Appl. Math. J. Chinese Univ.
2019, 34(1): 76-81

A note on the perturbed monomial mapping

QU Cheng-qin[1] ZHU Zhi-wei[2] ZHOU Zuo-ling[3]

Abstract. In this paper, we present a necessary and sufficient condition that the perturbed monomial mapping is ergodic on a sphere $S_{p^{-1}}(1)$, which is in a combination with Anashin's earlier results about the perturbed monomial ergodic mappings on a sphere $S_{p^{-r}}(1),\ r > 1$, completely solve a problem posed by A. Khrennikov about the ergodicity of a perturbed monomial mapping on a sphere.

§1 Introduction

The p-adic ergodic theorey is now in the focus of international research activities due to its theoretical significance and applied value in different areas, e.g., in mathematical physics, computer science, automata theorey, numerical analysis, cryptography, quantitative biology, genetics, etc.([1], [9]). Some of these cognitive models are described by random dynamical systems in the fields of p-adic numbers, see [5], [11]. In this paper we say that the function $f : \mathbb{Z}_p \to \mathbb{Z}_p$ is ergodic whenever f ergodic with respect to the Haar measure μ_p, which is normalized so that the measure of the whole space is 1. Earlier in [7] ergodicity of monomial mappings $x \to x^l$ on spheres $S_{p^{-r}}(1)$ of a radius p^{-r} with a center at 1 was studied: It was shown that for odd p and $r > 1$ the mapping is ergodic iff l is a generator of the group $(\mathbb{Z}/p^2\mathbb{Z})^*$. The following problem was put at the 2nd Intl Conference on p-adic Mathematical Physics by Professor Andrei Khrennikov (see also [7], [8], and [10]):

We know for which l and p the dynamical system $f(x) = x^l$ is ergodic on the sphere $S_{p^{-r}}(1)$. Let us consider the ergodicity of a perturbed system $f(x) = x^l + q(x)$ for some polynomial $q(x) \in \mathbb{Z}_p[x]$ such that all coefficients of $q(x)$ are p-adicaly smaller than p^r. This condition is necessary in order to guarantee that $S_{p^{-r}}(1)$ is invariant. For such a system to be ergodic, it is necessary that l is a generator of $(\mathbb{Z}/p^2\mathbb{Z})^*$. Is this a sufficient condition?

Anashin's general result on smooth dynamics on p-adic spheres [2] implies affirmative answer to the above problem when $r > 1$(see [2]), that is:

Received: 2016-08-27. Revised: 2017-05-01.
MR Subject Classification: 28A78, 28A80.
Keywords: perturbed monomial mapping, ergodic, $p-$adic integers.
Digital Object Identifier(DOI): https://doi.org/10.1007/s11766-019-3490-y.
Supported by the National Natural Science Foundation of China (10771075, 11371379).

Theorem 1. *The perturbed monomial mapping $f(x) = x^l + q(x)$, where $q(x) = p^{r+1}u(x)$ for some function $u \in \mathcal{B}$ (e.g., for a polynomial $u(x) \in \mathbb{Z}_p[x]$) is ergodic on the sphere $S_{p^{-r}}(1)$ (where $r > 1$) if and only if l is primitive modulo p^2.*

However, if $r = 1$, the above question is not answered using Anashin's methods. In this note we use a method to study the structures on $\mathbb{Z}/p^n\mathbb{Z}$ inductively in desJardins and Zieve [3] and Fan and Liao [6], and prove the following theorem, which give an affirmative answer for the above problem if the radius p^{-1}, i.e., when $r = 1$.

Our main result is

Theorem 2. *The perturbed monomial mapping $f(x) = x^l + q(x)$, where $q(x) = p^2u(x)$ for some polynomial $u(x) \in \mathbb{Z}_p[x]$ is ergodic on the sphere $S_{p^{-1}}(1)$ if and only if l is a generator of $(\mathbb{Z}/p^2\mathbb{Z})^*$.*

Remark. Theorem 1 together with Theorem 2 completely solve the problem posed by A. Khrennikov more than 15 years ago.

