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Sharp upper bounds for the adjacency and the signless Laplacian spectral radius of graphs

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Abstract. Let G be a simple graph with n vertices and m edges. In this paper, we present some new upper bounds for the adjacency and the signless Laplacian spectral radius of graphs in which every pair of adjacent vertices has at least one common adjacent vertex. Our results improve some known upper bounds. The main tool we use here is the Lagrange identity.

§1 Introduction

Let $G = (V, E)$ be a connected graph with vertex set $V = \{v_1, v_2, \cdots, v_n\}$ and edge set $E = \{e_1, e_2, \cdots, e_m\}$. If v_i is adjacent to v_j, then we denote it by $v_i \sim v_j$. Let $d_i = d(v_i)$ be the degree of a vertex v_i in G, for $i = 1, 2, \ldots, n$. Let $\Delta(G) = \Delta, \delta(G) = \delta$ be the maximum degree and minimum degree of the vertices of G, respectively. If necessary, we assume that $\Delta = d_1 \geq d_2 \geq \cdots \geq d_n = \delta$. Let $D(G) = diag(d_1, d_2, \cdots, d_n)$ be the diagonal matrix of vertex degrees of G. The adjacency matrix of a graph G is $A(G) = (a_{ij})_{n\times n}$, where elements $a_{ij} = 1$ if $v_i \sim v_j$, and $a_{ij} = 0$ otherwise. The signless Laplacian matrix of G is defined to be $Q(G) = D(G) + A(G)$. Since both $Q(G)$ and $A(G)$ are real symmetric matrices, their eigenvalues are all real numbers. The largest eigenvalues of $A(G)$ and $Q(G)$, denoted by $\lambda(G)$ and $\mu(G)$ (abbreviated as λ and μ), are called the adjacency spectral radius and the signless Laplacian spectral radius of G, respectively. The average 2-degree of vertex v_i is defined as $m_i = m(v_i) = \frac{\sum\limits_{v_j \sim v_i} d(v_j)}{d(v_i)}$, abbreviated as $d_i m_i = \sum\limits_{j\sim i} d_j$.

A graph G is called a triangulation [1], if every pair of adjacent vertices of G has at least one common adjacent vertex. Undefined terminology and notation may refer to [2].

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Spectral graph theory is a fast growing branch of algebraic graph theory and it concerns an interwind tale of properties of graphs and spectrum of related matrices. The most studied problems within spectral graph theory are those lower and upper bounds for the spectral radius, Laplacian spectral radius and signless Laplacian spectral radius, as well as the characterization of extremal graphs achieving the bounds. There exist a lot of papers on this topic. Some recent results related to the theory of graph spectra may be found in [1, 3–23]. In this paper, we firstly review some classical upper bounds for the adjacency and the signless Laplacian spectral radius of graphs. Then we present some new sharp upper bounds for the spectral radius and the signless Laplacian spectral radius of graphs, which improve some known results listed in section 2.

§2 Some known sharp upper bounds for the adjacency and the signless Laplacian spectral radius of graphs

Some known sharp upper bounds for the adjacency and the signless Laplacian spectral radius of graphs are summarized in this section.

We start with a few sharp upper bounds for the adjacency spectral radius of graphs.

Hong [3] gave the following bound

$$\lambda \leq \sqrt{2m-n+1}. \tag{2-1}$$

Das and Kumar [4] showed that

$$\lambda \leq \sqrt{2m-(n-1)\delta+(\delta-1)\Delta}. \tag{2-2}$$

Stanley [5] obtained the following bound

$$\lambda \leq \frac{-1+\sqrt{8m+1}}{2}. \tag{2-3}$$

Hu [6] presented the bound as follows:

$$\lambda \leq \sqrt{2m-n-\delta+2}. \tag{2-4}$$

Shu and Wu [7] showed that

$$\lambda \leq \frac{d_i-1+\sqrt{(d_i+1)^2+4(i-1)(d_1-d_i)}}{2}. \tag{2-5}$$

Liu and Weng [8] showed the bound

$$\lambda \leq \frac{d_i-1+\sqrt{(d_i+1)^2+4\sum\limits_{j=1}^{i-1}(d_j-d_i)}}{2}. \tag{2-6}$$

