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Precise large deviations for sums of random vectors in a multidimensional size-dependent renewal risk model

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Abstract. Consider a multidimensional renewal risk model, in which the claim sizes $\{\boldsymbol{X}_k, k \geq 1\}$ form a sequence of independent and identically distributed random vectors with nonnegative components that are allowed to be dependent on each other. The univariate marginal distributions of these vectors have consistently varying tails and finite means. Suppose that the claim sizes and inter-arrival times correspondingly form a sequence of independent and identically distributed random pairs, with each pair obeying a dependence structure. A precise large deviation for the multidimensional renewal risk model is obtained.

§1 Introduction

Consider an insurer who simultaneously operates m ($m \geq 2$) lines of businesses with a common claim-number process. The claim sizes $\{\boldsymbol{X}_k = (X_{1,k}, \ldots, X_{m,k})^\top, k \geq 1\}$ form a sequence of independent and identically distributed (i.i.d.) random vectors with nonnegative components which are allowed to be dependent on each other. Denoted by $\tau_0 = 0$, $\tau_k = \sum_{i=1}^k \theta_i$, $k \geq 1$ the claim occurrence times where $\{\theta_k, k \geq 1\}$ are i.i.d. claim inter-arrival times with a common finite positive mean $1/\lambda$. Then, the number of claims by given time $t(\geq 0)$ is

$$N(t) = \sup\{k \geq 1\colon \tau_k \leq t\}.$$

In this way, the aggregate claims up to time $t(\geq 0)$ are given by the compound sum of the form

$$\boldsymbol{S}(t) = \sum_{k=1}^{N(t)} \boldsymbol{X}_k := \begin{pmatrix} \sum_{k=1}^{N(t)} X_{1,k} \\ \vdots \\ \sum_{k=1}^{N(t)} X_{m,k} \end{pmatrix}, \ t \geq 0, \tag{1}$$

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where, by convention, a summation over an empty index set produces an m-dimensional zero vector $\mathbf{0}$. Throughout this paper, for any two vectors $\boldsymbol{x}, \boldsymbol{y}$, the sum $\boldsymbol{x}+\boldsymbol{y}$, and vector inequalities such as $\boldsymbol{x} > \boldsymbol{y}$ are operated component-wisely.

If $\{\boldsymbol{X}_k, k \geq 1\}$ and $\{\theta_k, k \geq 1\}$ are mutually independent, then we obtain the standard multidimensional renewal risk model. Especially, for unidimensional case, it has been playing a fundamental role in classical and modern risk theory since it was introduced by Sparra Andersen [1] in the middle of the last century.

We are interested in the precise large deviations for a non-standard multidimensional renewal risk model $\boldsymbol{S}(t)$ under the assumption that the univariate marginal distributions of $\boldsymbol{X}_k$ are heavy-tailed while $\boldsymbol{X}_k$ and θ_k are size-dependent (see Assumption 2 for its accurate definition). Heavy-tailed distributions have become important building blocks in a wide variety of risk models. Evidence for heavy-tailed distributions is by now well documented in insurance theory. A well known class of heavy-tailed distributions is the class of consistently varying distributions. Recall that the distribution function (d.f.) F with support on $[0, \infty)$ is said to have a consistently varying tail and is denoted by $F \in \mathcal{C}$ if the tail $\bar{F} = 1 - F$ satisfies

$$\lim_{v \searrow 1} \liminf_{x \to \infty} \frac{\bar{F}(vx)}{\bar{F}(x)} = \lim_{v \nearrow 1} \limsup_{x \to \infty} \frac{\bar{F}(vx)}{\bar{F}(x)} = 1.$$

