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V-uniform ergodicity for fluid queues

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Abstract. In this paper, we show that a positive recurrent fluid queue is automatically V-uniformly ergodic for some function $V \geq 1$ but never uniformly ergodic. This reveals a similarity of ergodicity between a fluid queue and a quasi-birth-and-death process. As a byproduct of V-uniform ergodicity, we derive computable bounds on the exponential moments of the busy period.

§1 Introduction

A fluid queue (X_t, φ_t) is a two-dimensional continuous-time Markov process. The first component X_t, called the level, takes values continuously and represents the content of the fluid buffer at time t. The second component φ_t, called the phase, takes discrete values and corresponds to the state of an underling Markov process at time t. The level is controlled by the phase. A good explanation of the queue is a buffer or reservoir that is filled up with water and emptied out, wherein the content X_t varies linearly with time t, and the rate of variation depends on φ_t, the state of a continuous time Markov process evolving in the background. Fluid queue models with Markov modulated input rates have been widely used for the performance evaluation of telecommunication and computer systems. See, for example, Elwalid and Mitra (1991), and Van Foreest et al. (2003). Fluid queues have also appeared to be useful in the analysis of risk processes. See, for example, Badescu et al. (2005).

The joint stationary distribution of the level and the phase has been investigated by various approaches, see Stern and Elwalid (1991) for the spectral analysis method, Rogers (1994) for the Wiener-Hopf factorization method, and Ramaswami (1999) for the Markov-renewal approach. Ramaswami (1999), for the first time, developed effective numerical algorithms by investigating the similarity between fluid queues and discrete-time quasi-birth-and-death (QBD) processes (see e.g. Latouche and Ramaswami, 1999). Inspired by Ramaswami (1999), the PhD thesis of da Silva Soares (2005) continues to build the analogy between fluid queues and QBD processes.

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Stochastic stability has also been a central topic in the study of stochastic processes. The similarity of stability between discrete-time QBD processes and fluid queues has been recently investigated by Govorun et al. (2013), and characteristic inequalities have been established there for a fluid queue to be transient, null recurrent or positive recurrent.

In this paper, we investigate the convergence rate of the transition kernel function to its invariant probability distribution for a positive recurrent fluid queue. In Section 2, we give a mathematical description of the fluid queue, and state the main results of this paper. Our results show that positive recurrent fluid queues are V-uniformly ergodic for some $V \geq 1$ but never uniformly ergodic, which has been found for discrete-time or continuous-time QBD processes by Liu and Hou (2006), Dendievel et al. (2013) and Mao et al. (2014). In this sense, we build up the analogy of ergodicities between QBD processes and fluid queues. The proof of V-uniform ergodicity is deferred to Section 3. Following the proof of V-uniform ergodicity, we derive computable bounds on the exponential moments of the busy period. The conclusion of this paper is provided in Section 4.

§2 Mathematical model and main results

Let φ_t be a continuous-time Markov chain on the finite state space $\mathbb{E} := \{1, 2, \cdots, m\}$ with an irreducible infinitesimal generator T. A fluid queue is a continuous-time two-dimensional Markov process $\{(X_t, \varphi_t) : t \in \mathbb{R}_+\}$, given by

$$X_t = Y_t - \inf_{0 \leq s \leq t} Y_t,$$

where $Y_t = \int_0^t r_{\varphi_s} ds$. Its state space is $\mathbb{S} = \mathbb{R}_+ \times \mathbb{E} = \bigcup_{i=0}^{\infty} \ell(i)$ with $\ell(i) := \{(x, i), x \in \mathbb{R}_+, i \in \mathbb{E}\}$. The components X_t and φ_t are called the level and phase respectively. The numbers r_i, called the fluid rates, constitute a real-valued function on $\mathbb{E}$. We assume that the set $\mathbb{E}$ is partitioned into $\mathbb{E} = \mathbb{E}^- \bigcup \mathbb{E}^0 \bigcup \mathbb{E}^+$, where $\mathbb{E}^- = \{i \in \mathbb{E} : r_i < 0\}$, $\mathbb{E}^0 = \{i \in \mathbb{E} : r_i = 0\}$ and $\mathbb{E}^+ = \{i \in \mathbb{E} : r_i > 0\}$. Let $\mathbb{E}^* = \mathbb{E}^- \bigcup \mathbb{E}^0$ and $\ell(0)^* = \{(0, i) | i \in \mathbb{E}^*\}$. Note that a state $(0, i) \in \ell(0)^*$ behaves like one in a continuous-time Markov chain on a countable state space, since whenever the fluid queue hits a state $(0, i)$, it will stay there for a random length exponentially distributed with parameter $-T_{ii}$.