Oliveira et al. [9] showed that

$$\mu \leq \max_{1\leq i\leq n}\left\{\frac{d_i+\sqrt{d_i^2+8d_im_i}}{2}\right\}, \tag{2-7}$$

and

$$\mu \leq \max_{1\leq i\leq n}\left\{d_i+\sqrt{d_im_i}\right\}. \tag{2-8}$$

Chen and Wang [10] obtained the following bound

$$\mu \leq \frac{(\delta-1)^2+8[2m+\Delta^2-(n-1)\delta]}{2}. \qquad (2-9)$$

Maden et al. [11] showed that

$$\mu \leq \max_{v_i \sim v_j}\left\{\frac{d_i+2d_j-1+\sqrt{(d_i+2d_j-1)^2+4d_i}}{2}\right\}. \qquad (2-10)$$

Li and Pan [12] gave the following bound

$$\mu \leq \frac{\Delta+\delta-1+\sqrt{(\Delta+\delta-1)^2+8[2m-(n-1)\delta]}}{2}. \qquad (2-11)$$

Li and Pan [13] presented the following bound

$$\mu \leq \max_{1\leq i\leq n}\left\{\sqrt{2d_i(d_i+m_i)}\right\}. \qquad (2-12)$$

Fan and Weng [14] gave the following bound

$$\mu \leq \Delta-\frac{s}{4}+\sqrt{(\Delta-\frac{s}{4})^2+(1+t)\Delta+s(n-1)-\Delta^2}, \qquad (2-13)$$

where t (resp. s) denotes the maximum number of common neighbors of a pair of adjacent vertices (resp. nonadjacent distinct vertices) of G.

Su et al. [15] gave the bound that

$$\mu \leq \min_{1\leq i\leq n}\left\{\frac{d_1+2d_i-1+\sqrt{(2d_i-d_1+1)^2+8(i-1)(d_1-d_i)}}{2}\right\}. \qquad (2-14)$$

Cui et al. [16] showed that

$$\mu \leq \min_{1\leq i\leq n}\left\{\frac{d_1+2d_i-1+\sqrt{(2d_i-d_1+1)^2+8\sum_{k=1}^{i-1}(d_k-d_i)}}{2}\right\}. \qquad (2-15)$$

§3 New sharp upper bounds for the adjacency and the signless Laplacian spectral radius of graphs

Throughout this section, let G be a simple graph with n vertices and m edges and K_n be a complete graph with n vertices. Recall that the line graph G^L of a graph G is the graph whose vertices are the edges of G, with two vertices in G^L adjacent whenever the corresponding edges in G have exactly one vertex in common. Observe that every edge $e_{ij} \in E(G)$ corresponding to a vertex $v_{ij} \in V(G^L)$. Let $d_{ij} = d_{ij}(G^L)$, $m_{ij} = m_{ij}(G^L)$ be the degree and average 2-degree of vertex v_{ij} in G^L.

Now, we will establish some new sharp upper bounds for the adjacency and the signless Laplacian spectral radius of graphs in terms of d_i, m_i, Δ, δ.

Lemma 3.1. ([1], Lemma 2.1) *Let G be a simple graph with n vertices and m edges, and Δ, δ be the maximum and minimum degree of G, respectively. Then*

$$d_i m_i \leq 2m-(n-1)\delta+(\delta-1)\Delta.$$

Lemma 3.2. ([17], Lemma 3.1) *Let G be a simple graph and G^L be its line graph, then*

$$\mu(G) = 2 + \lambda(G^L),$$

where $\mu(G)$ and $\lambda(G^L)$ are spectral radius of $Q(G)$ and $A(G^L)$, respectively.

Let us recall the famous Lagrange's identity.