More discussions of the heavy-tailed distributions can be found in Bingham et al. [6], Cline and Samorodnitsky [9] and Embrechts et al. [11]. Here we recall a useful functional index. For a d.f. F, we set

$$\bar{F}_*(v) = \liminf_{x \to \infty} \frac{\bar{F}(vx)}{\bar{F}(x)}, \ \bar{F}^*(v) = \limsup_{x \to \infty} \frac{\bar{F}(vx)}{\bar{F}(x)}, \ v > 0,$$

and set

$$J_F^+ = -\lim_{v \to \infty} \frac{\log \bar{F}_*(v)}{\log v}, \ J_F^- = -\lim_{v \to \infty} \frac{\log \bar{F}^*(v)}{\log v},$$

and without any confusion we simply call $J_F^\pm$ the upper/lower Matuszewska index of F. For details of Matuszewska indices, see Bingham et al. [6]. If $F \in \mathcal{C}$, then from Proposition 2.2.1 in Bingham et al. [6], we know that, for any $p > J_F^+$, there are positive constants C and D, such that the inequality

$$\frac{\bar{F}(vx)}{\bar{F}(x)} \leq C v^{-p} \tag{2}$$

holds for all $x \geq D/v$ and $0 < v \leq 1$.

The precise large deviations of $\boldsymbol{S}(t)$ for unidimensional case have been well developed in literature: Baltrūnas et al. [4], Kaas and Tang [14], Klüppelberg and Mikosch [15], Ng et al. [21] and Tang et al. [25]. However, the classic unidimensional models cannot provide the whole picture for effects of different businesses to an insurer's solvency since many of the risk models are genuinely multidimensional. In recent years, the large deviation problem of multidimensional risk models has been studied by some scholars: Feng et al. [12], Lu [19] and Wang and Wang [27,28]. However, in all these references, the authors put the several types of aggregated risks together, that makes the problems to be reduced to unidimensional risk model when all the individual risks are i.i.d. or each claim event just induces one type of claim though different claims may depend on each other. As mentioned above, such situations are far from reality. Assuming that $\{\theta_k, k \geq 1\}$ and $\{\boldsymbol{X}_k, k \geq 1\}$ are mutually independent, Chen et al. [8], Shen et al. [22] and Shen and Tian [23] consider those problems in a different way which is natural in practice. They investigate the precise large deviations for sums of risk vectors component-wisely instead of adding the different types of aggregated risks together.

The independence between $\{\theta_k, k \geq 1\}$ and $\{\boldsymbol{X}_k, k \geq 1\}$ may be unreasonable in many applications. If the deductible applied to each loss is raised, then the claim sizes would decrease and the inter-arrival time will increase, since small losses will be retained by the insured. During the last decade, many scholars addressed this issue by proposing some non-standard unidimensional renewal risk models in which the claim size and its corresponding inter-arrival time are dependent. For details about the assumptions of such non-standard unidimensional renewal risk models, the reader is referred to Asimit and Badescu [2], Badescu et al. [3], Bi and Zhang [5], Chen and Yuen [7], Cossette et al. [10], Li et al. [18] and references therein. In this paper, we study the precise large deviations of $\boldsymbol{S}(t)$ in an m-dimensional ($m \geq 2$) size-dependent renewal risk model, in which claim sizes $\{\boldsymbol{X}_k, k \geq 1\}$ and inter-arrival times $\{\theta_k, k \geq 1\}$ correspondingly form a sequence of i.i.d random pairs, with each pair obeying a dependent structure described via the conditional distribution of the inter-occurrence time given the subsequent claim sizes being large. The results obtained here extend in Chen and Yuen [7], Shen et al. [22] and Shen and Tian [23].

The rest of this paper consists of three sections. Section 2 introduces the risk model of study and presents the large deviation. Section 3 states some lemmas and Section 4 presents the proof of the main result by establishing corresponding asymptotic lower and upper bounds.

§2 Main results and applications

In what follows, for two positive functions $a(\cdot)$ and $b(\cdot)$, we write $a(x) \lesssim b(x)$ or $b(x) \gtrsim a(x)$ if $\limsup_{x\to\infty} a(x)/b(x) \leq 1$, and $a(x) = o(b(x))$ if $\lim_{x\to\infty} a(x)/b(x) = 0$. Also a limit relation with certain uniformity will be frequently used. For instance, for two positive bivariate functions $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$, we say that $a(x,t) \lesssim b(x,t)$, as $t \to \infty$, holds uniformly for $x \in \Delta_t \neq \emptyset$ if

$$\limsup_{t\to\infty} \sup_{x\in\Delta_t} \frac{a(x,t)}{b(x,t)} \leq 1.$$

Hereafter, the following notation will be used throughout this paper. Let $\mathbb{I} := \{1, \dots, m\}$. For a nonempty subset $\mathbb{I}_d = \{i_1, \dots, i_d\} \subseteq \mathbb{I}$, and $\boldsymbol{x}_{\mathbb{I}_d} := (x_i, i \in \mathbb{I}_d)^\top$ is a d-dimensional subvector. As for m-dimensional vectors, we may omit the subscript $\mathbb{I}_m$ without any confusion.