Since T is assumed to be irreducible, it has a stationary probability vector $\boldsymbol{\pi}$, which is the unique solution of the following system

$$\begin{cases} \boldsymbol{\pi} T = \mathbf{0}, \\ \boldsymbol{\pi} \mathbf{e} = 1, \end{cases}$$

where $\boldsymbol{e} = (1, \cdots, 1)^T$ is a column vector of m dimensions.

The following proposition, taken from Govorun et al. (2013), gives a complete characterization of transience and recurrence for the fluid queue.

Proposition 2.1 *Define $\mu = \sum_{i=1}^m \pi_i r_i$ to be the stationary mean drift. The fluid queue is positive recurrent, null recurrent or transient if and only if $\mu < 0$, $\mu = 0$ or $\mu > 0$, respectively.*

For any set $A \subset \mathbb{R}_+$, define the transition kernel by

$$P^t((x,i);(A,j)) = P\{X_t \in A, \varphi_t = j | X_0 = x, \varphi_0 = i\}.$$

If $\mu < 0$, then we have $\lim_{t\to\infty} P^t((x,i);(A,j)) = \beta((A,j))$. The unique limit $\beta((A,j))$ is the invariant probability measure of the transition kernel, which can be explicitly determined, see e.g. Govorun et al. (2013). Obviously, this elementwise convergence is equivalent to the following convergence

$$\lim_{t\to\infty} \|P^t((x,i);\cdot) - \beta\| = 0 \tag{1}$$

for all x, i, where $\|\alpha\| = \sup_{|f|\le 1} |\alpha(f)|$ denotes the usual total variation norm for a signed measure α. Note that the fluid process is called ergodic if (1) holds.

Naturally we may ask if we can show stronger convergence than the known ordinary ergodicity for the fluid queue. To answer this question, we consider V-uniform ergodicity (see, e.g. Down et al. 1995) as follows

$$\|P^t((x,i);\cdot) - \beta\|_V \le DV(x,i)e^{-\rho t}, \quad t \ge 0,$$

where $V \ge 1$ is a measurable function on $\mathbb{S}$, D and ρ are constants such that $D < \infty$ and $\rho > 0$, and $\|\alpha\|_V = \sup_{|f|\le V} |\alpha(f)|$ is referred to as the V-norm. Note that when $V \equiv 1$, $\|\alpha\|_V = \|\alpha\|$, and V-uniform ergodicity is the usual uniform ergodicity or strong ergodicity. Actually, for the fluid queue we can have the explicit expression of the V-norm

$$\begin{aligned}\|P^t((x,i);\cdot) - \beta\|_V &= \sum_{j\in\mathbb{E}} \int_0^\infty |p^t((x,i);(y,j)) - \tilde{\beta}(y,j)| V(y,j) dy \\ &\quad + \sum_{j\in\mathbb{E}_-} |P^t((x,i);(0,j)) - \beta(0,j)| V(y,j),\end{aligned}$$

where $p^t((x,i);(y,j))$ and $\tilde{\beta}(y,j)$ are the densities of the kernel functions $P^t((x,i);(A,j))$ and $\beta(A,j)$, respectively.

We are now in position to state our answers to the above question, which are main results in this paper.

Theorem 2.2 *Suppose that the fluid queue is positive recurrent, i.e. $\mu < 0$. Then*

(i) the fluid queue is V-uniformly ergodic for some function $V \ge 1$, but

(ii) the fluid queue is not uniformly ergodic.