Lemma 3.3. *Let n be positive integer, $a_1, a_2, \cdots, a_n$, $b_1, b_2, \cdots, b_n$ be real numbers. Then*

$$(\sum_{j=1}^{n} a_j^2)(\sum_{j=1}^{n} b_j^2) = (\sum_{j=1}^{n} a_j b_j)^2 + \sum_{1\le j<k\le n} (a_j b_k - a_k b_j)^2.$$

Lemma 3.4. ([17] Theorem 3.2) *Let G be a simple graph, d_i, m_i be the degree and average 2-degree of vertex v_i in G. Then $d_{ij}(G^L) = d_i + d_j - 2$ and*

$$m_{ij}(G^L) = \frac{d_i(d_i + m_i - 4) + d_j(d_j + m_j - 4) + 4}{d_i + d_j - 2}.$$

Theorem 3.1. *Let G be a triangulation with n vertices and m edges, $A = (a_{ij})_{n\times n}$ be its adjacency matrix, and λ be the largest eigenvalue. Then*

$$\lambda(G) \le \max_{1\le i\le n} \Big\{\frac{1+\sqrt{4d_i m_i - 4d_i + 1}}{2}\Big\}.$$

Moreover, if any two adjacent vertices of G have exactly one common adjacent vertex, and $x_k = x_j$ for any $i, j, k \in \{1, 2, \ldots, n\}$ satisfying $j < k$, $a_{ij} = a_{ik} = 1$, $a_{jk} = 0$, then equality holds if and only if for each i, $d_i m_i - d_i = k$, where k is a positive integer. Especially if $G \cong K_3$, then equality holds.

Proof. Since A is an irreducible nonnegative matrix, by the famous Perron-Frobenius Theorem, the largest eigenvalue λ is simple and there exists a unique positive unit eigenvector $X = (x_1, x_2, \cdots, x_n)^T$ corresponding to λ.
Clearly, $X^T X = x_1^2 + x_2^2 + \cdots + x_n^2 = 1$. Then, from $\lambda X = AX$ it follows that

$$\lambda = \lambda X^T X = X^T AX = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} x_i x_j.$$

Since A is symmetric matrix, we get

$$\sum_{1\le i<j\le n} a_{ij}(x_i - x_j)^2 = \sum_{i=1}^{n} d_i x_i^2 - 2\sum_{1\le i<j\le n} a_{ij} x_i x_j = \sum_{i=1}^{n} d_i x_i^2 - \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} x_i x_j.$$

That is,

$$\sum_{1\le i<j\le n} a_{ij}(x_i - x_j)^2 = \sum_{i=1}^{n} d_i x_i^2 - \lambda. \quad (1)$$

For each i, we have $Ax_i = \sum\limits_{j=1}^{n} a_{ij}x_j = \lambda x_i$. Thus we have

$$\lambda^2 x_i^2 = (\sum_{j=1}^{n} a_{ij}x_j)^2.$$

Let $a_j = \sqrt{a_{ij}}, b_j = \sqrt{a_{ij}}x_j$. By Lemma 3.3, we have

$$\begin{aligned}
\lambda^2 x_i^2 = (\sum_{j=1}^{n} a_{ij}x_j)^2 &= (\sum_{j=1}^{n} a_{ij})(\sum_{j=1}^{n} a_{ij}x_j^2) - \sum_{1\le j<k\le n} (\sqrt{a_{ij}}\sqrt{a_{ik}}x_k - \sqrt{a_{ik}}\sqrt{a_{ij}}x_j)^2 \\
&= d_i \sum_{j=1}^{n} a_{ij}x_j^2 - \sum_{1\le j<k\le n} (\sqrt{a_{ij}}\sqrt{a_{ik}}x_k - \sqrt{a_{ik}}\sqrt{a_{ij}}x_j)^2 \\
&= d_i \sum_{j=1}^{n} a_{ij}x_j^2 - \sum_{1\le j<k\le n} [(\sqrt{a_{ij}}\sqrt{a_{ik}}(x_k - x_j)]^2 \\
&= d_i \sum_{j=1}^{n} a_{ij}x_j^2 - \sum_{\substack{1\le j<k\le n \\ a_{ij}=a_{ik}=1}} (x_k - x_j)^2.
\end{aligned}$$