Keep in mind that $\{(\boldsymbol{X}_k^\top, \theta_k), k \geq 1\}$ are i.i.d. copies of $(\boldsymbol{X}^\top = (X_1, \dots, X_m), \theta)$, with dependent components $\boldsymbol{X}$ and θ. For the convenience of expression, we state the following assumptions regarding the claim sizes $\{\boldsymbol{X}_k, k \geq 1\}$ and the counting process $N(t)$.

Assumption 1. The random vector $\boldsymbol{X}$ has a finite mean vector $\mathrm{E}\boldsymbol{X} = \boldsymbol{\mu} = (\mu_1, \dots, \mu_m)^\top$ and univariate marginal d.f.s $F_i \in \mathcal{C}$ (of X_i), $i \in \mathbb{I}$, and a joint survival function $\bar{\boldsymbol{F}}(\boldsymbol{x}) = \mathrm{P}(X_i >$

$x_i, i \in \mathbb{I})$ which is governed by a survival copula $\hat{C}(\cdot, \ldots, \cdot)$ satisfying

$$\hat{C}(u_i, i \in \mathbb{I}) \le g(m) \prod_{i \in \mathbb{I}} u_i, \ (u_i, i \in \mathbb{I})^\top \in [0,1]^m, \tag{3}$$

where $g(\cdot) \ge 1$ is a finite positive function.

Assumption 2. There exists a nonnegative random variable θ^* with finite mean such that θ conditional on $(X_i > x_i), i \in \mathbb{I}$, is stochastically bounded by θ^* for all large x_i, $i \in \mathbb{I}$, i.e., there exists some $\boldsymbol{x}_0 = (x_{1,0}, \ldots, x_{m,0})^\top$ such that it holds for all $\boldsymbol{x} > \boldsymbol{x}_0$ and $t \in [0, \infty)$ that

$$\mathrm{P}(\theta > t | X_i > x_i) \le \mathrm{P}(\theta^* > t), \ i \in \mathbb{I}. \tag{4}$$

Remark 1. It is easy to see $\bar{\boldsymbol{F}}(\boldsymbol{x}) = \hat{C}(\bar{F}_i(x_i), i \in \mathbb{I})$ due to the Sklar's Theorem(Nelsen [20]). Hence, Assumption 1 implies that $X_i, i \in \mathbb{I}$ are widely upper orthant dependent (WUOD), which is an important dependence structure introduced by Wang et al. [26] and covers some common negative dependence and positive dependence structures. As noted in Remark 3.1 of Ko and Tang [16], copulas in the Frank family of the form

$$\widehat{C}(u_i, \, i \in \mathbb{I}) = -\frac{1}{\theta} \ln \left(1 + \frac{\prod_{i \in \mathbb{I}} (\mathrm{e}^{-\theta u_i} - 1)}{(\mathrm{e}^{-\theta} - 1)^{n-1}} \right), \ \theta > 0,$$

or copulas in the Clayton family of the form

$$\widehat{C}(u_i, \, i \in \mathbb{I}) = \left(1 - n + \sum_{i \in \mathbb{I}} u_i^{-\theta} \right)^{-1/\theta}, \ \theta > 0,$$

result in that random variables $X_i, i \in \mathbb{I}$ are WUOD with $g(m) \equiv M$ where M is a positive constant. For other copulas that satisfy Assumption 1, we refer the reader to Remark 2 of Shen et al. [22] and Section 3 of Wang et al. [26].