Remark 2.3 Recently, there are a lot of research works on ergodicities for Markov processes with switching, see e.g. Cloez and Hairer (2015) for exponential ergodicity for Markov processes with random switching, and see, e.g. Shao and Xi (2013), Shao (2015), Yin and Xi (2010) for strong ergodicity or exponential ergodicity for regime-switching diffusion processes. Their models are much more involved and complicated than the fluid model considered in this paper. However, there is some difference between our model and results with theirs.

(1) The fluid queue considered in this paper has a reflecting boundary on the level zero, while their models evolves in the entire plane. The particular boundary can result in essential changes of stability for the processes.

(2) This paper investigates exponential ergodicity in V-norm, while theirs consider exponential ergodicity in total norm or Wasserstein distance.

§3 V-uniform ergodicity

We will identify V-uniform ergodicity for the fluid queue by finding an appropriate drift function towards a petite set for the extended generator. Specifically, we will adopt a result in Down et al. (1995), which is presented by Proposition 3.2 below. The criterion involves ψ-irreducibility, aperiodicity and a drift function towards a petite set. We now recall some related definitions, and please refer to Down et al. (1995) for more details. Let Φ_t be a continuous-time Markov process on a locally compact, separable metric space X endowed with the transition kernel $P^t(x, A)$ and the Borel field $\mathscr{B}(X)$, whose extended generator is denoted by $\mathcal{A}$. See, for example, Davis (1993) for more details about the extended generator. A function f is said to be in the domain of $\mathcal{A}$, i.e. $f \in \mathcal{D}(\mathcal{A})$, if there exists a function $g : X \to \mathbb{R}$ such that the process C_t^f, defined by

$$C_t^f = f(\Phi_t) - f(\Phi_0) - \int_0^t g(\Phi_s)ds, \tag{2}$$

is a local martingale. We then define the extended generator $\mathcal{A}$ by letting $\mathcal{A}f = g$. Define $\tau_A = \inf\{t \geq 0 : \Phi_t \in A\}$ to be the first hitting time on set A.

The following lemma identifies the extended generator for the fluid queue. The identification of the generator is useful for studying other problems such as Poisson equation (e.g., Jiang et al. 2014) and central limit theorems (e.g., Liu and Zhang 2015). Define the weakly infinitesimal generator $\mathcal{B}$ of the fluid queue (X_t, φ_t) by

$$\mathcal{B}f(x, i) = \lim_{h\to 0} \frac{E_{(x,i)}[f(X_h, \varphi_h)] - f(x, i)}{h},$$

where $E_{(x,i)}[\cdot]$ simply denotes the conditional expectation of the fluid queue with the initial state (x, i). Similarly, in the subsequent analysis, we will use $P_{(x,i)}(\cdot)$ to simply denote the conditional probability with the initial state (x, i).

Lemma 3.1 *Let (X_t, φ_t) be the fluid queue, and let $f(x, i)$ be a function on $\mathbb{S}$ such that f is partially differentiable about the first component x.*

(i) For $x > 0, i \in \mathbb{E}$ or $x = 0, i \in \mathbb{E}^+$

$$\mathcal{B}f(x, i) = r_i \frac{df(x, i)}{dx} + \sum_{j\in\mathbb{E}} T_{ij} f(x, j), \tag{3}$$

and for $i \in \mathbb{E}^*$

$$\mathcal{B}f(0, i) = \sum_{j\in\mathbb{E}} T_{ij} f(0, j). \tag{4}$$

(ii) If the partial derivative $\frac{\partial f(x,i)}{\partial x}$ is continuous in x, then $f \in \mathcal{D}(\mathcal{A})$ and $\mathcal{A}f = \mathcal{B}f$.