§2 Proof of Theorem

Let $p \geq 3$ be a prime and let $f \in \mathbb{Z}_p[x]$ be a polynomial with coefficients in $\mathbb{Z}_p$. The dynamics of f on $\mathbb{Z}_p$ is determined by those of its induced finite dynamics on $\mathbb{Z}/p^n\mathbb{Z}$ ([3, 6]). The idea to study these finite dynamics inductively comes from DesJardins and Zieve [3]. It allows Fan and Liao [6] to give the decomposition theorem for any polynomial in $\mathbb{Z}_p[x]$. Let us briefly recall some basic definitions and facts which are useful in proving our main theorem. For details, see [3] or [6]. Let $n \geq 1$ be a positive integer. Denote by f_n the induced mapping of f on $\mathbb{Z}/p^n\mathbb{Z}$, i.e.,

$$f_n(x \pmod{p^n}) = f(x) \pmod{p^n}$$

The dynamical behaviors of f are linked to those of f_n. One is the following.

Theorem 2.1 ([3], [6]). *Let $f \in \mathbb{Z}_p[x]$ and $E \subset \mathbb{Z}_p$ be a compact $f-$invariant set. Then $f : E \to E$ is minimal if and only if $f_n : E/p^n\mathbb{Z}_p \to E/p^n\mathbb{Z}_p$ is minimal for each $n \geq 1$.*

Assume that $\sigma = (x_1, \cdots, x_k) \subset \mathbb{Z}/p^n\mathbb{Z}$ is a cycle of f_n of length k (also called $k-$cycle), i.e.,

$$f_n(x_1) = x_2, \cdots, f_n(x_i) = x_{i+1}, \cdots, f_n(x_k) = x_1.$$

Let $X := \bigcup_{i=1}^k X_i$, where

$$X_i := \{x_i + p^n t; t = 0, \cdots, p-1\} \subset \mathbb{Z}/p^{n+1}\mathbb{Z}.$$

Then

$$f_{n+1}(X_i) \subset X_{i+1} (1 \leq i \leq k-1),\ f_{n+1}(X_k) \subset X_1.$$

In the following we shall study the behavior of the finite dynamics f_{n+1} on the $f_{n+1}-$invariant set X and determine all cycles in X of f_{n+1}, which will be called lifts of σ. Remark that the length of any lift $\widetilde{\sigma}$ of σ is a multiple of k.

Let $g := f^k$ be the k-*th* iterate of f. Then, any point in σ is fixed by g^n, the n-*th* induced map of g. For $x \in \sigma$, denote

$$a_n(x) := g'(x) = \prod_{j=0}^{k-1} f'(f^j(x)). \tag{2.1}$$

$$b_n(x) := \frac{g(x)-x}{p^n} = \frac{f^k(x)-x}{p^n}. \tag{2.2}$$

The coefficient $a_n(x) \pmod p$ is always constant on X_i and the coefficient $b_n(x) \pmod p$ is also constant on X_i but under the condition $a_n(x) \equiv 1 \pmod p$. For simplicity, sometimes we shall write a_n and b_n without mentioning x. The values on the cycle $\sigma = (x_1, \cdots, x_k)$ of the functions a_n and b_n are important for our purpose. Using a_n and b_n, we can show that $g_{n+1} : X_i \to X_i$ is conjugate to a linear map. If $a_n \equiv 1 \pmod p$ and $b_n \not\equiv 0 \pmod p$, then f_{n+1} restricted to X preserves a single cycle of length pk. In this case we say σ grows. According to 1) and 2) of Proposition 2.5 of [6], we have

Theorem 2.2 ([6]). *Assume that $n \geq 2$, if $\sigma = (x_1, \cdots, x_k) \subset \mathbb{Z}/p^n\mathbb{Z}$ grows, then its lift also grows, and the lift of the lift will grow and so on.*

So, the clopen set

$$\mathbb{X} = \bigcup_{i=1}^{k} (x_i + p^n\mathbb{Z}_p)$$

is a minimal set.

Let $g(x) = x^l$, we have the following conclusion:

Theorem 2.3 ([3], [6]). *Let $p \geq 3$ and $r \geq 1$. Then, the monomial dynamical system $g(x) = x^l$ is minimal on the circle $S_{p^{-r}}(1)$ if and only if l is a generator of $(\mathbb{Z}/p^2\mathbb{Z})^*$.*

Assume that $f(x) = x^l + q(x)$, where $q(x) = p^2u(x)$ for some polynomial $u(x) \in \mathbb{Z}_p[x]$. Let $x_0 = 1 + p \in S_{p^{-1}}(1)/p^2\mathbb{Z}$ and $x_i = f^i(x_0)$, $i = 1, \cdots, p-1$, then $x_i \in S_{p^{-1}}(1)/p^2\mathbb{Z}$ for all $i = 1, \cdots, p-1$, and denote $\sigma = (x_0, \cdots, x_{p-1}) \subset \mathbb{Z}/p^2\mathbb{Z}$. In order to show main theorem, we need the following lemmas.