It is worth mentioning that Shen et al. [22] assumed

$$\hat{C}_{\mathbb{I}_d}(u_i, i \in \mathbb{I}_d) \le M u_j \hat{C}_{\mathbb{I}_d \setminus \{j\}}(u_i, i \in \mathbb{I}_d \setminus \{j\}), \ (u_i, i \in \mathbb{I}_d)^\top \in [0,1]^d, j \in \mathbb{I}_d, \tag{5}$$

where $\hat{C}_{\mathbb{I}_d}$ is a d-dimensional marginal copula of $\hat{C}$ and $M \ge 1$ is a positive constant. Clearly, (5) implies (3), i.e., $\hat{C}_{\mathbb{I}_d}$ satisfying (5) means random variables $X_i, i \in \mathbb{I}$ are WUOD.

Remark 2. If $m = 1$, the dependence structure specified in (4) implies that the marginal vector (X_i, θ) obeys a parallel bidimensional dependence structure which is actually our main motivation to propose Assumption 2. Such popular dependence structure in (X_i, θ) termed as size-dependent, is first established by Chen and Yuen [7] and has been extensively applied in both risk analysis and probability theory, see Bi and Zhang [5], Fu and Shen [13] and Shen et al. [24], among others.

We can now state our main result.

Theorem 1. *Consider the model (1), in which pairs of claim sizes and its corresponding inter-arrival times* $(\boldsymbol{X}_k^\top, \theta_k), k \ge 1$, *are i.i.d. as a generic pair* $(\boldsymbol{X}^\top, \theta)$ *satisfying Assumptions 1 and 2. Then, for any* $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_m)^\top > \boldsymbol{0}$,

$$\mathrm{P}\left(\boldsymbol{S}(t) - \boldsymbol{\mu}\lambda t > \boldsymbol{x}\right) \sim (\lambda t)^m \prod_{i=1}^m \bar{F}_i(x_i), \ t \to \infty$$

holds uniformly for all $\boldsymbol{x} \geq \boldsymbol{\gamma} t$*, i.e.,*

$$\lim_{t\to\infty} \sup_{\boldsymbol{x}\geq\boldsymbol{\gamma}t} \left| \frac{\mathrm{P}\left(\boldsymbol{S}(t) - \boldsymbol{\mu}\lambda t > \boldsymbol{x}\right)}{(\lambda t)^m \prod_{i=1}^m \bar{F}_i(x_i)} - 1 \right| = 0.$$

§3 Lemmas

According to Assumption 2, introduce m independent and nonnegative random variables $\theta_i^{*m}, i \in \mathbb{I}$ that are independent of all other sources of randomness and identically distributed as θ conditional on $(X_i > x_i), i \in \mathbb{I}$, respectively. We construct an m-delayed renewal counting process $\{N^{*m}(t), t \geq 0\}$ with inter-arrival times $\tau_1^{*m} = \theta_1^{*m}$, $\tau_2^{*m} = \theta_1^{*m} + \theta_2^{*m}, \ldots, \tau_m^{*m} = \sum_{i=1}^m \theta_i^{*m}, \ldots, \tau_n^{*m} = \sum_{i=1}^m \theta_i^{*m} + \sum_{i=m+1}^n \theta_i$, for $2 \leq m \leq n$. Note that the distribution of $\{N^{*m}(t), t \geq 0\}$ depends on $\boldsymbol{x}$ through the conditions $(X_1 > x_1), \ldots, (X_m > x_m)$.

We first show a lemma concerning these m-delayed renewal counting processes.

Lemma 1. *In addition to Assumption 2, assume that* $\mathrm{E}\theta = 1/\lambda \in (0, \infty)$ *and* $\mathrm{E}\theta^2 < \infty$. *Then, for every* $\delta > 0$ *and any* $\boldsymbol{\gamma} > \mathbf{0}$,

$$\lim_{t\to\infty} \sup_{\boldsymbol{x}\geq\boldsymbol{\gamma}t} \mathrm{P}\left(\left| \frac{N^{*m}(t) - \lambda t}{t} \right| > \delta \right) = 0, \ \ m \geq 2.$$

Proof. Going along the same lines of the proofs of Lemma 2.1 in Chen and Yuen [7] and Lemma 3.4 in Bi and Zhang [5] but with some obvious modifications, we can prove lemma 1 immediately. □

The following lemma is a restatement of Theorem 1(i) of Kočetova et al. [17].