Proof. (i) To determine the infinitesimal generator $\mathcal{B}$, we need to calculate the expectation $E_{(x,i)}[f(X_h, \varphi_h)]$. Let N_h be the number of state changes of the phase process φ_t in the time interval $[0, h]$. Define $I_k(h) = E_{(x,i)}[f(X_h, \varphi_h), N_h = k], k = 0, 1$. For $x > 0$, and $i \in \mathbb{E}$, utilizing the law of total probability, we have

$$E_{(x,i)}[f(X_h, \varphi_h)] = \sum_{k=0}^{\infty} E_{(x,i)}[f(X_h, \varphi_h), N_h = k] = I_0(h) + I_1(h) + \hat{I}_2(h), \tag{5}$$

where $\hat{I}_2(h)=\sum_{k=2}^{\infty}E_{(x,i)}[f(X_h,\varphi_h),N_h=k]$.

We first consider $I_0(h)$. For this case, the phase process φ_t stays at the initial state i during the time interval $[0,h]$, which implies that the level process X_t will hit the state of $x+r_ih$ at time h. Moreover, the probability that the phase process φ_t stays at i in the interval $[0,h]$ equals $e^{-|T_{ii}|h}$. Hence we have

$$\begin{aligned} I_0(h) &= f(x+r_ih,i)e^{-|T_{ii}|h} \\ &= f(x+r_ih,i)(1+T_{ii}h+o(h)) \\ &= f(x,i)+f(x,i)T_{ii}h+\frac{df(x,i)}{dx}r_ih+o(h), \end{aligned} \tag{6}$$

where the notation $o(h)$ has the usual meaning $\lim_{h\to 0}o(h)/h=0$.

Now we consider $I_1(h)$. For this case, it only permits the phase process φ_t to jump once during the time interval $[0,h]$. By performing the decomposition according to the first jump time of the phase process φ_t, we have

$$I_1(h) = \sum_{j\neq i}\int_0^h f(x+r_is+r_j(h-s),j)|T_{ii}|e^{-|T_{ii}|s}\frac{T_{ij}}{|T_{ii}|}e^{-|T_{jj}|(h-s)}ds.$$

By the Mean Value Theorem for Integrals, we know that there exists some point $\eta\in[0,h]$, dependent on j and h, such that

$$I_1(h)=\sum_{j\neq i}f(x+r_i\eta+r_j(h-\eta),j)T_{ij}e^{-|T_{ii}|\eta}e^{-|T_{jj}|(h-\eta)}h. \tag{7}$$

Now we consider $\hat{I}_2(h)$. It is routine to write

$$\hat{I}_2(h) = \sum_{j\in\mathbb{E}}\int_{A_h(x)} f(y,j)P_{(x,i)}(X_h\in(y,y+dy),\varphi_h=j,N_h\geq 2),$$

where $A_h(x)$ denotes the set of values that the fluid process X_t takes at time h. For any given x, it is easy to see $A_h(x)\leq[0,x+rh]$ where $r=\max_{i\in\mathbb{E}}|r_i|$. Define $\hat{f}_h(x,j)=\max\{f(y,j):0\leq y\leq x+rh\}$ to be the maximal value of $f(y,j)$ in the interval $[0,x+rh]$. Hence we can have

$$\begin{aligned} \hat{I}_2(h) &\leq \sum_{j\in\mathbb{E}}\hat{f}_h(x,j)\int_0^{x+rh}P_{(x,i)}(X_h\in(y,y+dy),\varphi_h=j,N_h\geq 2) \\ &= \sum_{j\in\mathbb{E}}\hat{f}_h(x,j)P_{(x,i)}(X_h\in[0,x+rh],\varphi_h=j,N_h\geq 2), \end{aligned}$$

which implies $\hat{I}_2(h)=o(h)$ since $P_{(x,i)}(N_h\geq 2)=o(h)$. Substituting this, (6) and (7) into (5), we obtain (3) easily.

For $x=0$, and $i\in\mathbb{E}^+$, it can be similarly proved that $f(x,i)$ also satisfies (3). While for $x=0,i\in\mathbb{E}^*$, noting the difference in analyzing $I_0(h)$, we have (4).

