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Perfect matchings on a type of lattices with toroidal boundary

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Abstract. Enumeration of perfect matchings on graphs has a longstanding interest in combinatorial mathematics. In this paper, we obtain some explicit expressions of the number of perfect matchings for a type of Archimedean lattices with toroidal boundary by applying Tesler's crossing orientations to obtain some Pfaffian orientations and enumerating their Pfaffians.

§1 Introduction

Let $G = (V, E)$ be a graph. A perfect matching of G is a set of independent edges covering every vertex of G exactly once. It is also called a close-packed dimer in statistical physics and a Kekulé structure in organic chemistry. Lovász [8] pointed out that counting perfect matchings of graphs not only is an intriguing mathematical problem but also has found plenty of applications in physics and chemistry. And counting the number of perfect matchings in general graphs (even in bipartite graphs) is #P-complete [15]. It makes sense to seek special classes for which the problem can be solved exactly. Many mathematicians, physicists and chemists have given lots of their attention to counting perfect matchings of graphs, as seen, for example, [2, 7, 11, 12, 17, 19, 20, 21, 23].

Kepler [4] proved only 11 Archimedean lattices in the nature. And the Archimedean lattices have attracted the most attention in lattice perfect matchings statistics, such as the quadratic lattice [3, 10, 13, 22], hexagonal lattice [5, 6], kagomé lattice [16], etc. with the different boundary condition.

The 8.8.4 bulk lattice shown in Figure 1, denoted by $G(m, n)$, is one type of Archimedean lattices. It is obtained from an $m \times n$ quadratic lattice graph by replacing each vertex by a quadrangle and two adjacent quadrangles connected with an edge such that all new finite faces are 8-polygons. As shown in Figure 1, we label the vertices of $G(m, n)$ with degree two by a_i, a_i^*, b_k and b_k^*, $i = 1, 2, \cdots, m$ and $k = 1, 2, \cdots, n$. Denote by $G^c(m, n)$ the 8.8.4 lattice

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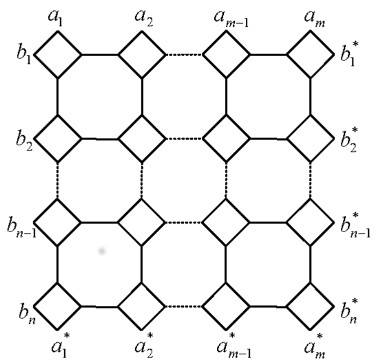

Figure 1: A 8.8.4 lattice.

graph with cylindrical boundary condition, which is obtained from $G(m,n)$ by adding extra edges (b_i, b_i^*) for $1 \le i \le n$. For $0 \le r \le m-1$, denote by $G^k(m,n,r)$ the 8.8.4 lattice graph with Klein bottle boundary condition, which is obtained from $G^c(m,n)$ by adding extra edges $(a_i, a^*_{m+1+r-i})$, where $i = 1, 2, \ldots, m$ and $m+1+r-i$ is modulo m. And denote by $G^t(m,n,r)$ the 8.8.4 lattice graph with toroidal boundary condition, which is obtained from $G^c(m,n)$ by adding extra edges (a_i, a^*_{i+r}), where $i = 1, 2, \ldots, m$ and $i+r$ is modulo m.

Yan, Yeh, Zhang [18] got some explicit expressions of the perfect matching number of the bipartite $G^c(2m,n)$. Lu et al. [9] have obtained explicit expressions of the perfect matching number of $G^k(m,n,0)$ by enumerating 4-Pfaffians.

In this paper, we apply Tesler's crossing orientation [14] to get some Pfaffian orientations of some 8.8.4 lattice graphs with toroidal boundary condition. Furthermore, we obtain explicit expressions of their perfect matching number by enumerating Pfaffians according to the Pfaffian orientations.

§2 Pfaffian orientation

Let G be a graph with vertex set $V(G) = \{1, 2, \cdots, 2p\}$. For an orientation $\overrightarrow{G}$ of G there corresponds a $2p \times 2p$ skew adjacent matrix $A = [a_{uv}]$ of $\overrightarrow{G}$, where $a_{uv} = a_{vu} = 0$ when uv is not an edge in G, and otherwise $a_{uv} = 1$ if the edge uv is directed in $\overrightarrow{G}$ from u to v and $a_{uv} = -1$ if the edge uv is directed in $\overrightarrow{G}$ from v to u. Let $M = \{\{u_1, v_1\}, \cdots, \{u_p, v_p\}\}$ be a perfect matching of G, define the signed weight of M as

$$w(M) = \operatorname{sgn}\begin{pmatrix} 1 & 2 & \cdots & 2p-1 & 2p \\ u_1 & v_1 & \cdots & u_p & v_p \end{pmatrix} \cdot a_{u_1v_1} \cdots a_{u_pv_p},$$

where sgn denotes the sign of the permutation expressed in 2-line notation. The Pfaffian of the matrix A is defined as

$$\operatorname{Pf}(A) = \sum_M w(M).$$

Lemma 2.1. *[8]. If A is a skew symmetric matrix, then* $\det(A) = (\mathrm{Pf}(A))^2$.