Lemma 2.1 ([7]). *l is a generator of $(\mathbb{Z}/p^2\mathbb{Z})^*$ if and only if l is a generator of $(\mathbb{Z}/p\mathbb{Z})^*$.*

Lemma 2.2. *If l is a generator of $(\mathbb{Z}/p^2\mathbb{Z})^*$, then $\sigma \subset S_{p^{-1}}(1)/p^2\mathbb{Z}$ is a cycle of f_2 of length p.*

Proof. By a simple calculation, we get

$$\begin{aligned}
f(1+p) &\equiv (1+p)^l \pmod{p^2} \equiv 1 + lp \pmod{p^2}, \\
f^2(1+p) &\equiv (1+p)^{l^2} \pmod{p^2} \equiv 1 + l^2p \pmod{p^2}, \\
f^3(1+p) &\equiv (1+p)^{l^3} \pmod{p^2} \equiv 1 + l^3p \pmod{p^2}, \\
&\cdots \\
f^{p-1}(1+p) &\equiv (1+p)^{l^{p-1}} \pmod{p^2} \equiv 1 + l^{p-1}p \pmod{p^2}.
\end{aligned}$$

By Lemma 2.1 we know that the set

$$\{l, l^2, l^3, \cdots, l^{p-1}\} \pmod p$$

is a cyclic permutation of $\{1, 2, 3, \cdots, p-1\}$, which means that $\sigma \subset S_{p^{-1}}(1)/p^2\mathbb{Z}$ is a cycle of f_2. This completes the proof of Lemma 2.2.

Lemma 2.3. *If l is a generator of $(\mathbb{Z}/p^2\mathbb{Z})^*$, then*

$$a_2(x) \equiv 1 \pmod p, \quad b_2(x) \neq 0 \pmod p$$

for all $x \in \sigma$.

Proof. By Lemma 2.1 we know that l is a generator of $(\mathbb{Z}/p\mathbb{Z})^*$, which implies

$$l \neq 1 \pmod p, \ l^{p-1} = 1 + \tau p (\tau \neq 0). \tag{2.3}$$

$$l^p \equiv 1 \pmod p, \ l^p \equiv 1 \pmod{p^2}, \tag{2.4}$$

and

$$l^{p-1} + l^{p-2} + \cdots + l + 1 \equiv 0 \pmod p. \tag{2.5}$$

Taking $x_0 = 1 + p \in S_{p^{-1}}(1)/p^2\mathbb{Z}$, we have

$$\begin{aligned} a_2(x_0) &\equiv \prod_{j=0}^{p-1} f'(f^j(1+p)) \pmod{p^2} \\ &\equiv l^p \prod_{j=0}^{p-1} (f^j(1+p))^{l-1} \pmod{p^2} \\ &\equiv l^p \prod_{j=0}^{p-1} (1 + l^j p))^{l-1} \pmod{p^2} \\ &\equiv l^p (1 + (l^{p-1} + l^{p-2} + \cdots + l + 1)p)^{l-1} \pmod{p^2} \\ &\equiv l^p \pmod{p^2} \equiv 1 \pmod{p^2}. \end{aligned}$$

On the other hand, we have $u(1+p) \equiv u(1) \pmod p$. By a direct calculation, we can get

$$\begin{aligned} f(1+p) &\equiv (1+p)^l + p^2 u(1) \pmod{p^3}, \\ f^2(1+p) &\equiv (1+p)^{l^2} + (l+1)p^2 u(1) \pmod{p^3}, \\ f^3(1+p) &\equiv (1+p)^{l^3} + (l^2+l+1)p^2 u(1) \pmod{p^3}, \\ &\cdots \\ f^{p-1}(1+p) &\equiv (1+p)^{l^{p-1}} + (l^{p-1} + \cdots + l^2 + l + 1)p^2 u(1) \pmod{p^3}. \end{aligned}$$

Therefore, we have

$$\begin{aligned} b_2(x_0) &= \tfrac{f^{p-1}(x_0) - (x_0)}{p^2} \\ &\equiv \tfrac{(1+p)^{l^{p-1}} - (1+p)}{p^2} \pmod p \\ &\equiv \tfrac{l^{p-1}-1}{p} + \tfrac{l^{p-1}(l^{p-1}-1)}{2} \pmod p. \end{aligned} \tag{2.6}$$