(ii) Define $O_m=[0,m)\times\mathbb{E}$ for any $m\geq 0$ and $T_m=\tau_{\mathbb{S}-O_m}=\inf\{t\geq 0:(X_t,\varphi_t)\notin O_m\}$. The stopped process $(X_{t\wedge m},\varphi_{t\wedge m})$ is defined by

$$(X_{t\wedge m},\varphi_{t\wedge m})=\begin{cases}(X_t,\varphi_t), & t<T_m,\\ \Delta_m, & t\geq T_m,\end{cases}$$

where Δ_m is a point not in the state space of (X_t,φ_t). Similar to the analysis of part (i) with special attention to the restriction in the region O_m, we can show that

$$\lim_{h\to 0}\frac{E_{(x,i)}[f(X_{h\wedge T_m},\varphi_{h\wedge m})]-f(x,i)}{h}=\mathcal{B}f(x,i).$$

According to Kushner (1967) or Meyn and Tweedie (1993), we see that the following Dynkin formula holds

$$E_{(x,i)}[f(X_{t\wedge T_m},\varphi_{t\wedge m})]-f(x,i)=E_{(x,i)}\left[\int_0^{t\wedge T_m}\mathcal{B}f(x_s,\varphi_s)ds\right].$$

This implies that if $\frac{\partial f(x,i)}{\partial x}$ is continuous in x, process C_t^f, defined by (2), is a local martingale with respect to the natural σ-algebra $\mathcal{F}_t$, since $C_{t\wedge T_m}^f$ is uniformly integrable. Thus we obtain the second assertion.

Let $\mathbf{a}$ be a sampling distribution on $(0,\infty)$ and let $\nu_{\mathbf{a}}$ be a nontrivial measure on $\mathscr{B}(X)$. A nonempty set $C\in\mathscr{B}(X)$ is called $\nu_{\mathbf{a}}$-petite if

$$K_{\mathbf{a}}(x,\cdot)\geq\nu_{\mathbf{a}}(\cdot),\quad x\in C, \tag{8}$$

where $K_{\mathbf{a}}(x,\cdot)=\int_0^\infty P^t(x,.)\mathbf{a}(dt)$ for $x\in X$.

Proposition 3.2 *For a ψ-irreducible, aperiodic Markov process Φ_t, if there exist constants $b,c>0$, a petite set C in $\mathscr{B}(X)$ and a function $V\geq 1$, such that*

$$\mathcal{A}V\leq -cV+b\mathbb{I}_C$$

holds, then Φ_t is V-uniformly ergodic.

To avoid introducing too many notations, we do not give the definitions of ψ-irreducibility and aperiodicity, for which please refer to Down et al. (1995). We note two obvious facts that an ergodic fluid queue is ψ-irreducible and aperiodic, and that the state space $\mathbb{S}$ of a fluid queue is locally compact and separable. In the following, we will show that $\ell(0)^*$ is a petite set.

Lemma 3.3 *For the positive recurrent fluid queue, $\ell(0)$ is a petite set.*

Proof. Let $\mathbf{a}(dt)=\lambda e^{-t}dt$. Since the fluid queue is positive recurrent, the invariant distribution π exists. Hence for any state $i\in\mathbb{E}$, there exist a finite time $t(i)$ such that for any $t\geq t(i)$

$$P^t((0,i);B)\geq\frac{\pi(B)}{2},$$

which implies that

$$K_{\mathbf{a}}((0,i);B)\ =\ \int_0^\infty P^s((0,i);B)e^{-s}ds\geq\int_{t(i)}^\infty\frac{\pi(B)}{2}e^{-t}dt\geq\frac{e^{-\hat{t}}}{2}\pi(B),$$

where $\hat{t}=\max_{i\in\mathbb{E}}t(i)<\infty$. Hence $\ell(0)$ is a petite set.

The following spectral properties of the key matrix $K(s)=T+sR$ are taken from Section 4 in Asmussen (1994), where $R=diag(r_i:i\in\mathbb{E})$ is a diagonal matrix.

