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Reductivity and bundle shifts

XU An-jian†

Abstract. For the Hardy space $H_E^2(R)$ over a flat unitary vector bundle E on a finitely connected domain R, let T_E be the bundle shift as [3]. If $\mathcal{B}$ is a reductive algebra containing every operator $\psi(T_E)$ for any rational function ψ with poles outside of R, then $\mathcal{B}$ is self adjoint.

§1 Introduction

In this paper, let $\mathcal{H}$ be a complex separable Hilbert space, and $B(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$. A unital subalgebra $\mathcal{A}$ of $B(\mathcal{H})$ is called transitive if it has only trivial invariant subspaces. The transitive algebra problem ask if every transitive algebra $\mathcal{A} \subset B(\mathcal{H})$ is strongly dense in $B(\mathcal{H})$. An operator is called transitive if every operator algebra containing it is transitive. It is Arveson [5] who stated explicitly the problem first, and he developed a main tool for studying the transitive algebra problem. In the same paper, Arveson proved that the unilateral shift with multiplicity one is transitive. Richter [13] proved that Dirichlet shift is transitive. And Nordgren [10] generalized Arveson's result to unilateral shifts with finite multiplicities. Cheng, Guo and Wang [7] proved the coordinate multiplication operators on functional Hilbert spaces with complete Nevanlinna-Pick kernels are transitive. The invariant subspace problem ask if a singly generated algebra acing on a Hilbert space $\mathcal{H}$ is transitive.

A weakly closed subalgebra $\mathcal{B}$ of $B(\mathcal{H})$ is called reductive if all of its invariant subspaces are reducing. An operator is called reductive if every operator algebra containing it is reductive. The reductive algebra problem raised firstly in [12] asks if every reductive algebra $A \subset B(\mathcal{H})$ is self adjoint. An affirmative answer to this problem would imply a positive answer to the transitive algebra problem [12]. Nordgren and Rosenthal [11] proved that a unilateral shift with finite multiplicity is reductive. Cheng, Guo and Wang [7] showed that the coordinate multiplication operators on functional Hilbert spaces with complete Nevanlinna-Pick kernels are reductive.

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Both the transitive algebra problem and the reductive algebra problem are still unsolved.

Let R be a finite-connected bounded domain in the complex plane $\mathbb{C}$ whose boundary ∂R consists of $n+1$ nonintersecting analytic Jordan curves. The Hardy space $H^2(R)$ over R is defined to be the space of all analytic functions f on R such that the subharmonic functions $|f|^2$ are majorized by harmonic functions u. For a fixed point $t \in R$, there is a norm $\|\cdot\|$ on $H^2(R)$ defined by

$$\|f\| = \inf_u \{u(t)^{\frac{1}{2}}\},$$

where u is a harmonic majorant of $|f|^2$. Let m be the harmonic measure for the point t and $L^2(\partial R)$ be the square integrable complex-valued measurable function on ∂R defined with respect to m. $H^2(\partial R)$ is defined to be the set of function $f \in L^2(\partial R)$ such that $\int_{\partial R} f(z)g(z)dz = 0$ for every g that is analytic in a neighborhood of the closure of R. $H^2(\partial R)$ is a reproducing kernel Hilbert space, let k_λ be the reproducing kernel at $\lambda \in R$. As the case $R = \mathbb{D}$, $H^2(R)$ can be identified with $H^2(\partial R)$ by non-tangential limits. We define an operator T_z on $H^2(R)$ by $T_z f = zf$ for every $f \in H^2(R)$, and an operator N on $L^2(\partial R)$ by the same formula $Nf(z) = zf(z)$. It is easy to see that T_z is a pure subnormal operator and N is the minimal normal extension of T_z.

Similarly, for a Hilbert space $\mathcal{H}$, we can define an $\mathcal{H}$-valued Hardy space, $H^2_{\mathcal{H}}(R)$, which is the space of all $\mathcal{H}$-valued analytic functions $f : R \to \mathcal{H}$ such that the subharmonic functions $\|f(z)\|^2_{\mathcal{H}}$ are majorized by harmonic functions u on R. We define two corresponding operators, $(T_{\mathcal{H}} f)(z) = zf(z)$ for $f \in H^2_{\mathcal{H}}(R)$ and $z \in R$, and $N_{\mathcal{H}}$ on $L_{\mathcal{H}}(\partial R)$, $(N_{\mathcal{H}} f)(z) = zf(z)$ for $f \in L^2_{\mathcal{H}}(\partial R)$ and $z \in \partial R$. Now $H^2_{\mathcal{H}}(R)$ is a reproducing kernel Hilbert space, and we use $k^{\mathcal{H}}_\lambda \in B(\mathcal{H})$ to represent the reproducing kernel at $\lambda \in R$; that is, $\langle f(\lambda), h\rangle_{\mathcal{H}} = \langle f, k^{\mathcal{H}}_\lambda h\rangle_{H^2_{\mathcal{H}}(R)}$ for $f \in H^2_{\mathcal{H}}(R)$ and $h \in \mathcal{H}$. For more information about function theory on finitely connected domains, one can see [1,14,15].