Suppose M is a perfect matching of G. Define the sign of M relative to $\overrightarrow{G}$ as

$$\varepsilon_{\overrightarrow{G}}(M) = \mathrm{sgn}\begin{pmatrix} 1 & 2 & \cdots & 2p-1 & 2p \\ u_1 & v_1 & \cdots & u_p & v_p \end{pmatrix} \cdot (-1)^{\sharp \ edges \ oriented \ (v_k,u_k) \ in \ \overrightarrow{G}}.$$

We call $\overrightarrow{G}$ a Pfaffian orientation if every perfect matching of G has the same sign relative to $\overrightarrow{G}$. We say G is Pfaffian if it has a Pfaffian orientation. The significance of Pfaffian orientations stems from the fact that if G has such an orientation, then the number of perfect matchings of G (as well as other related problems) can be computed in polynomial time, namely $|\mathrm{Pf}(A)|$ is equal to the number of perfect matchings of G for the Pfaffian orientation $\overrightarrow{G}$ [8], and by Lemma 2.1, the number of perfect matchings of G can be computed efficiently.

It is easy to see that the torus can be obtained from a 4-polygon P with 4 sides p_1, p_2, p_1' and p_2' in order by passing p_1 and p_1', p_2 and p_2'. A *plane model* of a graph G which can be embedded on the torus is a drawing such that a planar subgraph of G containing all vertices are drawn in P and remains two parts of edges, denoted by E_1 and E_2, where the edges in E_1 are drawn through the sides p_1 and p_1' and disjoint from each other, the edges in E_2 are drawn through the the sides p_2 and p_2' and disjoint from each other.

Now, we draw the 8.8.4 lattice $G^t(m, n, r)$ on torus in a plane model as shown in Figure 2. Firstly, we draw a planar subgraph of $G^t(m, n, r)$ containing mn quadrangles in a 4-polygon $p_1p_2p_1'p_2'$ which is constructed by dotted lines. For convenience, we label the mn quadrangles by $0, 1, 2, \cdots, mn-1$, and label the four vertices of quadrangle i by $4i+1, 4i+2, 4i+3, 4i+4$. Thus the remain two parts of edges not in the 4-polygon are

$E_1 = \{\{4(in-1), 4(in-1)+1\} | i = 1, 2, \cdots, m\}$ and

$E_2 = \{\{4i+2, 4[(m-1)n+i]+3\} | i = 0, 1, \cdots, n-1\}$

$\cup \{\{4(j+1)n, 4(m-r+j)n+1\} | j = 0, 1, \cdots, r-1\}$.

The edges in E_i are drawn through the sides p_i and p_i' and disjoint from each other for $i = 1, 2$. Thus each edge of E_1 has exactly one crossing with each edge of E_2, where a crossing means that two edges pass through the same non-vertex point in the drawing. Let M be a perfect matching of $G^t(m, n, r)$, and $\kappa(M)$ be the crossing number in M. If $M \cap E_1 = \emptyset$ or $M \cap E_2 = \emptyset$, then $\kappa(M)$=0; if $M \cap E_1 \neq \emptyset$ and $M \cap E_2 \neq \emptyset$, then $\kappa(M) = |M \cap E_1| \times |M \cap E_2|$.

For an orientation of a plane graph, we say a face (except out face) of the graph is clockwise odd when its boundary has an odd number of edges pointing clockwise. Tesler [14] formed a *crossing orientation* of such a plane model of G embedding on the torus as follows. Orient the spanning planar subgraph in the 4-polygon so that all its faces are clockwise odd, and orient each edge $e \in E_i$ $(i = 1, 2)$ such that the face formed by e and certain edges in the 4-polygon along the boundary of the spanning plane subgraph is clockwise odd.

Tesler [14] also determined that if an orientation of a graph on torus is a crossing orientation, then every perfect matching M has sign

$$\varepsilon(M) = \varepsilon_0(-1)^{\kappa(M)}; \qquad (1)$$

where $\varepsilon_0 = \pm 1$ is the sign of a perfect matching with no crossing edges (when such exists).