Proposition 3.4 *(i) For $s\geq 0$, the matrix $K(s)$ has a real simple eigenvalue $k(s)$, which is larger than the real part of any other eigenvalue. The corresponding left-eigenvector $\boldsymbol{\nu}(s)$ and right-eigenvector $\boldsymbol{h}(s)$ can be chosen so that $\nu_i(s)>0$, $h_i(s)>0$ for all $i\in\mathbb{E}$ and $\boldsymbol{\nu}(s)\boldsymbol{e}=\boldsymbol{\nu}(s)\boldsymbol{h}(s)=1$.*

(ii) The function $k(s)$ is strictly convex with

$$k'(0)=\sum_{i\in\mathbb{E}}\pi_i r_i,\quad \lim_{s\to\infty}k(s)=\infty,\quad \lim_{s\to\infty}k'(s)=\max_{i\in\mathbb{E}^+}r_i,\quad \lim_{s\to-\infty}k'(s)=\min_{i\in\mathbb{E}^-}r_i.$$

Now we present the drift function for the extended generator and finish the proof of V-uniform ergodicity. For the need of our arguments, it is convenient to write (3) and (4) into the matrix form. Define $\tilde{R} = diag(\tilde{r}_i : i \in \mathbb{E})$ to be a diagonal matrix, where $\tilde{r}_i = r_i, i \in \mathbb{E}^+$ and $\tilde{r}_i = 0, i \in \mathbb{E}^*$. Then we have

$$\mathcal{B}\boldsymbol{f}(x) = R\frac{d\boldsymbol{f}(x)}{dx} + T\boldsymbol{f}(x), \quad x \geq 0, i \in \mathbb{E}, \tag{9}$$

$$\mathcal{B}\boldsymbol{f}(x) = \tilde{R}\frac{d\boldsymbol{f}(x)}{dx} + T\boldsymbol{f}(x), \quad x = 0, i \in \mathbb{E}, \tag{10}$$

where $\boldsymbol{f}(x) = (f(x,1), \cdots, f(x,m))^T$ is a column vector.

Theorem 3.5 *Suppose that the fluid queue is positive recurrent. Then for any $s \in (0, s_1)$, the following drift condition holds*

$$\mathcal{A}\boldsymbol{V}(x) \leq -\lambda(s)\boldsymbol{V}(x) + b(s)\mathbf{e}\mathbb{I}_{\ell(0)}(x), \tag{11}$$

with

$$\boldsymbol{V}(x) = d(s)e^{sx}\boldsymbol{h}(s), \ \lambda(s) = -k(s), \ b(s) = \max_{i\in\mathbb{E}}[(s\tilde{R} + T + \lambda(s)I)\boldsymbol{V}(0)]_i,$$

where I is an identity matrix, $\boldsymbol{h}(s), k(s)$ are given by Proposition 3.4, s_1 is the maximal solution of the equation $k(s) = 0$, and $d(s)$ is the minimal positive real number such that $d(s)\boldsymbol{h}(s) \geq \boldsymbol{e}$. Let s_0 be the unique solution of the equation $k'(s) = 0$, then $0 < s_0 < s_1$ and $\lambda(s) = -k(s)$ takes the maximal value at the point s_0.

Furthermore, the fluid queue is V-uniformly ergodic.

Proof. For any $s > 0$, define

$$\boldsymbol{V}(x) = d(s)e^{sx}\boldsymbol{h}(s),$$

where $d(s)$ is the minimal positive real number such that $d(s)\boldsymbol{h}(s) \geq \boldsymbol{e}$. It is obvious that $V(x,i)$ is continuously partially differentiable about the first component x, which implies that $V \in \mathcal{D}(\mathcal{A})$ and $\mathcal{A}V(x,i) = \mathcal{B}V(x,i)$.