Lemma 2. *Let the inter-arrival times* $\{\theta_k, k \geq 1\}$ *form a sequence of i.i.d. nonnegative random variables with a common mean* $1/\lambda \in (0, \infty)$. *Then, for every* $\alpha > \lambda$ *and some* $b > 1$,

$$\lim_{t\to\infty} \sum_{n>at} b^n \mathrm{P}(\tau_n \leq t) = 0.$$

The following lemma is a restatement of Theorem 2.1 of Shen et al. [22] and forms the basis for the proof of Theorem 1. For the sake of completeness, we give the proof here and it will show that the result still holds without (5) and the tail equivalence between marginal d.f.s of claim sizes, while such assumptions are used in Shen et al. [22].

Lemma 3. *Let* $\{\boldsymbol{X}_k, k \geq 1\}$ *be a sequence of i.i.d. random vectors satisfying Assumption 1. Then for any* $\boldsymbol{\gamma} > \mathbf{0}$, *it holds uniformly for all* $\boldsymbol{x} \geq \boldsymbol{\gamma} n$, *that*

$$\mathrm{P}(\boldsymbol{S}_n - n\boldsymbol{\mu} > \boldsymbol{x}) \sim \sum_{j_1=1}^n \cdots \sum_{j_m=1}^n \mathrm{P}(X_{i,j_i} > x_i, i \in \mathbb{I}) \sim n^m \prod_{i=1}^m \bar{F}_i(x_i), \ \ n \to \infty, \tag{6}$$

where $\boldsymbol{S}_n = (S_{i,n}, i \in \mathbb{I})^\top := \sum_{k=1}^n \boldsymbol{X}_k = (\sum_{k=1}^n X_{i,k}, i \in \mathbb{I})^\top$.

Proof. Note that for any $\boldsymbol{\gamma} > \mathbf{0}$, it holds uniformly for all $\boldsymbol{x} \geq \boldsymbol{\gamma} n$, that

$$\sum_{j_1=1}^{n} \cdots \sum_{j_m=1}^{n} \mathrm{P}\left(X_{i,j_i} > x_i, i \in \mathbb{I}\right) \sim n^m \prod_{i=1}^{m} \bar{F}_i(x_i) \tag{7}$$

due to Assumption 1. Hence, we just need to show the relation between $\mathrm{P}(\boldsymbol{S}_n - n\boldsymbol{\mu} > \boldsymbol{x})$ and $n^m \prod_{i=1}^{m} \bar{F}_i(x_i)$ in (6).

For any $\rho > 1$, via a procedure similar to the proof of the asymptotic lower bound of Theorem 1 in Shen et al. [22] but with some minor modifications, we have

$$\liminf_{n\to\infty} \inf_{\boldsymbol{x} \geq \boldsymbol{\gamma} n} \frac{\mathrm{P}\left(\boldsymbol{S}_n - n\boldsymbol{\mu} > \boldsymbol{x}\right)}{n^m \prod_{i=1}^{m} \bar{F}_i(\rho x_i)} \geq 1$$

due to (7). This implies

$$\liminf_{n\to\infty} \inf_{\boldsymbol{x} \geq \boldsymbol{\gamma} n} \frac{\mathrm{P}\left(\boldsymbol{S}_n - n\boldsymbol{\mu} > \boldsymbol{x}\right)}{n^m \prod_{i=1}^{m} \bar{F}_i(x_i)} \geq \lim_{\rho \searrow 1} \liminf_{n\to\infty} \inf_{\boldsymbol{x} \geq \boldsymbol{\gamma} n} \frac{n^m \prod_{i=1}^{m} \bar{F}_i(\rho x_i)}{n^m \prod_{i=1}^{m} \bar{F}_i(x_i)} = 1, \tag{8}$$

where, in the last step, we also used the fact that $F_i \in \mathcal{C},\ i \in \mathbb{I}$.