For the plane model of $G^t(m,n,r)$ as above, we construct a crossing orientation of it, which is shown in Figure 2. Firstly, we orient the edges in the 4-polygon such that each face of the quadrangles and octagons are clockwise odd. More specifically, orient the four edges of each quadrangle i anticlockwise except edge $\{4i+2, 4i+4\}$ such that every quadrangle is clockwise odd, orient the edge in $\{\{4i+3, 4(n+i)+2\} | i = 0, 1, \cdots, (m-1)n-1\}$ from $4i+3$ to $4(n+i)+2$, and orient each edge in $\{\{4i, 4i+1\} | i = 1, 2, \cdots, n-2, n+1, n+2, \cdots, mn-2\} \cup \{\{4in+1, 4(r+i+1)n\} | i = 0, 1, \cdots, m-r-1\}$ from $4i$ to $4i+1$ or from $4in+1$ to $4(r+i+1)n$, such that every octagon is clockwise odd.

Then, we orient the edges in E_1. For an edge $e = \{4(in-1), 4(in-1)+1\}$ in E_1. Let P be the path $4(in-1), 4(in-1)+1, 4(in-1)+2, 4[(i-1)n-1]+3, 4[(i-1)n-1]+1, 4[(i-1)n-1]+2, \cdots, 4(n-1)+2, 4n, 4(n-1)+3, \cdots, 4(r+1)n, 1, 2, 4, \cdots, 4(n-1), 4(n-2)+3, \cdots, 4(in-1)$ along the boundary of the subgraph in the 4-polygon. In order to make the face with the boundary cycle consisting of e and P clockwise odd, we orient the edge e from $4(in-1)$ to $4(in-1)+1$.

Finally, we orient the edges in E_2. For an edge e in E_2, if $e \in \{\{4i+2, 4[(m-1)n+i]+3\} | i = 0, 1, \cdots, n-1\}$, orient e from $4i+2$ to $4[(m-1)n+i]+3$. Then the face with the boundary cycle consisting of e and the path $4i+2, 4[(m-1)n+i]+3, 4[(m-1)n+i]+1, 4[(m-1)n+i], \cdots, 4(m-1)n+1, 4(m-1)n+2, 4(m-2)n+3, 4(m-2)n+1, \cdots, 4(m-r-1)n+3, 4(m-r-1)n+1, 4mn, 4mn-1, 4mn-3, \cdots, 4n-3, 4n-2, 4n, 4n-1, 4(2n-1)+2, \cdots, 4(r+1)n, 1, 2, 4, \cdots, 4i+2$, along the boundary of the subgraph of in the 4-polygon, is clockwise odd. For an edge $e \in \{\{4(j+1)n, 4(m-r+j)n+1\} | j = 0, 1, \cdots, r-1\}$, the orientation of e is from $4(j+1)n$ to $4(m-r+j)n+1$, which is decided in a similar discussion.

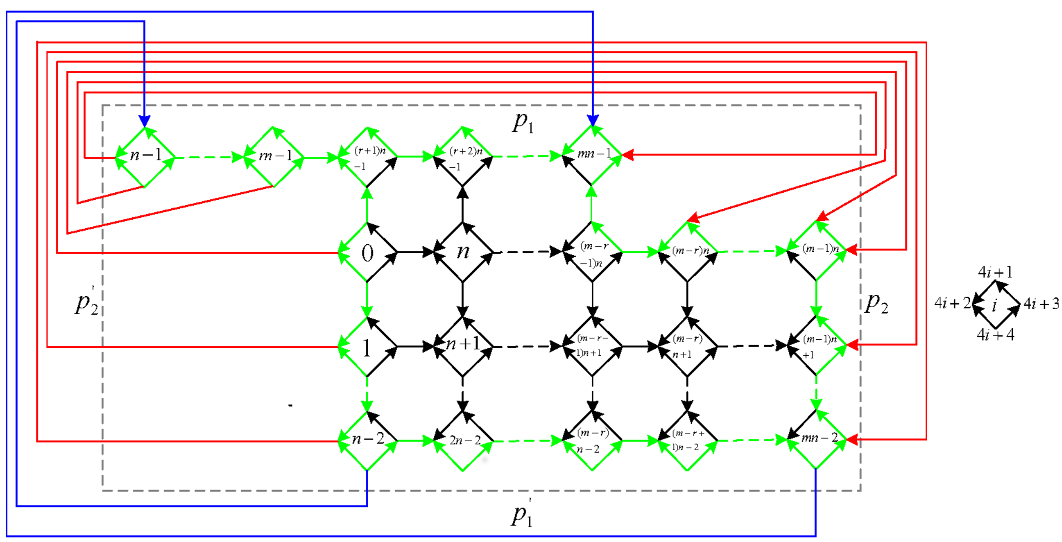

Figure 2: A plane model and a crossing orientation of $G^t(m,n,r)$

It is well known in graph theory that a graph is bipartite if and only if every cycle has even length. Clearly, if a bipartite graph G with a vertex bipartition (X, Y) has a perfect matching, then $|X| = |Y|$. It easy to check that $G^t(m, n, r)$ is non-bipartite if and only if m or $n + r$ is odd. In other words, either $m + n + r$ is odd or both m and $n + r$ are odd.