Let E be a Hermitian holomorphic vector bundle over R. A section of E is a holomorphic function f from R into E such that $p(f(z)) = z$ for all $z \in R$, where $p : E \to R$ is the projection map [9]. The set of all holomorphic sections of E is denoted by $\Gamma_a(E)$ where the subscript "a" represents "analytic". A unitary coordinate cover for E is a covering $\{U_s, \varphi_s\}$ with $\varphi_s : U_s \times \mathbb{C}^n \to E|_{U_s}$ such that for each s and $z \in U_s$, the fiber map $\varphi^z_s : \mathbb{C}^n \to E_z$, is unitary. The unitary coordinate cover $\{U_s, \varphi_s\}$ is said to be *flat* if the transition functions, $\varphi_{st} = \varphi_s^{-1}\varphi_t$ on $U_s \cap U_t$ for all s and t, are constant. A flat unitary vector bundle is a vector bundle with a flat unitary coordinate covering.

If E is a flat unitary vector bundle over the finitely-connected domain R with fiber $\mathcal{E}$ and coordinate covering $\{U_s, \varphi_s\}$ and f is a holomorphic section of E, then for $z \in U_s \cap U_t$, the operator $(\varphi^z_t)^{-1}\varphi^z_s$ is unitary so that $\|(\varphi^z_t)^{-1} f(z)\| = \|(\varphi^z_s)^{-1} f(z)\|$. This means that there is a function on R defined by $h^E_f(z) = \|(\varphi^z_s)^{-1} f(z)\|_E$, where $z \in U_s$. One defines $H^2_E(R)$ to be the space of holomorphic sections f of E such that $(h^E_f(z))^2$ is majorized by a harmonic function, then $H^2_E(R)$ is a Hilbert space. $H^2_E(R)$ is invariant under multiplication by any bounded analytic function on R. The operator T_E on $H^2_E(R)$, defined by $(T_Ef)(z) = zf(z)$ for $z \in R$, is called a *bundle shift* over R. These objects are studied by Abrahamse and Douglas [3]. In the paper, they proved a Beurling-type theorem for a bundle shift over a multiply-connected domain.

Lemma 1 ([3]). *Let T_E is a bundle shift on $H^2_E(R)$. A closed subspace $\mathcal{M}$ of $H_E(R)$ is invariant for $Rat(T_E)$ if and only if $\mathcal{M} = \Theta H^2_F(R)$, where F is a flat unitary bundle over R and Θ, is an inner bundle map from F to E. Moreover, two subspaces $\Theta_1 H^2_{F_1}(R)$ and $\Theta_2 H^2_{F_2}(R)$ are equal if and only if F_1 and F_2 are equivalent flat unitary bundles over R and there exists a bundle map Φ from F_1 onto F_2 that establishes the equivalence and satisfies $\Theta_1 = \Theta_2\Phi$.*

Let $Rat(T_E)$ denote the algebra of all $r(T_E)$, where r is a holomorphic rational function on R with poles outside of R. It is proved in [2,8] that $Rat(T_E)$ is reductive.

Finally, let $\mathcal{J}_E$ be the subalgebra of $B(H^2_E(R))$ of all operators T_Φ, where every Φ is a bundle map on E which extends to an open set containing the closure of R.

§2 Main results and proofs

Let $\mathcal{H}$ denote an infinite dimensional Hilbert space, and $B(\mathcal{H})$ be the algebra of bounded linear operators on $\mathcal{H}$. Let $\mathcal{A}$ be a subalgebra of $B(\mathcal{H})$, and let n be a positive integer.