Summing over all i, we obtain

$$\begin{aligned}
\sum_{i=1}^{n} \lambda^2 x_i^2 &= \sum_{i=1}^{n} d_i \sum_{j=1}^{n} a_{ij}x_j^2 - \sum_{i=1}^{n} \sum_{\substack{1\le j<k\le n \\ a_{ij}=a_{ik}=1}} (x_k - x_j)^2 \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}d_i x_j^2 - \sum_{i=1}^{n} \sum_{\substack{1\le j<k\le n \\ a_{ij}=a_{ik}=1}} (x_k - x_j)^2 \\
&= \sum_{j=1}^{n}\sum_{j\sim i} d_i x_j^2 - \sum_{i=1}^{n} \sum_{\substack{1\le j<k\le n \\ a_{ij}=a_{ik}=1}} (x_k - x_j)^2 \\
&= \sum_{j=1}^{n} d_j m_j x_j^2 - \sum_{i=1}^{n} \sum_{\substack{1\le j<k\le n \\ a_{ij}=a_{ik}=1}} (x_k - x_j)^2 \\
&= \sum_{i=1}^{n} d_i m_i x_i^2 - \sum_{i=1}^{n} \sum_{\substack{1\le j<k\le n \\ a_{ij}=a_{ik}=1}} (x_k - x_j)^2.
\end{aligned}$$

Noting that G is a triangulation, we have

$$\begin{aligned}
\sum_{i=1}^{n} \sum_{\substack{1\le j<k\le n \\ a_{ij}=a_{ik}=1}} (x_k - x_j)^2 &\ge \sum_{i=1}^{n} \sum_{\substack{1\le j<k\le n \\ a_{ij}=a_{ik}=1, a_{jk}=1}} (x_k - x_j)^2 \\
&\ge \sum_{\substack{1\le j<k\le n \\ a_{jk}=1}} (x_k - x_j)^2 = \sum_{1\le j<k\le n} a_{jk}(x_k - x_j)^2. \qquad (2)
\end{aligned}$$

The first equality in (2) holds if and only if $x_k = x_j$, for any $i, j, k \in \{1, 2, \ldots, n\}$ satisfying $j < k$, $a_{ij} = a_{ik} = 1$, and $a_{jk} = 0$. The second equality in (2) holds if and only if any two adjacent vertices of G have exactly one common adjacent vertex, or $x_k = x_j$ for any adjacent vertices (v_j, v_k) with more than one common adjacent vertex.

From (1) and (2), we have

$$\sum_{i=1}^{n} \sum_{\substack{1\le j<k\le n \\ a_{ij}=a_{ik}=1}} (x_k - x_j)^2 \ge \sum_{j=1}^{n} d_j x_j^2 - \lambda.$$

Thus, we have

$$\begin{aligned}\sum_{i=1}^{n} \lambda^2 x_i^2 &= \sum_{i=1}^{n} d_i m_i x_i^2 - \sum_{i=1}^{n} \sum_{\substack{1\le j<k\le n \\ a_{ij}=a_{ik}=1}} (x_k - x_j)^2 \\ &\le \sum_{i=1}^{n} d_i m_i x_i^2 - \sum_{j=1}^{n} d_j x_j^2 + \lambda.\end{aligned}$$

This yields the following relations

$$\sum_{i=1}^{n} (\lambda^2 - \lambda + d_i - d_i m_i) x_i^2 \le 0.$$

Then there must exist $i \in \{1, 2, \ldots, n\}$ such that $\lambda^2 - \lambda + d_i - d_i m_i \le 0$. Equality holds if and only if $d_i m_i - d_i = \lambda^2 - \lambda$ be a positive integer.