Denote $\overrightarrow{G}$ the crossing orientation of $G^t(m, n, r)$, and denote $\overrightarrow{G}_{\overleftarrow{E}_1}$ the orientation of $G^t(m, n, r)$ obtained from the crossing orientation $\overrightarrow{G}$ by reversing the orientation of all the edges in E_1.

Claim: (1) *If $m + n + r$ is odd, then the orientation $\overrightarrow{G}$ of $G^t(m, n, r)$ is a Pfaffian orientation*;

(2) *If both m and $n + r$ are odd, then the orientation $\overrightarrow{G}_{\overleftarrow{E}_1}$ of $G^t(m, n, r)$ is a Pfaffian orientation.*

Proof. It suffices to show that all perfect matchings in $\overrightarrow{G}$ and $\overrightarrow{G}_{\overleftarrow{E}_1}$ have the same sign, respectively.

The subgraph of $G^t(m, n, r)$ obtained from it by deleting all edges in $E_1 \cup E_2$ is a bipartite graph, since it contains no odd cycles. Denote the bipartite graph by (X, Y). Noting that all vertices of $G^t(m, n, r)$ are covered by mn quadrangles and the combination of perfect matchings of the quadrangles is a perfect matching of $G^t(m, n, r)$. Hence, $|X| = |Y|$. For an edge $e_1 = uv \in E_1$, the ends u and v of the edge e_1 either both contained in X or both contained in Y if and only if a $u - v$ path in the bipartite graph (X, Y) has even length, and if and only if $n + r$ is odd. Similarly, for an edge $e_2 = wz \in E_2$, both ends w and z of the edge e_2 either both contained in X or both contained in Y if and only if m is odd. Let E_X and E_Y be the sets of edges with both ends in X and Y, respectively. Suppose M is a perfect matching of $G^t(m, n, r)$, then we have $|M \cap E_X| = |M \cap E_Y|$ since $|X| = |Y|$.

(1) If $m + n + r$ is odd, we first consider the case for m is even and $n + r$ is odd. Then $|M \cap E_1| = |M \cap E_X| + |M \cap E_Y| = 2|M \cap E_X|$. Hence, the number of times edges in M cross each other is

$$\kappa(M) = |M \cap E_1| \cdot |M \cap E_2| = 2|M \cap E_X| \cdot |M \cap E_2|.$$

By Equation (1), the sign of M of $G^t(m, n, r)$ related to $\overrightarrow{G}$ is

$$\varepsilon(M) = \varepsilon_0(-1)^{\kappa(M)} = \varepsilon_0.$$

Therefore, the signs of all perfect matchings of $G^t(m, n, r)$ related to $\overrightarrow{G}$ are the same. For m is odd and $n + r$ is even, it can be verified that all perfect matchings of $G^t(m, n, r)$ related to $\overrightarrow{G}$ have the same sign by a similar discussion as above.

(2) If both m and $n + r$ are odd, then $|M \cap E_1| + |M \cap E_2| = |M \cap E_X| + |M \cap E_Y| = 2|M \cap E_X|$. Therefore, the number of times edges in M cross each other is

$$\kappa(M) = |M \cap E_1| \cdot |M \cap E_2| = |M \cap E_1| \cdot (2|M \cap E_X| - |M \cap E_1|).$$

Thus, $\kappa(M) + |M \cap E_1| = 2|M \cap E_1| \cdot |M \cap E_X| - (|M \cap E_1| - 1) \cdot |M \cap E_1|$ is even. And note that $\overrightarrow{G}_{\overleftarrow{E}_1}$ is obtained from the crossing orientation of $G^t(m, n, r)$ by reversing the orientation of all the edges in E_1. By Equation (1) and the definition of the sign of a perfect matching, the sign of M of $G^t(m, n, r)$ related to $\overrightarrow{G}_{\overleftarrow{E}_1}$ is

$$\varepsilon(M) = \varepsilon_0(-1)^{\kappa(M)} \cdot (-1)^{|M \cap E_1|} = \varepsilon_0.$$

Therefore, the signs of all perfect matchings of $G^t(m,n,r)$ related to $\overrightarrow{G}_{\overleftarrow{E}_1}$ are the same. □

§3 The number of perfect matchings of $G^t(m,n,r)$

In the section, we enumerate the number of perfect matchings of the non-bipartite $G^t(m,n,r)$ according Pfaffian orientations of $G^t(m,n,r)$ which is described in the Claim in section 2 by enumerating their Pfaffians. For the $4mn$ vertices of $G^t(m,n,r)$ labeled by $1,2,\cdots,4mn$ as shown in Figure 2, let X be the adjacency matrix of $G^t(m,n,r)$ with the vertex labels and the Pfafian orientation. In order to obtain our results, we introduce the following lemma. Denote the skew block circulant matrix

$$\begin{bmatrix} V_0 & V_1 & V_2 & \cdots & V_{m-1} \\ -V_{m-1} & V_0 & V_1 & \cdots & V_{m-2} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ -V_1 & -V_2 & \cdots & -V_{m-1} & V_0 \end{bmatrix},$$

over the complex number field by $scirc(V_0,V_1,\cdots,V_{m-1})$.