Lemma 2 ([11]). *If $\mathcal{B}^{(n)}$ is a reductive algebra for every positive integer n, then $\mathcal{B}$ is self adjoint.*

Definition 1. A closed linear manifolds $\mathcal{M} \subset \mathcal{H}^{(n)}$ is called an invariant graph subspace for $\mathcal{B}^{(n)}$ if it is invariant for $\mathcal{B}^{(n)}$ and there exist (n-1) linear transformations $T_1, \cdots, T_{n-1}$ on a linear manifold $\mathcal{D}$ of $\mathcal{H}$ distinct from $\{0\}$, such that

$$\mathcal{M} = \{(x, T_1x, \cdots, T_{n-1}x) : x \in \mathcal{D}\},$$

where $\mathcal{H}^{(n)}$ denotes the direct sum of n copies of $\mathcal{H}$. A linear transformation T is called a graph transformation for $\mathcal{B}$ if for some n, T is one of the T_i's in an invariant graph subspace for $\mathcal{B}^{(n)}$.

The study of the reductivity of $\mathcal{B}^{(n)}$ reduces to the study invariant graph subspaces.

Definition 2. A linear transformation T is said to have a compression spectrum if there exists $\lambda \in \mathbb{C}$ such that the range of $T - \lambda$ is not dense in H.

Lemma 3. *Every densely defined invariant graph transformation T for $Rat(T_E)$ on $H^2_E(R)$ has a compression spectrum.*

Proof. Only one thing needs to be observed after noticing that the bundles $R \times \mathbb{C}^n$ and E extend to a trivial and a flat unitary bundle over the closure of R and these extensions are similar [3]. If Φ is a bundle map from $clos(R) \times \mathbb{C}^n$ to the extension of E, establishing the similarity of $T_\mathcal{H}$ and T_E, then Φ induces a module isomorphism denoted by $\widetilde{\Phi}$ from $H^2_{\mathbb{C}^n}(R)$ to $H^2_E(R)$ conjugating $\mathcal{J}_E(R)$ and $\mathcal{J}_{\mathcal{R}\times\mathbb{C}^n}(R)$. Thus the similarity not only takes $T_{\mathbb{C}^n}$ to T_E, but also a linear transformation $\tilde{T}$ on $T_{\mathbb{C}^n}$ to a linear transformation T on T_E, and $\mathcal{J}_E(R) \otimes M_n(\mathbb{C})$

to $\mathcal{J}_E(R)$. If $\mathcal{M}$ is invariant for T_E, then $\widetilde{\Phi}^{-1}\mathcal{M}$ denoted by $\widetilde{\mathcal{M}}$ is invariant for $T_{\mathbb{C}^n}$. T has a compression spectrum if and only if $\widetilde{T}$ has a compression spectrum since $\widetilde{\Phi}$ maps a densely set to a densely set. So we need only to prove that every densely defined graph transformation $\widetilde{T}$ on $H^2_{\mathbb{C}^n}(R)$ has a compression spectrum, this is a result in the proof of the second Lemma in [10].

Remark 1. The above lemma shows that operator algebras containing bundle shifts are strongly dense in $B(H^2_E(R))$.

Lemma 4 ([11]). *If $\mathcal{B}$ is a reductive algebra and T is a closed linear transformation commuting with $\mathcal{B}$ such that the range of T is contained in the direct sum of the kernel of T and the orthogonal complement of its domain, then T commutes with $\mathcal{B}^*$.*

Lemma 5. *T is a closed linear transformation with a dense domain $\mathcal{D}$ in $H^2_E(R)$. If T commutes with T_E, then there are invertible bundle maps Θ and Γ such that $\Theta H^2_E(R) \subset \mathcal{D}$ and $Tf = \Theta^{-1}\Gamma f$ for $f \in \mathcal{D}$. On the other side, the operator defined by $Tf = \Theta^{-1}\Gamma f$ commutes with T_E and is closable, furthermore, its closure commutes with $Rat(T_E)$ also.*

Proof. $\mathcal{M} = \{(f, Tf) | f \in \mathcal{D}\}$ is invariant under the action of (T_E, T_E) defined by $(S, S)(f, Tf) = (Sf, STf)$ since T commutes with T_E. So there exist a flat unitary bundle F over R and an inner bundle map from F to E such that $\mathcal{M} = \Theta H^2_F(R)$. It follows that there exist bundle maps Θ_1, Θ_2 such that for every $f \in \mathcal{D}$, there is a unique $f_1 \in H^2_F(R)$ satisfying

$$f \oplus Tf = \Theta f_1 = \Theta_1 f_1 \oplus \Theta_2 f_1.$$

The density of $\mathcal{D}$ implies that F and E are equivalent flat unitary vector bundles over R, and so we can take $F = E$.