This implies that

$$\lambda \le \frac{1 + \sqrt{4d_i m_i - 4d_i + 1}}{2}.$$

Thus, we have

$$\lambda \le \max_{1\le i\le n} \left\{ \frac{1 + \sqrt{4d_i m_i - 4d_i + 1}}{2} \right\}.$$

It follows from the above proof that if any two adjacent vertices of G have exactly one common adjacent vertex, and $x_k = x_j$ for any $i, j, k \in \{1, 2, \ldots, n\}$ satisfying $j < k$, $a_{ij} = a_{ik} = 1$, $a_{jk} = 0$, then equality holds if and only if for each i, $d_i m_i - d_i = k$, where k is a positive integer. Especially if $G \cong K_3$, then equality holds in Theorem 3.1. □

Remark 3.1. If G is regular of degree r, then $d_i m_i - d_i = r^2 - r$ for all $i \in \{1, 2, \ldots, n\}$. We assert that there exist some non-regular graphs that satisfy $d_i m_i - d_i = k$ for all i (see Fig.1).

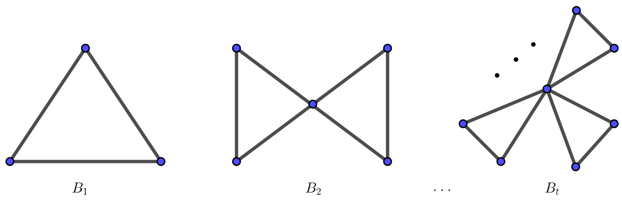


Fig.1 The graphs B_1, $B_2, \ldots, B_t$

By direct calculations, $d_i m_i - d_i = 2t$ for the graph B_t with $n = 2t+1$, $(t = 2, 3, \cdots)$ for all $i \in \{1, 2, \ldots, n\}$.

Remark 3.2. Two graphs G_1 and G_2 (see Fig.2) are presented to illustrate that the upper bound in Theorem 3.1 is better than the bounds in [3–7] in some cases. Furthermore, it can be observed from Table 1 that our bound in Theorem 3.1 is very close to λ.

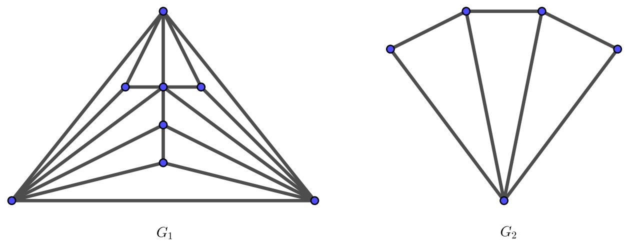


Fig.2 Graphs G_1 and G_2

Table 1: The value of some classic upper bounds for the above two graphs

	λ	Theorem 3.1	(2-1)	(2-2)	(2-3)	(2-4)	(2-5)	(2-6)
G_1	4.7990	5.1098	5.3852	5.1962	5.5208	5.1962	5.4641	5.0000
G_2	2.9354	3.0000	3.1623	3.1623	3.2749	3.0000	3.2361	3.0000

From Table 1, we see the bound in Theorem 3.1 is the best in the above mentioned upper bounds (2-1)–(2-5) for G_1 and G_2. But the upper bound 5.0000 in (2-6) which is better than the upper bound 5.1098 obtained from Theorem 3.1 for G_1, and the upper bound 3.0000 in (2-6) which is the same as the one obtained from Theorem 3.1 for G_2.

Corollary 3.1. *Let G be a triangulation with n vertices and m edges, then*

$$\lambda \leq \frac{1 + \sqrt{4[2m - (n-1)\delta + (\delta - 1)\Delta] - 4\delta + 1}}{2},$$

where λ is the spectral radius of G, with equality holds if $G \cong K_3$.