Lemma 3.1. *[1]. Let $V=scirc(V_0,V_1,\cdots,V_{m-1})$ be a skew block circulant matrix over the complex number field, where all V_t are $n\times n$ matrices, $t=0,1,\cdots,m-1$. Then*

$$\det(V)=\prod_{t=0}^{m-1}\det(F_t),$$

where $F_t=V_0+V_1\varepsilon_t+V_2\varepsilon_t^2+\cdots+V_{m-1}\varepsilon_t^{m-1}$, and $\varepsilon_t=\cos\frac{(2t+1)\pi}{m}+i\sin\frac{(2t+1)\pi}{m}$.

Theorem 3.2. *If $m+n+r$ is odd, then the number of perfect matchings of $G^t(m,n,r)$ can be expressed by*

$$|M(G^t(m,n,r))|=\prod_{t=0}^{m-1}\left(\frac{1}{2^n}\left[\left(5+\sqrt{17-4\beta_{2t}}\right)^n+\left(5-\sqrt{17-4\beta_{2t}}\right)^n\right]+\beta_t^n\beta_{rt}\right)^{\frac{1}{2}},$$

where $\beta_{kt}=2\cos\frac{(2t+1)k\pi}{m}$.

Proof. For convenience, we first introduce some notations. Let B^T be the transpose of matrix B. Define 4×4 matrices

$$H_1=\begin{bmatrix} 0 & 1 & -1 & 0 \\ -1 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \end{bmatrix},\ H_2=\begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\1&0&0&0 \end{bmatrix},\ H_3=\begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&1&0&0\\0&0&0&0 \end{bmatrix}$$

and $n\times n$ matrices

$$I_n=\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix},\ J_n=\begin{bmatrix} 0&1&0&\cdots&0\\0&0&1&\cdots&0\\ \vdots&\vdots&\vdots&\ddots&\vdots\\0&0&0&\cdots&1\\0&0&0&\cdots&0 \end{bmatrix}.$$

When $m+n+r$ is odd, then the adjacency matrix X corresponding to $\overrightarrow{G}$ can be written in the following form:

$$X = scirc(V_0, V_1, \cdots, V_{m-1}),$$

where

$$V_i = [0]_{4n\times 4n}, \text{for } i \notin \{0, 1, r, m-r, m-1\},$$

$$V_0 = H_1 \bigotimes I_n + H_2 \bigotimes J_n - H_2^T \bigotimes J_n^T,$$

$$V_1 = V_{m-1}^T = H_3 \bigotimes I_n,$$

$$V_r = V_{m-r}^T = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & 0 \\ 0 & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \end{bmatrix}_{4n\times 4n},$$

and V_i represents the adjacent relations between the vertex set $\{4kn+1,\ 4k+2,\ \cdots,\ 4(k+1)n\}$ and vertex set $\{4(k+i)n+1,\ 4(k+i)+2,\ \cdots,\ 4(k+i+1)n\}$ for $i = 0,\ 1,\ \cdots,\ m-1$ and $0 \le k+i \le m-1$, and $-V_i$ represents the adjacent relations between vertex set $\{4kn+1,\ 4k+2,\ \cdots,\ 4(k+1)n\}$ and vertex set $\{4(k+i-m)n+1,\ 4(k+i-m)+2,\ \cdots,\ 4(k+i-m+1)n\}$ for $i = 1,\ 2,\ m-1$ and $m \le k+i \le 2m-1$. (In the others cases, the adjacency matrix X of $G^t(m,n,r)$ can be obtained in a similar way. We note the differences of X by boldface.) So by Lemma 3.1, we have that

$$\det(X) = \prod_{t=0}^{m-1} \det(F_t),$$

where

$$\begin{aligned} \det(F_t) &= \det(V_0 + V_1\varepsilon_t - V_1^T\varepsilon_t^{-1} + V_r\varepsilon_t^r - V_r^T\varepsilon_t^{-r}) \\ &= \begin{vmatrix} 0 & 1 & -1 & 0 & & & & & & & \varepsilon_t^r \\ -1 & 0 & -\varepsilon_t^{-1} & -1 & & & & & & & \\ 1 & \varepsilon_t & 0 & -1 & & & & & & & \\ 0 & 1 & 1 & 0 & 1 & & & & & & \\ & & & -1 & & & & & & & \\ & & & & & \ddots & & & & & \\ & & & & & & & 1 & & & \\ & & & & & & -1 & 0 & 1 & -1 & 0 \\ & & & & & & & -1 & 0 & -\varepsilon_t^{-1} & -1 \\ & & & & & & & 1 & \varepsilon_t & 0 & -1 \\ -\varepsilon_t^{-r} & & & & & & & 0 & 1 & 1 & 0 \end{vmatrix}. \end{aligned}$$