The density of the range of Θ_1 shows that Θ_1 is a surjective bundle map and the fibre of the bundle is finitely dimensional, so it is invertible, i.e., there exists a bundle map Θ_1^{-1} from E to F such that $\Theta_1\Theta_1^{-1} = I_E$ and $\Theta_1^{-1}\Theta_1 = I_F$, hence $\mathcal{D} \supset \Theta_1^{-1} H^2_F(R)$. Then $Tf = \Theta^{-1}\Gamma f$. It is clear that whenever a closable linear transformation commutes with a bounded operator A, then its closure also commutes with A.

It is obvious that the operator defined by $Tf = \Theta^{-1}\Gamma f$ for $f \in \mathcal{D}$ commutes with $Rat(T_E)$. The left is to prove T is closable. Let $\{f_n\}$ be a sequence in $\mathcal{D}$ that converges to 0 such that $\{Tf_n\}$ converges. We must show that $\lim_{n\to\infty} Tf_n = 0$. By choosing an appropriate subsequence, so we can assume that $\{f_n\}$ and $\{Tf_n\}$ both converge pointwise a.e. on the boundary of R. Now $\Theta^{-1}\Gamma$ has a radial limit at almost every point of R. For almost every $z \in R$,

$$\lim_{n\to\infty} Tf_n(z) = \lim_{n\to\infty} \Theta^{-1}(z)\Gamma(z)f(z) = \Theta^{-1}(z)\Gamma(z) \lim_{n\to\infty} f_n(z) = 0.$$

Hence $\lim_{n\to\infty} Tf_n(z) = 0$, and T is closable. The graph of T is a subspace invariant under $T_E \oplus T_E$, it implies the closure of T commutes with $Rat(T_E)$.

The quotient representation and the closability property are very important in the transitive algebra problem and the reductive algebra problem. The quotient representation of a function $f \in H^2(\mathbb{D})$ by two bounded analytic functions [4] is the key to prove the reductive algebra problem for the shift in [5]. The closability property was studied in [6].

Lemma 6. *If $\mathcal{U}$ is a reductive algebra on $H^2_E(R)$ containing the bundle shift T_E, $\mathcal{M} = \{(x, Tx) : x \in \mathcal{D}\}$ is a nonzero invariant graph subspace for $\mathcal{U}^{(2)}$. Then $\mathcal{M}$ contains a nonzero reducing subspace of $\mathcal{U}^{(2)}$.*

Proof. It is clear that T commutes with $\mathcal{U}$. There exists $\lambda \in \mathbb{C}$ such that the range $Ran(T-\lambda)$ of $T-\lambda$ is not dense in $H^2_E(R)$ by Lemma 3. $Ran(T-\lambda)$ is invariant under $\mathcal{U}$, so is its closure $\overline{Ran(T-\lambda)}$. The orthogonal completement $\overline{Ran(T-\lambda)}^{\perp}$ of $\overline{Ran(T-\lambda)}$ in $H^2_E(R)$ is also invariant under $\mathcal{U}$ and so T_E since $\mathcal{U}$ is reductive. Then there is a flat unitary bundle F over R and an inner bundle map from F to E such that $\overline{Ran(T-\lambda)}^{\perp} = \Theta H^2_F(R)$. $\mathcal{D}_0 = \mathcal{D} \cap \overline{Ran(T-\lambda)}^{\perp}$ is dense in $\overline{Ran(T-\lambda)}^{\perp}$ by Lemma 1. So $\mathcal{D}_0$ is nonzero.

Now define $\mathcal{M}_0 = \{(x, Tx) : x \in \mathcal{D}_0\}$. It is clear that $\mathcal{M}_0$ is a closed subspace of $\mathcal{M}$ and invariant under $\mathcal{U}^{(2)}$. Furthermore, $(T-\lambda)|_{\mathcal{D}_0}$ is closed linear transformation commuting with $\mathcal{U}$ whose range is orthogonal to $\mathcal{D}_0$. So $(T-\lambda)|_{\mathcal{D}_0}$ commutes with $\mathcal{U}^*$. $\mathcal{M}_0$ is invariant under $(\mathcal{U}^*)^{(2)}$ and so reduces $\mathcal{U}^{(2)}$.

