n} \cdot \langle \frac{\partial (f+\tau h)}{\partial x_1}, \frac{\partial (f+\tau h)}{\partial x_2}, \ldots, \frac{\partial (f+\tau h)}{\partial x_n} \rangle_{E_n} \\ & = \langle x_1 \cdot \frac{\partial (f+\tau h)}{\partial x_1}, \ldots, x_1 \cdot \frac{\partial (f+\tau h)}{\partial x_n}, x_2 \cdot \frac{\partial (f+\tau h)}{\partial x_1}, \ldots, x_2 \cdot \frac{\partial (f+\tau h)}{\partial x_n}, \\ & \quad \ldots, x_n \cdot \frac{\partial (f+\tau h)}{\partial x_1}, \ldots, x_n \cdot \frac{\partial (f+\tau h)}{\partial x_n} \rangle_{E_n} \\ & = \langle x_1 \cdot \frac{\partial f}{\partial x_1} + x_1 \cdot \frac{\partial (\tau h)}{\partial x_1}, \ldots, x_1 \cdot \frac{\partial f}{\partial x_n} + x_1 \cdot \frac{\partial (\tau h)}{\partial x_n}, x_2 \cdot \frac{\partial f}{\partial x_1} + x_2 \cdot \frac{\partial (\tau h)}{\partial x_1}, \\ & \quad \ldots, x_2 \cdot \frac{\partial f}{\partial x_n} + x_2 \cdot \frac{\partial (\tau h)}{\partial x_n}, \ldots, x_n \cdot \frac{\partial f}{\partial x_1} + x_n \cdot \frac{\partial (\tau h)}{\partial x_1}, \ldots, x_n \cdot \frac{\partial f}{\partial x_n} + x_n \cdot \frac{\partial (\tau h)}{\partial x_n} \rangle_{E_n}. \end{aligned}$$

Denote that

$$x_1 \cdot \frac{\partial(\tau h)}{\partial x_1} = \tau \cdot \eta_1(x),\ x_1 \cdot \frac{\partial(\tau h)}{\partial x_2} = \tau \cdot \eta_2(x),\ \ \ldots,\ x_1 \cdot \frac{\partial(\tau h)}{\partial x_n} = \tau \cdot \eta_n(x),$$

$$x_2 \cdot \frac{\partial(\tau h)}{\partial x_1} = \tau \cdot \eta_{n+1}(x),\ \ldots,\ x_i \cdot \frac{\partial(\tau h)}{\partial x_j} = \tau \cdot \eta_{(i-1)\cdot n+j}(x),\ \ldots,\ x_n \cdot \frac{\partial(\tau h)}{\partial x_n} = \tau \cdot \eta_{n^2}(x) = \tau \cdot \eta_r(x),$$

where $r = n^2, (i, j = 1, 2, \ldots, n)$. Then

$$M_n \cdot J(f) = \langle g_1(x), g_2(x), \ldots, g_r(x) \rangle_{E_n},$$
$$M_n \cdot J(f + \tau h) = \langle w_1(x), w_2(x), \ldots, w_r(x) \rangle_{E_n},$$

where

$$x_i \cdot \frac{\partial f}{\partial x_j} = g_{(i-1)\cdot n+j}(x), x_i \cdot \frac{\partial (f + \tau h)}{\partial x_j} = w_{(i-1)\cdot n+j}(x), (i, j = 1, 2, \ldots, r).$$

Notice that $h(x) \in M_n^k$, then $\eta_i(x) \in M_n^k$, thus $\eta_i(x) \in M_n^k \subset M_n \cdot J(f)$. And

$$M_n \cdot J(f) = \langle g_1(x), g_2(x), \ldots, g_r(x) \rangle_{E_n},$$

then there exists $a_{ij}(x) \in E_n, (i, j = 1, 2, \ldots, r)$, such that

$$\eta_i(x) = \sum_{j=1}^{r} a_{ij} \cdot g_j(x). \tag{1}$$

Because $h(x) \in M_n^k$ is sufficiently small, we have $x_i \cdot \frac{\partial(\tau h)}{\partial x_j} \in M_n^k$ is sufficiently small for all $\tau \in [0, 1]$. It means that: in (3.1), for each $i, a_{ij}(x)$ is also small enough. That is, $a_{ij}(0)$ is sufficiently small $(i, j = 1, 2, \ldots, r)$. Since

$$w_i(x) = g_i(x) + \tau h_i(x), \quad (i, j = 1, 2, \ldots, r), \tag{2}$$

we have

$$w_i(x) = g_i(x) + \tau \sum_{j=1}^{r} a_{ij}(x) \cdot g_j(x), \quad (i, j = 1, 2, \ldots, r).$$