Partition matrix F_t into nine blocks by splitting the rows into three groups: the first row; the middle $4n-2$ rows; and the last row. Split the columns in the same way. And by the properties of the determinant, we have

$$\begin{aligned}\det(F_t) &:= \begin{vmatrix} 0 & S_1 & \varepsilon_t^r \\ S_2 & S_3 & S_4 \\ -\varepsilon_t^{-r} & S_5 & 0 \end{vmatrix}_{4n\times 4n} = \begin{vmatrix} 0 & S_1 & 0 \\ S_2 & S_3 & S_4 \\ -\varepsilon_t^{-r} & S_5 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 & \varepsilon_t^r \\ S_2 & S_3 & S_4 \\ -\varepsilon_t^{-r} & S_5 & 0 \end{vmatrix} \\ &= \begin{vmatrix} 0 & S_1 & 0 \\ S_2 & S_3 & S_4 \\ 0 & S_5 & 0 \end{vmatrix} + \begin{vmatrix} 0 & S_1 & 0 \\ 0 & S_3 & S_4 \\ -\varepsilon_t^{-r} & S_5 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 & \varepsilon_t^r \\ S_2 & S_3 & S_4 \\ 0 & S_5 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 & \varepsilon_t^r \\ 0 & S_3 & S_4 \\ -\varepsilon_t^{-r} & S_5 & 0 \end{vmatrix} \\ &= \begin{vmatrix} 0 & S_1 & 0 \\ S_2 & S_3 & S_4 \\ 0 & S_5 & 0 \end{vmatrix} + \varepsilon_t^{-r}\begin{vmatrix} S_1 & 0 \\ S_3 & S_4 \end{vmatrix} - \varepsilon_t^{r}\begin{vmatrix} S_2 & S_3 \\ 0 & S_5 \end{vmatrix} + |S_3|.\end{aligned}$$

Set $A_n(t) = \begin{vmatrix} 0 & S_1 & 0 \\ S_2 & S_3 & S_4 \\ 0 & S_5 & 0 \end{vmatrix}$, $B_n(t) = \begin{vmatrix} S_1 & 0 \\ S_3 & S_4 \end{vmatrix}$, $C_n(t) = \begin{vmatrix} S_2 & S_3 \\ 0 & S_5 \end{vmatrix}$ and $D_n(t) = |S_3|$, then we have

$$\det(F_t) = A_n(t) + \varepsilon_t^{-r} B_n(t) - \varepsilon_t^{r} C_n(t) + D_n(t).$$

Let $A'_n(t)$ be the determinant of the matrix which is obtained from $A_n(t)$ by deleting the first row and the first column. Expand $A_n(t)$ by the first four rows and $A'_n(t)$ by the first three rows. We get a recurrences for $A_n(t)$ and $A'_n(t)$ as follows:

$$\begin{cases} A_n(t) = 4A_{n-1}(t) + (\varepsilon_t - \varepsilon_t^{-1})A'_{n-1}(t), & \text{for } n \geq 2, \\ A'_n(t) = (\varepsilon_t^{-1} - \varepsilon_t)A_{n-1}(t) + A'_{n-1}(t), & \text{for } n \geq 2, \\ A_1(t) = 4, A'_1(t) = \varepsilon_t^{-1} - \varepsilon_t. \end{cases} \tag{2}$$

By Equation (2), we have

$$\begin{aligned} &A_{n-1}(t) = 4A_{n-2}(t) + (\varepsilon_t - \varepsilon_t^{-1})A'_{n-2}(t), \\ &A'_{n-2}(t) = \frac{A_{n-1}(t) - 4A_{n-2}(t)}{\varepsilon_t^{-1} - \varepsilon_t}, \\ &A'_{n-1}(t) = (\varepsilon_t^{-1} - \varepsilon_t)A_{n-2}(t) + A'_{n-2}(t) = \left[(\varepsilon_t^{-1} - \varepsilon_t)A_{n-2}(t) + \frac{A_{n-1}(t) - 4A_{n-2}(t)}{\varepsilon_t^{-1} - \varepsilon_t}\right]. \end{aligned}$$