Proof. By Lemma 3.1 and Theorem 3.1, we have

$$\lambda \le \max_{1\le i\le n}\left\{\frac{1+\sqrt{4[2m-(n-1)\delta+(\delta-1)\Delta]-4d_i+1}}{2}\right\}.$$

Thus, we obtain

$$\lambda \le \frac{1+\sqrt{4[2m-(n-1)\delta+(\delta-1)\Delta]-4\delta+1}}{2}.$$

By direct calculations, equality holds if $G \cong K_3$. □

Lemma 3.5. *If G is a triangulation, then its line graph G^L is also a triangulation.*

Proof. Suppose that G^L is not a triangulation, then there exist two adjacent vertices $v_{xy}, v_{xz} \in V(G^L)$, which have no common adjacent vertex, i.e., no edges are adjacent to both two edges xy and xz in G. Hence, G has no subgraphs H_1, H_2 as follows (see Fig.3). Then, the vertice x and y in G have no common adjacent vertex, a contradiction. □

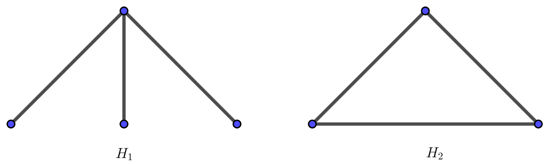


Fig.3 Graphs H_1 and H_2

Corollary 3.2. *Let G be a triangulation, μ be its signless Laplacian spectral radius. Then*

$$\mu \le 2+\max_{v_i\sim v_j}\left\{\frac{1+\sqrt{4d_i^2+4d_im_i+4d_j^2+4d_jm_j-20d_i-20d_j+25}}{2}\right\},$$

with equality holds if $G \cong K_3$.

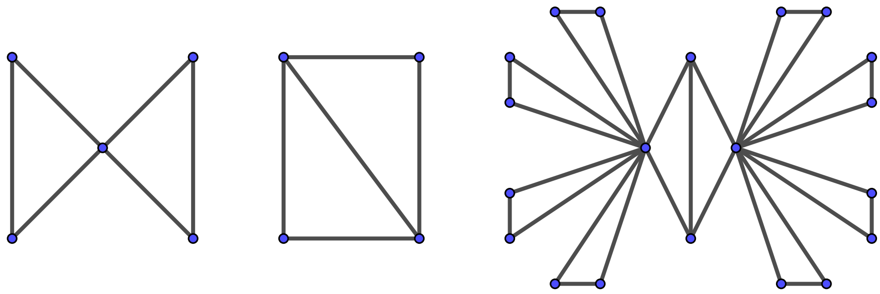
Proof. Let G^L be the line graph of G. By Lemma 3.4, for any $v_{ij} \in V(G^L)$ we have $d_{ij}(G^L) = d_i + d_j - 2$ and $m_{ij}(G^L) = \frac{d_i(d_i+m_i-4)+d_j(d_j+m_j-4)+4}{d_i+d_j-2}$.

By Lemma 3.5, the line graph of triangulation is also a triangulation. Using Lemma 3.2 and Theorem 3.1, we have

$$\mu \le 2+\max_{v_i\sim v_j}\left\{\frac{1+\sqrt{4d_i^2+4d_im_i+4d_j^2+4d_jm_j-20d_i-20d_j+25}}{2}\right\}.$$

If $G \cong K_3$, it is easy to check that equality holds. □

Remark 3.3. The upper bound in Corollary 3.2 is better than the upper bounds for the signless Laplacian spectral radius in [9–14] in some cases. Three graphs G_3, G_4 and G_5 are given as follows.