Similarly, going along the same lines of the proofs of the asymptotic upper bound of Theorem 1 in Shen et al. [22] but with some minor modifications, we have

$$\limsup_{n\to\infty} \sup_{\boldsymbol{x} \geq \boldsymbol{\gamma} n} \frac{\mathrm{P}\left(\boldsymbol{S}_n - n\boldsymbol{\mu} > \boldsymbol{x}\right)}{n^m \prod_{i=1}^{m} \bar{F}_i(x_i)} \leq 1.$$

This, coupled with (8), yields (6). □

§4 Proof of Theorem 1

In the following, every limit relation is understood as valid uniformly for all $\boldsymbol{x} \geq \boldsymbol{\gamma} t$ as $t \to \infty$. Trivially, Theorem 1 amounts to the conjunction of

$$\mathrm{P}\left(\boldsymbol{S}(t) - \boldsymbol{\mu}\lambda t > \boldsymbol{x}\right) \gtrsim (\lambda t)^m \prod_{i=1}^{m} \bar{F}_i(x_i) \tag{9}$$

and

$$\mathrm{P}\left(\boldsymbol{S}(t) - \boldsymbol{\mu}\lambda t > \boldsymbol{x}\right) \lesssim (\lambda t)^m \prod_{i=1}^{m} \bar{F}_i(x_i). \tag{10}$$

First, we show the asymptotic lower bound. For $0 < \delta < 1$ and $\nu > 1$,

$$\begin{aligned}
&\mathrm{P}\left(\boldsymbol{S}(t) - \boldsymbol{\mu}\lambda t > \boldsymbol{x}\right) \\
&\geq \sum_{(1-\delta)\lambda t \leq n \leq (1+\delta)\lambda t} \mathrm{P}\left(\boldsymbol{S}_n - \boldsymbol{\mu}\lambda t > \boldsymbol{x}, N(t) = n\right) \\
&\geq \sum_{(1-\delta)\lambda t \leq n \leq (1+\delta)\lambda t} \mathrm{P}\left(\boldsymbol{S}_n - \boldsymbol{\mu}\lambda t > \boldsymbol{x}, N(t) = n, \max_{1 \leq j_i \leq n} X_{i,j_i} > \nu x_i, i \in \mathbb{I}\right)
\end{aligned}$$

Hence, by Bonferroni's inequality,

$$\begin{aligned}
&\mathrm{P}\left(\boldsymbol{S}_n-\boldsymbol{\mu}\lambda t>\boldsymbol{x}, N(t)=n, \max_{1\le j_i\le n} X_{i,j_i}>\nu x_i, i\in\mathbb{I}\right)\\
&\ge \sum_{1\le j_1,\dots,j_m\le n} \mathrm{P}\left(\boldsymbol{S}_n-\boldsymbol{\mu}\lambda t>\boldsymbol{x}, N(t)=n, X_{i,j_i}>\nu x_i, i\in\mathbb{I}\right)\\
&\quad-\sum_{1\le j_1,\dots,j_m,p_1\le n, j_1\ne p_1} \mathrm{P}\left(N(t)=n, X_{1,p_1}>\nu x_1, X_{i,j_i}>\nu x_i, i\in\mathbb{I}\right)\\
&\quad-\cdots\\
&\quad-\sum_{1\le j_1,\dots,j_m,p_m\le n, j_m\ne p_m} \mathrm{P}\left(N(t)=n, X_{m,p_m}>\nu x_m, X_{i,j_i}>\nu x_i, i\in\mathbb{I}\right)\\
&=: P_0-P_1-\cdots-P_m.
\end{aligned}$$

We set $\boldsymbol{S}_{n,(j_1,j_2,\dots,j_k)}=\boldsymbol{S}_n-\boldsymbol{X}_{j_1}-\cdots-\boldsymbol{X}_{j_k}$. By applying the i.i.d. assumption of the sequence $\{(\boldsymbol{X}_k^\top,\theta_k), k\ge 1\}$, we have