So we have

$$\begin{cases} A_n(t) = 5A_{n-1}(t) - (\varepsilon_t^2 + \varepsilon_t^{-2} + 2)A_{n-2}(t), & \text{for } n \geq 3, \\ A_1(t) = 4, A_2(t) = 18 - \varepsilon_t^2 - \varepsilon_t^{-2}. \end{cases} \tag{3}$$

Let $D'_{n-1}(t)$ be the determinant of the matrix which obtained from $D_n(t)$ by deleting the first three rows and the first three columns. Expand $D_n(t)$ by the first three rows, and $D'_n(t)$ by the first four rows. We get a recurrence in $D_n(t)$ and $D'_n(t)$ as follows:

$$\begin{cases} D_n(t) = D_{n-1}(t) + (\varepsilon_t^{-1} - \varepsilon_t)D'_{n-1}(t), & \text{for } n \geq 2, \\ D'_n(t) = (\varepsilon_t - \varepsilon_t^{-1})D_{n-1}(t) + 4D'_{n-1}(t), & \text{for } n \geq 2, \\ D_1(t) = 1, D'_1(t) = \varepsilon_t - \varepsilon_t^{-1}. \end{cases} \tag{4}$$

By the same deduction on Equation (4) as Equation (2), we have

$$\begin{cases} D_n(t) = 5D_{n-1}(t) - (\varepsilon_t^2 + \varepsilon_t^{-2} + 2)D_{n-2}(t), & \text{for } n \geq 3, \\ D_1(t) = 1, D_2(t) = 3 - \varepsilon_t^{-2} - \varepsilon_t^2. \end{cases} \tag{5}$$

Then the characteristic Equations of (3) and (5) are the same, i.e.

$$\lambda^2 - 5\lambda + (\varepsilon_t^2 + \varepsilon_t^{-2} + 2) = 0. \tag{6}$$

Hence $\lambda = \frac{5 \pm \sqrt{17 - 4\varepsilon_t^2 - 4\varepsilon_t^{-2}}}{2}$ are the characteristic values of Equation (6). Therefore

$$A_n(t) = a_1 \left(\frac{5 + \sqrt{17 - 4\varepsilon_t^2 - 4\varepsilon_t^{-2}}}{2} \right)^n + a_2 \left(\frac{5 - \sqrt{17 - 4\varepsilon_t^2 - 4\varepsilon_t^{-2}}}{2} \right)^n,$$

$$D_n(t) = d_1 \left(\frac{5 + \sqrt{17 - 4\varepsilon_t^2 - 4\varepsilon_t^{-2}}}{2} \right)^n + d_2 \left(\frac{5 - \sqrt{17 - 4\varepsilon_t^2 - 4\varepsilon_t^{-2}}}{2} \right)^n.$$

Since $A_1(t) = 4, A_2(t) = 18 - \varepsilon_t^2 - \varepsilon_t^{-2},\ D_1(t) = 1$ and $D_2(t) = 3 - \varepsilon_t^{-2} - \varepsilon_t^2$, we have that $a_1 = d_2 = \frac{3 + \sqrt{17 - 4\varepsilon_t^2 - 4\varepsilon_t^{-2}}}{2\sqrt{17 - 4\varepsilon_t^2 - 4\varepsilon_t^{-2}}}$ and $a_2 = d_1 = \frac{-3 + \sqrt{17 - 4\varepsilon_t^2 - 4\varepsilon_t^{-2}}}{2\sqrt{17 - 4\varepsilon_t^2 - 4\varepsilon_t^{-2}}}$. Thus,

$$\begin{aligned} A_n(t) &= \frac{1}{2^{n+1}\sqrt{17 - 4\beta_{2t}}} \Bigg[\Big(3 + \sqrt{17 - 4\beta_{2t}} \Big) \Big(5 + \sqrt{17 - 4\beta_{2t}} \Big)^n \\ &\quad + \Big(-3 + \sqrt{17 - 4\beta_{2t}} \Big) \Big(5 - \sqrt{17 - 4\beta_{2t}} \Big)^n \Bigg], \\ D_n(t) &= \frac{1}{2^{n+1}\sqrt{17 - 4\beta_{2t}}} \Bigg[\Big(-3 + \sqrt{17 - 4\beta_{2t}} \Big) \Big(5 + \sqrt{17 - 4\beta_{2t}} \Big)^n \\ &\quad + \Big(3 + \sqrt{17 - 4\beta_{2t}} \Big) \Big(5 - \sqrt{17 - 4\beta_{2t}} \Big)^n \Bigg], \end{aligned}$$

where $\beta_{kt} = \varepsilon_t^k + \varepsilon_t^{-k} = 2\cos\frac{k(2t+1)\pi}{m}$.