G_3 G_4 G_5

Fig.4 Graphs G_3, G_4 and G_5

Table 2: The value of some classic upper bounds for the above three examples

	μ	Corollary 3.2	(2-7)	(2-8)	(2-9)	(2-10)	(2-11)	(2-12)	(2-13)
G_3	5.5616	5.7016	6.4721	6.8284	6.8443	7.3723	6.2749	6.9282	6.9211
G_4	5.2361	5.3723	5.5311	5.6458	5.6235	6.0000	5.4641	5.6569	6.0000
G_5	11.4784	12.2596	13.3066	14.6904	16.0000	19.1853	13.8815	25.2982	17.1322

From Table 2, we see that the bound in Corollary 3.2 is the best in the above mentioned upper bounds (2-7)–(2-13) for G_3, G_4 and G_5. But both bounds in (2-14) and (2-15) are sharp for G_3, G_4, and 13.0000, 13.0000 for G_5. Moreover, by applying the upper bounds in (2-6) plus two with $i = 5, 2, 5$ to the line graphs of G_3, G_4, G_5 (degree sequence $(4,4,4,4,2,2), (4,3,3,3,3), (11^4, 10^{16}, 4^1, 2^8)$), respectively, we can find the upper bound of μ are 5.7016, 5.2361, 12.3523, that the second one is better than the upper bounds 5.3723, the first one is the same as 5.7016, the last one is larger than the upper bound 12.2596 obtained from Corollary 3.2, respectively.

Corollary 3.3. *Let G be a triangulation with n vertices and m edges, and G^L be its line graph with n^L vertices and m^L edges. Then*

$$\mu(G) \leq 2 + \frac{1 + \sqrt{4[2m^L - (m-1)\delta^L + (\delta^L - 1)\Delta^L] - 4\delta^L + 1}}{2},$$

where $\Delta^L = \max\limits_{v_i \sim v_j} \Big\{ d_i + d_j - 2 \Big\}$, $\delta^L = \min\limits_{v_i \sim v_j} \Big\{ d_i + d_j - 2 \Big\}$ *are the maximum and minimum degrees of the vertices of G^L, respectively, with equality holds if $G \cong K_3$.*

Proof. Let G^L be the line graph of G. We note from the definition of line graph, $n^L = m$ and $m^L = \frac{\sum_{v_i \sim v_j} (d_i+d_j-2)}{2}$ be the number of vertices and edges of G^L, respectively. By Corollary 3.1, we have

$$\lambda(G^L) \le \frac{1 + \sqrt{4[2m^L - (m-1)\delta^L + (\delta^L - 1)\Delta^L] - 4\delta^L + 1}}{2}.$$

By Lemma 3.2, we obtain

$$\mu \le 2 + \frac{1 + \sqrt{4[2m^L - (m-1)\delta^L + (\delta^L - 1)\Delta^L] - 4\delta^L + 1}}{2}.$$

It is straightforward to verify that if equality holds, the line graph of G must be K_3. This implies that G is K_3. □

Theorem 3.2. *Let G be a triangulation with n vertices and m edges. If each edge of G at least belongs to t ($t \ge 1$) triangles, then*

$$\lambda \le \max_{1\le i\le n} \left\{ \frac{t + \sqrt{4d_i m_i - 4td_i + t^2}}{2} \right\}.$$

Moreover, if any two adjacent vertices of G have exactly t common adjacent vertices, and $x_k = x_j$ for any $i, j, k \in \{1, 2, \ldots, n\}$ satisfying $j < k$, $a_{ij} = a_{ik} = 1$, $a_{jk} = 0$, then equality holds if and only if for each i, $d_i m_i - d_i = k$, where k is a positive integer. Especially if $G \cong K_{t+2}$, then equality holds.

Proof. By Theorem 3.1, if each edge of G at least belongs to t ($t \ge 1$) triangles, we have

$$\begin{aligned}\sum_{i=1}^{n} \sum_{\substack{1\le j<k\le n\\ a_{ij}=a_{ik}=1}} (x_k - x_j)^2 &\ge \sum_{i=1}^{n} \sum_{\substack{1\le j<k\le n\\ a_{ij}=a_{ik}=1, a_{jk}=1}} (x_k - x_j)^2 \\ &\ge t \sum_{\substack{1\le j<k\le n\\ a_{jk}=1}} (x_k - x_j)^2 = t \sum_{1\le j<k\le n} a_{jk}(x_k - x_j)^2 \qquad (3)\end{aligned}$$

and

$$\begin{aligned}\sum_{i=1}^{n} \lambda^2 x_i^2 &= \sum_{i=1}^{n} d_i m_i x_i^2 - \sum_{i=1}^{n} \sum_{\substack{1\le j<k\le n\\ a_{ij}=a_{ik}=1}} (x_k - x_j)^2 \\ &\le \sum_{i=1}^{n} d_i m_i x_i^2 - t\sum_{j=1}^{n} d_j x_j^2 + t\lambda.\end{aligned}$$