According to (2.4) and (2.5),we have

$$\frac{l^{p-1} - 1}{p} = \tau, \quad \frac{l^{p-1}(l^{p-1} - 1)}{2} = \frac{(1 + \tau p)\tau}{2} p, \tag{2.7}$$

where $\frac{(1+\tau p)\tau}{2} \in \mathbb{Z}$. It follows from (2.6) and (2.7) that

$$b_2(x_0) \equiv \tau \neq 0 \pmod p.$$

Finally, from [3] and [6] we know that $a_n(x) \pmod p$ is always constant on the cycle $\sigma = (x_1, \cdots, x_k)$ and the $b_n(x) \pmod p$ is also constant on the cycle $\sigma = (x_1, \cdots, x_k)$ under the condition $a_n(x) \equiv 1 \pmod p$, therefore

$$a_2(x) \equiv 1 \pmod p, \quad b_2(x) \neq 0 \pmod p$$

for all $x \in \sigma$, this completes the proof of Lemma 2.3.

According to Theorem 3.2 of [4], we have

Lemma 2.4([4]). *A compatible mapping $f : \mathbb{Z}_p \to \mathbb{Z}_p$ is ergodic on the sphere $S_{p^{-r}}(y)$ if and only if it induces the residue ring $\mathbb{Z}/p^{r+1}\mathbb{Z}$ a mapping which acts on the subset*

$$S_{p^{-r}}(y) \pmod{p^{k+1}} = \{y + p^r s + p^{r+1}\mathbb{Z} : s = 1, 2, \cdots, p-1\} \subset \mathbb{Z}/p^{k+1}\mathbb{Z}$$

as a permutation with a single cycle, for all $k = r, r+1, \cdots$.

Proof of Theorem 2. By Lemma 2.1-2.4 we obtain Theorem 2. This completes the proof of Theorem 2.

Acknowledgements. The author would like to express their deep gratitude to the referees for reading carefully the manuscript and making many valuable suggestions.

References

[1] S Albeverio, A Yu Khrennikov, P Kloeden. *Memory retrieval as a p-adic dynamical system*, Biosystems, 1999, 49: 105-115.

[2] V S Anashin. *Ergodic transformations in the space of p-adic integers, p-adic mathematical physics*, 3-24, AIP Conf Proc, 826, Amer Inst Phys, Melville, NY, 2006.

[3] D L DesJardins, M E Zieve. *Polynomial mappings* $\pmod{p^n}$, arXiv: math/0103046v1.

[4] H Diao, C E Silva. *Digraph representations of rational functions over the p−adic numbers*, $p-$Adic Numbers, Ultrametric Analysis, and Applications, 2011, 3: 23-38.

[5] D Dubischar, V M Gundlach, A Yu Khrennikov, O Steinkamp. *Attractors of random dynamical systems over p-adic numbers and a model of noisy cognitive processes*, Phys D, 1999, 130: 1-12.

[6] A H Fan, L M Liao. *On minimal decomposition of p-adic polynomial dynamical systems*, Adv Math, 2011, 228: 2116-2144.

[7] M Gundlach, A Khrennikov, K-O Lindahl. *On ergodic behavior of p-adic dynamical systems*, Infin Dimens Anal Quantum Probab Relat Top, 2001, 4: 569-577.

[8] M Gundlach, A Khrennikov, K-O Lindahl. *Ergodicity on p-adic sphere*, In German Open Conference on Probability and Statistics, University of Hamburg, 2000:61.

[9] A Khrennikov. *Non-Archimedean Analysis: Quantum Paradoxes, Dynamical Systems and Biological Models*, Kluwer, 1997.

[10] A Yu Khrennikov, K -O Lindahl, M Gundlach. *Ergodicity in the p-adic framework,* In S Albeverio, N Elander, W N Everitt, and P Kurasov, editors, Operator Methods in Ordinary and Partial Differential Equations (S Kovalevski Symproium, Univ of Stockholm, June 2000), volume 132 of Operator Methods: Advances and Applications, Birkhauser, Basel-Boston-Berlin, 2002.

[11] K O Lindahl. *On Markovian properties of the dynamics on attractors of random dynamical systems over p-adic numbers*, Reports from MASDA, Vaxjo University, June 1999.

[1] Department of Applied Mathematics, South China University, Guangzhou 510640, China.
Email: chengqinqu@tom.com

[2] School of Mathematics and Statistics, Zhaoqing University, Zhaoqing 526061, China.
Email: zhiweizhu@zqu.edu.cn

[3] Lingnan College, Sun Yat-sen University, Guangzhou 510275, China.
Email: lnszzl@mail.sysu.edu.cn