For $x > 0, i \in \mathbb{E}$, from (9) we have

$$\begin{aligned} \mathcal{A}\boldsymbol{V}(x) &= d(s)Re^{sx}\boldsymbol{h}(s) + Td(s)e^{sx}\boldsymbol{h}(s) = d(s)e^{sx}(sR+T)\boldsymbol{h}(s) = d(s)e^{sx}K(s)\boldsymbol{h}(s) \\ &= k(s)\boldsymbol{V}(x). \end{aligned} \tag{12}$$

For $x = 0, i \in \mathbb{E}$, from (10), we can derive

$$\mathcal{A}\boldsymbol{V}(0) = (s\tilde{R} + T)\boldsymbol{V}(0) \leq b(s)\boldsymbol{e} - \lambda(s)\boldsymbol{V}(0).$$

Obviously, $k(0) = 0$. From Proposition 3.4, we know that the function $k(s)$ is strictly convex and analytic for $s \geq 0$, $k'(0) < 0$, $\lim_{s\to\infty} k(s) = \infty$, and $\lim_{s\to\infty} k'(s) > 0$. These facts guarantee that there exists a unique pair of points s_0 and s_1, such that $0 < s_0 < s_1 < \infty$, $k(s_1) = 0$, $k'(s_0) = 0$ and $k(s)$ takes the minimal value at the point $s = s_0$. Moreover, we have $k(s) < 0$ for any $0 < s < s_1$.

Therefore, for any $0 < s < s_1$, if we let $\lambda(s) = -k(s)$ and choose specific function $d(s)$ such that $d(s)e^{sx}\boldsymbol{h}(s) \geq \mathbf{e}$, then we can get the desired drift inequality (11). By Proposition 3.2 and Lemma 3.3, we obtain that an ergodic fluid queue is V-uniformly ergodic.

Remark 3.6 The drift condition $PV \leq \eta V + b\mathbb{I}_C$ is established in Dendievel et al. (2013)

for discrete-time QBD process. Both drift functions involve the spectral analysis for their own different key matrices. The arguments are quite different but the essential ideas are similar.

For a fluid queue, an important issue is to analyze the busy period related quantities. Let $D_{(x,i)}$ be the excursion time length that starts from the initial state $X_0 = x$, and $\varphi_0 = i \in \mathbb{E}^+$, and ends at the first return time to level 0. When $x > 0$, $D_{(x,i)}$ can be explained as the time needed to finish the service when the initial level is at x and the phase is at i. When $x = 0$ and $i \in \mathbb{E}^+$, $D_{(x,i)}$, which is simply denoted by D_i, is the classical busy period. An explicit expression of the exponential moments of the busy period D_i is given in Asmussen (1994), which however involves a functional inversion and may appear too complicated to be useful for computational purposes. As a byproduct of V-uniform ergodicity, we now present easily applicable upper bounds on $D_{(x,i)}$ and D_i.

Corollary 3.7 *Suppose that the fluid queue is positive recurrent. Let $\lambda_0 = \lambda(s_0)$. Then for any $x \geq 0$, $i \in \mathbb{E}$ and any positive number $\lambda \in (0, \lambda_0]$,*

$$E[e^{\lambda D_{(x,i)}}] \leq d(s_2)e^{s_2 x}h_i(s_2), \tag{13}$$

where the parameters of $d(s)$, $h_i(s)$, $\lambda(s)$ and s_0 have been defined in Theorem 3.5, and s_2 is the smaller one of the two solutions such that $k(s) = \lambda$. Moreover, for $i \in \mathbb{E}^+$,

$$E[e^{\lambda D_i}] \leq d(s_2)h_i(s_2). \tag{14}$$

Proof. By Theorem 6.1 in Down et al. (1995), for any $\lambda \leq \lambda_0$ and $x \geq 0, i \in \mathbb{E}$, we have

$$E[e^{\lambda D_{(x,i)}}] = E_{(x,i)}[e^{\lambda \tau_{\ell(0)}}] \leq V(x,i) = d(s)e^{sx}h_i(s) < \infty,$$

where s is a point such that $\lambda(s) = \lambda$. Actually for any $\lambda \in (0, \lambda_0]$, there are two points, denoted by s_2 and s_3, such that

$$0 < s_2 \leq s_0 \leq s_3 < s_1, \quad \lambda = \lambda(s_2) = \lambda(s_3),$$

where $s_2 = s_3$ if and only if $\lambda = \lambda_0$. In order to derive a better (smaller) bound on $E[e^{\lambda D_{(x,i)}}]$, we choose the smaller point s_2. Hence we have (13).