The first equality in (3) holds if and only if $x_k = x_j$, for any $i, j, k \in \{1, 2, \ldots, n\}$ satisfying $j < k$, $a_{ij} = a_{ik} = 1$, and $a_{jk} = 0$. The second equality in (3) holds if and only if any two adjacent vertices of G have exactly t common adjacent vertices, or $x_k = x_j$ for any adjacent vertices (v_j, v_k) with more than t common adjacent vertices.

Thus, we get

$$\sum_{i=1}^{n}(\lambda^2 - t\lambda + td_i - d_i m_i)x_i^2 \le 0.$$

Then there must exists $i \in \{1, 2, \ldots, n\}$ such that

$$\lambda^2 - t\lambda + td_i - d_i m_i \le 0.$$

This implies that

$$\lambda \le \frac{t + \sqrt{4d_i m_i - 4td_i + t^2}}{2}.$$

Hence,

$$\lambda \le \max_{1\le i\le n}\left\{\frac{t + \sqrt{4d_i m_i - 4td_i + t^2}}{2}\right\}.$$

Similarly with the proof of Theorem 3.1, if any two adjacent vertices of G have exactly t common adjacent vertices, and $x_k = x_j$ for any $i, j, k \in \{1, 2, \ldots, n\}$ satisfying $j < k$, $a_{ij} = a_{ik} = 1$, $a_{jk} = 0$, then equality holds if and only if for each i, $d_i m_i - d_i = k$, where k is a positive integer. Especially if $G \cong K_{t+2}$, then equality holds. □

Corollary 3.4. *Let G be a triangulation with n vertices and m edges. If each edge of G at least belongs to $t(t \ge 1)$ triangles, then*

$$\mu \le 2 + \max_{v_i\sim v_j}\left\{\frac{t + \sqrt{4d_i^2 + 4d_i m_i + 4d_j^2 + 4d_j m_j - (16+4t)d_i - (16+4t)d_j + 16 + 8t + t^2}}{2}\right\},$$

where μ is the signless Laplacian spectral radius.

Proof. Let G^L be the line graph of G. Noting that to every triangle in G, there corresponds a triangle in G^L. It follows that each edge of G^L also at least belongs to t triangles. By Theorem 3.2 and Lemma 3.4, we have

$$\lambda(G^L) \le \max_{v_{ij}\in V(G^L)}\left\{\frac{t + \sqrt{4d_{ij}m_{ij} - 4td_{ij} + t^2}}{2}\right\}.$$

Recall that,

$$d_{ij}(G^L) = d_i + d_j - 2, m_{ij}(G^L) = \frac{d_i(d_i + m_i - 4) + d_j(d_j + m_j - 4) + 4}{d_i + d_j - 2}.$$

Therefore,

$$\lambda(G^L) \leq \max_{v_i \sim v_j} \Big\{ \frac{t + \sqrt{4d_i^2 + 4d_i m_i + 4d_j^2 + 4d_j m_j - (16+4t)d_i - (16+4t)d_j + 16 + 8t + t^2}}{2} \Big\}.$$

Thus, by Lemma 3.2, we have

$$\mu \leq 2 + \max_{v_i \sim v_j} \Big\{ \frac{t + \sqrt{4d_i^2 + 4d_i m_i + 4d_j^2 + 4d_j m_j - (16+4t)d_i - (16+4t)d_j + 16 + 8t + t^2}}{2} \Big\}.$$

□

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