This is the matrix equation

$$\begin{pmatrix} w_1(x) \\ w_2(x) \\ \vdots \\ w_r(x) \end{pmatrix} = [I + \tau(a_{ij}(x))]_{r\times r} \cdot \begin{pmatrix} g_1(x) \\ g_2(x) \\ \vdots \\ g_r(x) \end{pmatrix}.$$

Here the matrix $I_{r\times r}$ satisfies $\det I_{r\times r} = 1 \neq 0$, since $a_{ij}(x)$ is sufficiently small, we have $\tau \cdot a_{ij}(x)$ is sufficiently small for all $\tau \in [0, 1]$, then $a_{ij}(0)$ is sufficiently small,

$$\det[I_{r\times r} + \tau \cdot (a_{ij}(0))]_{r\times r} \neq 0, (i, j = 1, 2, \ldots, r).$$

The matrix $[I + \tau \cdot (a_{ij}(x))]_{r\times r}$ is reversible. According to Definition 2.1,

$$\langle w_1(x), w_2(x), \ldots, w_r(x) \rangle_{E_n} = \langle g_1(x), g_2(x), \ldots, g_r(x) \rangle_{E_n},$$
$$M_n \cdot J(f + \tau h) = M_n \cdot J(f).$$

Then $M_n^k \subset M_n \cdot J(f)$ if and only if $M_n^k \subset M_n \cdot J(f + \tau h)$. □

By Theorem 3.1, we can get the following corollary.

Corollary 3.2. *Let $M_n^k \subset M_n \cdot J(f)$. If $h(x) \in M_n^k$ is sufficiently small, then the algebraic equation*

$$-h(x) = \sum_{i=1}^{n} X_i(x) \cdot (\frac{\partial f(x)}{\partial x_i} + t \cdot \frac{\partial h(x)}{\partial x_i})$$

is solvable, where $X_i(x) \in M_n$, $i = 1, 2, \ldots, n$, $t \in [0, 1]$.

Proof. For $t \in [0,1], t \cdot h(x)$ is sufficiently small since $h(x) \in M_n^k$ is sufficiently small. According to Theorem 3.1, there exist $h(x) \in M_n^k \subset M_n \cdot J(f)$ and $X_i(x) \in M_n (i = 1, 2, \ldots, n)$, such that

$$-h(x) = \sum_{i=1}^{n} X_i(x) \cdot (\frac{\partial f(x)}{\partial x_i} + t \cdot \frac{\partial h(x)}{\partial x_i}) \in M_n \cdot J(f + t \cdot h).$$

That is, the algebraic equation

$$-h(x) = \sum_{i=1}^{n} X_i(x) \cdot (\frac{\partial f(x)}{\partial x_i} + t \cdot \frac{\partial h(x)}{\partial x_i}), \quad X_i(x) \in M_n, \quad i = 1, 2, \ldots, n, \quad t \in [0, 1]$$

has a solution. □

Lemma 3.3. *Let $F(t, x) = f(x) + t \cdot h(x)$ be a function germ, where $t \in [0, 1]$, $h(x) \in M_n^k$ and $h(x)$ is sufficiently small. Then there exists a vector field*

$$X = \frac{\partial}{\partial t} + \sum_{i=1}^{n} X_i(x) \cdot \frac{\partial}{\partial x_i},$$

such that $X \cdot F = 0$.