Now consider the pathwise ordering of the fluid queue. It is easy to know that the fluid queue is of the pathwise ordering for any fixed phase, i.e., for two sample paths of the fluid queue starting at two different states with the same phase but different levels, the path with higher level at the starting time $t = 0$ is always higher than the one with lower initial level at any time $t \geq 0$. Hence we have

$$E[e^{\lambda D_i}] = E_{(0,i)}[e^{\lambda \tau_{\ell(0)}}] \leq \inf_{x>0} E_{(x,i)}[e^{\lambda \tau_{\ell(0)}}] = d(s_2)h_i(s_2)$$

for any state $i \in \mathbb{E}^+$ and $\lambda \leq \lambda_0$.

Now we end this section by presenting the proof of Theorem 2.2.

Proof of Theorem 2.2. The proof of (i) is presented in the proof of Theorem 3.5.

Now we prove (ii). We apply a result in Liu and Hou (2008) for uniform ergodicity. Let Φ_t be an ergodic continuous-time Markov process with an atom x_0 such that the process will sojourn there for an exponentially distributed random time whenever the Markov process hits x_0. Then from Liu and Hou (2008), we know that Φ_t is uniformly ergodic if and only if $\sup_{x \in X} E_x[\tau_{\{x_0\}}] < \infty$, where $\tau_{\{x_0\}}$ is the first hitting time on the state x_0. If the fluid queue is positive recurrent, the set $\ell(0)^*$ is not empty. Denote by $(0,i^*)$ an arbitrarily chosen state in $\ell(0)^*$, which is a desired atom according to the discussions in the beginning of Section 2.

Let $g(x,i)=E_{(x,i)}[\tau_{\ell(0)}]$ for $1\le i\le m$ and $\mathbf{g}(x)=(g(x,1),\cdots,g(x,m))^T$. Theorem 3.3 in Kulkarni and Tzenova (2002) has shown that for $x>0$,

$$\mathbf{g}(x)=\sum_{j=1}^{m_-}a_j\boldsymbol{\phi}_j e^{\lambda_j x}-\frac{\boldsymbol{e}x}{\mu}+\boldsymbol{b},$$

where for all $j\in\mathbb{E}_-$, λ_j and $\boldsymbol{\phi}_j$, are such that $Re(\lambda_j)<0$ and $K(\lambda)\boldsymbol{\phi}=\mathbf{0}$, $\boldsymbol{b}$ is any solution to the linear system $T\boldsymbol{b}=-(-\frac{1}{\mu}R+I)\boldsymbol{e}$, and the coefficients a_j are given by the solution to $\sum_{j=1}^{m_-}a_j\phi_{ij}+b_i=0$ for $i\in\mathbb{E}_-$. Hence,

$$\sup_{(x,i)\in\mathbb{S}}E_{(x,i)}[\tau_{(0,i^*)}]\ge\sup_{(x,i)\in\mathbb{S}}E_{(x,i)}[\tau_{\ell(0)}]=\infty,$$

from which we obtain that the fluid queue is not uniformly ergodic.

§4 Conclusion

We show that a positive recurrent fluid queue is always V-uniformly ergodic for some $V\ge 1$ and derive computable bounds on the busy period in terms of the Lyapunov drift condition method. By investigating the mean first hitting times, we further show that a positive recurrent fluid queue is never uniformly ergodic. This reveals the similarity of ergodicities between a fluid queue and a QBD process.

As we have noted in Remark 2.3, the fluid queue considered in this paper can be seen as a rather simplified model of the Markov processes with random switching or regime-switching diffusion processes with modification on the boundary. It is interesting to investigate stability for more general switching processes with a reflecting boundary, which will be our future research.

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