Theorem 1. *If $\mathcal{B}$ is a reductive algebra on $H^2_E(R)$ containing $Rat_E(R)$, then $\mathcal{B}^{(2)}$ is a reductive algebra on $(H^2_E(R))^{(2)}$.*

Proof. Let $\mathcal{M}$ be an invariant subspace of $\mathcal{B}^{(2)}$, and $\mathcal{N}$ be the closed linear span of all reducing subspace of $\mathcal{B}^{(2)}$ contained in $\mathcal{M}$. Then it is clear that $\mathcal{N} \subset \mathcal{M}$ is a reducing subspace of $\mathcal{B}^{(2)}$ also. The left is to show $\mathcal{M} = \mathcal{N}$. If it is not true, let $\mathcal{M}_0$ be the orthogonal complement of $\mathcal{N}$ in $\mathcal{M}$, which is nontrivial. $\mathcal{M}_0$ is an invariant subspace of $\mathcal{B}^{(2)}$ also. Moreover, if $(0, f) \in \mathcal{M}_0$, then $f = 0$. So $\mathcal{M}_0$ is the graph of a linear transformation T on $H^2_E(R)$, i.e.,

$$\mathcal{M}_0 = \{(f, Tf) : f \in \mathcal{D}_0\}.$$

It follows that $\mathcal{M}_0$ contains a nontrivial reducing subspace of $\mathcal{B}^{(2)}$ by Lemma 6, which contradicts that $\mathcal{M}_0$ contains no reducing subspace of $\mathcal{B}^{(2)}$.

Theorem 2. *If $\mathcal{B}$ is a reductive algebra on $H^2_E(R)$ containing $Rat_E(R)$, then $\mathcal{B}$ is self adjoint.*

Proof. $\mathcal{B}^{(2)}$ is reductive by Theorem 1, and so $\mathcal{B}^{(4)}$ is reductive also. It follows that $\mathcal{B}^{(2^n)}$ is reductive by Theorem 1, which shows $\mathcal{B}^{(n)}$ is reductive for any positive integer n, and so $\mathcal{B}$ is self adjoint by Lemma 2.

References

1 M B Abrahamse. *Toeplitz Operators in Multiply Connected Regions*, Amer J Math, Vol 96, No 2 (1972), pp: 261-297.

2 M B Abrahamse, J J Bastian. *Bundle Shifts and Ahlfors Functions*, Proc Amer Math Soci, Vol 72, No 1(1978), pp: 106-148.

3 M B Abrahamse, R G Douglas. *A Class of Subnormal Operators Related to Multiply-Connected Domains*, Adv Math, Vol 19(1976), pp: 106-148.

4 L V Ahlfors. *Bounded Analytic Functions*, Duke Math J, Vol 14, No 1(1947), pp: 1-11.

5 W B Arveson. *A density Theorem for Operator Algebras*, Duke Math J, Vol 34, No 4(1967), pp: 635-647.

6 H Bercovici, R G Douglas, C Foias, C Pearcy. *Confluent operator algebras and the closability property*, J Funct Anal, Vol 258 (2010), pp: 4122-4153.

7 G Z Cheng, K Y Guo, K Wang. *Tansitive Algebras and Reductive Algebras on Reproducing Analytic Hilbert Spaces*, J Func Anal, Vol 258 (2010), pp: 4229-4250.

8 R G Douglas, A J Xu. *Tansitivity and Bundle Shifts*, Invariant Subspaces of the Shift Operator, Contemp Math 638, pp: 287-297.

9 R G Douglas, K K Dineshi, A J Xu. *Generalized Bundle Shifts with Application to Toeplitz Operator on Bergman Spaces*, J Operator Theory, Vol 75, No 1(2016), pp: 3-19.

10 E A Nordgren. *Tansitive Operator Algebras*, J Math Anal App, Vol 32 (1967), pp: 639-643.

11 E A Nordgren, P Rosenthal. *Algebras Containing Unilateral Shifts or Finite-rank Operators*, Duke Math J, Vol 40 (1973), pp: 419-424.

12 H Radjavi, P Rosenthal. *A sufficient condition that an operator algebra be self adjoint*, Canad J Math, Vol 23 (1971), pp: 588-597.

13 S Richter. *Invariant Subspaces of the Dirichlet Shift*, J reine angew Math, Vol 386 (1988), pp: 205-220.

14 M Voichick. *Ideals and Invariant Subspaces of Analytic Functions*, Trans Amer Math, Vol 111, No 3(1964), pp: 493-512.

15 M Voichick. *Inner and Outer Functions on Riemann Surfaces*, Proc Amer Math Soc, Vol 16, No 6(1965), pp: 1200-1204.

† College of Science, Chongqing University of Technology, Chongqing 400065, China.