Expanding $B_n(t)$ and $C_n(t)$ by the first four rows, we obtain these recurrences for $B_n(t)$ and $C_n(t)$as follows:

$$\begin{cases} B_n(t) = (\varepsilon_t + \varepsilon_t^{-1})B_{n-1}(t), & \text{for } n \geq 2, \\ B_1(t) = \varepsilon_t + \varepsilon_t^{-1}, \end{cases} \tag{7}$$

$$\begin{cases} C_n(t) = (\varepsilon_t + \varepsilon_t^{-1})C_{n-1}(t), & \text{for } n \geq 2, \\ C_1(t) = -(\varepsilon_t + \varepsilon_t^{-1}). \end{cases} \tag{8}$$

And we can get the values of $B_n(t)$ and $C_n(t)$ by Equation (7) and (8), respectively, as following:

$$B_n(t) = -C_n(t) = \beta_t^n.$$

These implies that

$$\det(F_t) = \frac{1}{2^n} \Bigg[\Big(5 + \sqrt{17 - 4\beta_{2t}} \Big)^n + \Big(5 - \sqrt{17 - 4\beta_{2t}} \Big)^n \Bigg] + \beta_t^n \beta_{rt},$$

where $\beta_{kt} = 2\cos\frac{(2t+1)k\pi}{m}$.

And so, by Lemma 2.1, Theorem 3.2 follows. □

Theorem 3.3. *If both m and $n+r$ are odd, then the number of perfect matchings of $G^t(m,n,r)$ can be expressed by*

$$|M(G^t(m,n,r))| = \prod_{t=0}^{m-1} \left(\frac{1}{2^n} \Bigg[\Big(5 + \sqrt{17 - 4\beta_{2t}} \Big)^n + \Big(5 - \sqrt{17 - 4\beta_{2t}} \Big)^n \Bigg] + \beta_t^n \beta_{rt} \right)^{\frac{1}{2}},$$

where $\beta_{kt} = 2\cos\frac{(2t+1)k\pi}{m}$.

Proof. Let H_1, H_2, H_3, I_n be the same matrices as in the proof of Theorem 3.2, and

$$K_n = \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & -1 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

When both m and $n+r$ are odd, then the adjacency matrix X corresponding to $\overrightarrow{G}_{\overleftarrow{E}_1}$ can be written in the following form:

$$X = scirc(V_0, V_1, \cdots, V_{m-1}),$$

where V_i is the same as the case in Theorem 3.2 except $V_0 = H_1 \bigotimes I_n + H_2 \bigotimes \mathbf{K_n} - H_2^T \bigotimes \mathbf{K_n^T}$. So by Lemma 3.1, we always have that

$$\det(X) = \prod_{t=0}^{m-1} \det(F_t),$$

where

$$\begin{aligned} \det\,(F_t) &= det\,(V_0 + V_1\varepsilon_t - V_1^T\varepsilon_t^{-1} + V_r\varepsilon_t^r - V_r^T\varepsilon_t^{-r}) \\ &= \begin{vmatrix} 0 & 1 & -1 & 0 & & & & & & & \varepsilon_t^r \\ -1 & 0 & -\varepsilon_t^{-1} & -1 & & & & & & & \\ 1 & \varepsilon_t & 0 & -1 & & & & & & & \\ 0 & 1 & 1 & 0 & 1 & & & & & & \\ & & & -1 & & & & & & & \\ & & & & & \ddots & & & & & \\ & & & & & & & \mathbf{-1} & & & \\ & & & & & & \mathbf{1} & 0 & 1 & -1 & 0 \\ & & & & & & & -1 & 0 & -\varepsilon_t^{-1} & -1 \\ & & & & & & & 1 & \varepsilon_t & 0 & -1 \\ -\varepsilon_t^{-r} & & & & & & & 0 & 1 & 1 & 0 \end{vmatrix}. \end{aligned}$$

By the same discussion as Theorem 3.2, we have

$$\det(F_t) = \frac{1}{2^n}\left[\left(5+\sqrt{17-4\beta_{2t}}\right)^n + \left(5-\sqrt{17-4\beta_{2t}}\right)^n\right] - \beta_t^n\beta_{rt},$$

where $\beta_{kt} = 2\cos\frac{(2t+1)k\pi}{m}$.

And so, by Lemma 2.1, Theorem 3.3 follows. □

Remark. For bipartite graph $G^t(m,n,r)$, it can be proved that they are not Pfaffian. By now, we have not found an useful method to calculate the number of perfect matchings. By Tesler's method [14], we know that $|M(G^t(m,n,r))|$ can be counting by a linear combination of 4 Pfaffians. However, the expansions of the methods of this paper that using a linear combination of 4 Pfaffians does not work. Since we can get the values of $|\mathrm{Pf}(X_i)|$ by the same means as we used in Theorem 3.2, but the sign of $\mathrm{Pf}(X_i)$ is uncertain. Hence, for a bipartite graph $G^t(m,n,r)$, $|M(G^t(m,n,r))|$ is still open.

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