Proof. By Corollary 3.2, there exist $X_i(x) \in M_n (i = 1, 2, \ldots, n)$ satisfying the following algebraic equation

$$-h(x) = \sum_{i=1}^{n} X_i(x) \cdot (\frac{\partial f(x)}{\partial x_i} + t \cdot \frac{\partial h(x)}{\partial x_i}).$$

Hence, for $F(t, x) = f(x) + t \cdot h(x)$, if $t \in [0, 1]$ and $h(x) \in M_n^k$ is sufficiently small, there exist a vector field

$$X = \frac{\partial}{\partial t} + \sum_{i=1}^{n} X_i(x) \cdot \frac{\partial}{\partial x_i},$$

such that

$$\begin{aligned} X \cdot F &= \frac{\partial F}{\partial t} + \sum_{i=1}^{n} X_i(x) \cdot \frac{\partial F}{\partial x_i} = \frac{\partial (f(x) + t \cdot h(x))}{\partial t} + \sum_{i=1}^{n} X_i(x) \cdot \frac{\partial (f(x) + t \cdot h(x))}{\partial x_i} \\ &= h(x) + \sum_{i=1}^{n} X_i(x) \cdot (\frac{\partial f(x)}{x_i} + t \cdot \frac{\partial h(x)}{\partial x_i}) = 0. \end{aligned}$$

. □

Theorem 3.4. *Let $f \in E_n$ and $M_n^k \subset M_n \cdot J(f)$. Then the function germ g is right equivalent to function germ f, if $g - f \in M_n^k$, and $j^k g - j^k f \in P_n^k$ are sufficiently small.*

Proof. Let $g - f = h \in M_n^k$ and $F(t, x) = f(x) + t \cdot h(x)$, $t \in [0, 1]$. For sufficiently small $h(x) \in M_n^k$ and $M_n^k \subset M_n \cdot J(f)$, by Lemma 3.3, there exists a vector field

$$X = \frac{\partial}{\partial t} + \sum_{i=1}^{n} X_i(x) \cdot \frac{\partial}{\partial x_i},$$

such that $X \cdot F = 0$. That is,

$$\frac{\partial F}{\partial t} + \sum_{i=1}^{n} X_i(x) \cdot \frac{\partial F}{\partial x_i} = 0.$$

By Proposition 2.4, we have

$$\frac{\partial F}{\partial t} + \sum_{i=1}^{n} \frac{d\Phi_i(t,x)}{dt} \cdot \frac{\partial F}{\partial x_i} = 0,$$

that means

$$\frac{d}{dt}(F \circ \Phi(t,x)) = 0.$$

That is to say no matter what value t is, the derivative on t is 0, so the value of $F \circ \Phi(t,x)$ on t is a constant. Thus, for any $t_1, t_2 \in [0,1]$ and $t_1 \neq t_2$, we have $F \circ \Phi(t_1,x) = F \circ \Phi(t_2,x)$. Especially, when $t_1 = 0, t_2 = 1$, we have $F(0,\Phi(0,x)) = F(1,\Phi(1,x))$. By $F(t,x) = f(x) + t \cdot h(x)$, then $F(0,x) = f(x), F(1,x) = f(x) + h(x) = g(x)$. Hence, $g = f \circ \Phi(1,x)$. This implies that g is isomorphic (right equivalence) to f . □

§4 Example

As an application of Theorem 3.4, we illustrate the following example.

Example 4.1. Let $N_t(x,y) = xy(x - ty)$ be a two variable function family, $t \in (1,+\infty)$. For all $t_0, t_1 \in (1,+\infty)$, $t_0 \neq t_1$, $N_{t_1}(x,y)$ is right equivalent to $N_{t_0}(x,y)$.

Proof. Since

$$M_2 = \langle x,y\rangle_{E_n},\ M_2^3 = \langle x^3, x^2y, xy^2, y^3\rangle_{E_2} = \langle g_1,g_2,g_3,g_4\rangle_{E_2}$$

and $N_{t_0} = x^2y - t_0xy^2$, we have $J(N_{t_0}) = \langle 2xy - t_0y^2, x^2 - 2t_0xy\rangle_{E_{2+1}}$ and

$$\begin{aligned} & M_2 \cdot J(N_{t_0}) \\ &= \langle x,y\rangle_{E_2} \cdot \langle 2xy - t_0y^2, x^2 - 2t_0xy\rangle_{E_{2+1}} \\ &= \langle 2x^2y - t_0xy^2, x^3 - 2t_0x^2y, 2xy^2 - t_0y^3, x^2y - 2t_0xy^2\rangle_{E_{2+1}}. \end{aligned}$$

Let

$$\begin{aligned} & M_2 \cdot J(N_{t_0}) \\ &= \langle 2x^2y - t_0xy^2, x^3 - 2t_0x^2y, 2xy^2 - t_0y^3, x^2y - 2t_0xy^2\rangle_{E_{2+1}} \\ &= \langle x^3 - 2t_0x^2y, 2x^2y - t_0xy^2, x^2y - 2t_0xy^2, 2xy^2 - t_0y^3\rangle_{E_{2+1}} \\ &= \langle f_1,f_2,f_3,f_4\rangle_{E_{2+1}}, \\ & \quad M_2^3 = \langle x^3, x^2y, xy^2, y^3\rangle_{E_2} = \langle g_1,g_2,g_3,g_4\rangle_{E_2}. \end{aligned}$$

This means

$$\begin{cases} g_1 = x^3 = f_1 + 2t_0x^2y = f_1 + 2t_0g_2 \\ g_2 = x^2y = \frac{1}{2}f_2 + \frac{1}{2}t_0xy^2 = \frac{1}{2}f_2 + \frac{1}{2}t_0g_3 \\ g_3 = xy^2 = f_3 + 2t_0xy^2 = f_3 + 2t_0g_3 \\ g_4 = y^3 = -\frac{1}{t_0}f_4 + \frac{2}{t_0}xy^2 = -\frac{1}{t_0}f_4 + \frac{2}{t_0}g_3. \end{cases}$$

That is ,

$$\begin{cases} g_1 - 2t_0 g_2 = f_1 \\ g_2 - \frac{1}{2} t_0 g_3 = \frac{1}{2} f_2 \\ g_3 - 2t_0 g_3 = f_3 \\ g_4 - \frac{2}{t_0} g_3 = -\frac{1}{t_0} f_4. \end{cases}$$

Therefore,

$$\begin{cases} f_1 = g_1 - 2t_0 g_2 \\ f_2 = 2g_2 - t_0 g_3 \\ f_3 = g_3 - 2t_0 g_3 \\ f_4 = 2g_3 - t_0 g_4. \end{cases}$$

The matrix representation of the above equations are

$$\begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix} = \begin{pmatrix} 1 & -2t_0 & 0 & 0 \\ 0 & 2 & -t_0 & 0 \\ 0 & 0 & 1-2t_0 & 0 \\ 0 & 0 & 2 & -t_0 \end{pmatrix} \begin{pmatrix} g_1 \\ g_2 \\ g_3 \\ g_4 \end{pmatrix}.$$

For $t_0 \in (1, +\infty)$,

$$\begin{vmatrix} 1 & -2t_0 & 0 & 0 \\ 0 & 2 & -t_0 & 0 \\ 0 & 0 & 1-2t_0 & 0 \\ 0 & 0 & 2 & -t_0 \end{vmatrix} = 2t_0(2t_0 - 1) \neq 0.$$

Thus, the matrix $\begin{pmatrix} 1 & -2t_0 & 0 & 0 \\ 0 & 2 & -t_0 & 0 \\ 0 & 0 & 1-2t_0 & 0 \\ 0 & 0 & 2 & -t_0 \end{pmatrix}$ is invertible. By Definition 2.1, we have

$$M_n \cdot J(N_{t_0}) = M_n^3.$$

Therefore, we can draw the conclusion that $N_{t_0}(x, y) = xy(x - t_0 y)$ is 3-determinacy for any $t_0 \in (1, +\infty)$ by Mather's theorem. That is, $g(x, y) \in E_2$ and $g(x, y) - N_{t_0}(x, y) \in M_2^3$, then g is right equivalent to N_{t_0}. But we can not draw the conclusion that N_{t_0} and N_{t_1} are right equivalent for all $t_0, t_1 \in (1, +\infty), t_0 \neq t_1$. However, according to Theorem 3.4, we can show that $N_{t_0}(x, y)$ is right equivalent to $N_{t_1}(x, y)$ for the family of functions $N_t(x, y) = xy(x - ty)$, $t \in (1, +\infty)$, $t_0, t_1 \in (1, +\infty)$, and $t_0 \neq t_1$.

In fact, for any $t_0, t_1 \in (1, +\infty)$, $t_0 \neq t_1$, if $|t_0 - t_1|$ is small enough, then

$$j^3 N_{t_1}(x, y) - j^3 N_{t_0}(x, y) = N_{t_1}(x, y) - N_{t_0}(x, y) = xy^2(t_0 - t_1) \in P_2^3$$

is sufficiently small. We have $N_{t_1}(x, y)$ is right equivalent to $N_{t_0}(x, y)$ by Theorem 3.4. □

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