$$\begin{aligned}
P_0 &= \sum_{1\le j_1,\dots,j_m\le n} \mathrm{P}\left(\boldsymbol{S}_n-\boldsymbol{\mu}\lambda t>\boldsymbol{x}, N(t)=n, X_{i,j_i}>\nu x_i, i\in\mathbb{I}\right)\\
&\ge \sum_{\substack{1\le j_s\ne j_l\le n, s\ne l\\ s,l=1,\dots,m}} \mathrm{P}\left(\boldsymbol{S}_n-\boldsymbol{\mu}\lambda t>\boldsymbol{x}, N(t)=n, X_{i,j_i}>\nu x_i, X_{ij_k}\ge 0, k\ne i, i\in\mathbb{I}\right)\\
&\ge \sum_{\substack{1\le j_s\ne j_l\le n, s\ne l\\ s,l=1,\dots,m}} \mathrm{P}\left(\boldsymbol{S}_{n,(j_1,\dots,j_m)}-\boldsymbol{\mu}\lambda t>(1-\nu)\boldsymbol{x}, N(t)=n, X_{i,j_i}>\nu x_i, i\in\mathbb{I}\right)\\
&= \sum_{\substack{1\le j_s\ne j_l\le n, s\ne l\\ s,l=1,\dots,m}} \Big\{\mathrm{P}\left(\boldsymbol{S}_{n,(j_1,\dots,j_m)}-\boldsymbol{\mu}\lambda t>(1-\nu)\boldsymbol{x}, N(t)=n \,\middle|\, X_{i,j_i}>\nu x_i, i\in\mathbb{I}\right)\\
&\qquad\qquad \times \mathrm{P}\left(X_{i,j_i}>\nu x_i, i\in\mathbb{I}\right)\Big\}\\
&\ge n(n-1)\cdots(n-m+1)\mathrm{P}\left(\boldsymbol{S}_{n,(1,\dots,m)}-\boldsymbol{\mu}\lambda t>(1-\nu)\boldsymbol{x}, N^{*m}(t)=n\right)\prod_{i=1}^m \bar{F}_i(\nu x_i).
\end{aligned}$$

By choosing positive δ small enough such that $(1-\delta)\mu_i\lambda-\mu_i\lambda>(1-\nu)\gamma_i$ for $i\in\mathbb{I}$, then it follows from the laws of large numbers for the partial sums $S_{i,n}, i\in\mathbb{I}$, $n\ge 1$,

$$\mathrm{P}\left(\boldsymbol{S}_{\lfloor(1-\delta)\lambda t\rfloor,(1,\dots,m)}-\boldsymbol{\mu}\lambda t>(1-\nu)\boldsymbol{x}\right)=1, \tag{11}$$

where $\lfloor y\rfloor$ denotes the integer part of y. Thus, relation (11) coupled with Lemma 1 yields

$$\begin{aligned}\sum_{(1-\delta)\lambda t\le n\le(1+\delta)\lambda t} P_0 \ge&\lfloor(1-\delta)\lambda t\rfloor\cdots(\lfloor(1-\delta)\lambda t\rfloor-m+1)\prod_{i=1}^m\bar F_i(\nu x_i)\\&\times \mathrm{P}\left(\boldsymbol{S}_{\lfloor(1-\delta)\lambda t\rfloor,(1,\ldots,m)}-\boldsymbol{\mu}\lambda t>(1-\nu)\boldsymbol{x},\left|\frac{N^{*m}(t)}{\lambda t}-1\right|\le\delta\right)\\ \ge&\lfloor(1-\delta)\lambda t\rfloor\cdots(\lfloor(1-\delta)\lambda t\rfloor-m+1)\prod_{i=1}^m\bar F_i(\nu x_i)\\&\times\left(\mathrm{P}\left(\boldsymbol{S}_{\lfloor(1-\delta)\lambda t\rfloor,(1,\ldots,m)}-\boldsymbol{\mu}\lambda t>(1-\nu)\boldsymbol{x}\right)-\mathrm{P}\left(\left|\frac{N^{*m}(t)}{\lambda t}-1\right|>\delta\right)\right)\\ \gtrsim&((1-\delta)\lambda t)^m\prod_{i=1}^m\bar F_i(\nu x_i).\end{aligned}$$

Then, by the condition $F_i\in\mathcal{C}$ for $i\in\mathbb{I}$, we can see that

$$\lim_{\delta\searrow0}\lim_{\nu\searrow1}\liminf_{t\to\infty}\inf_{\boldsymbol{x}\ge\boldsymbol{\gamma}t}\frac{\sum_{(1-\delta)\lambda t\le n\le(1+\delta)\lambda t}P_0}{(\lambda t)^m\prod_{i=1}^m\bar F_i(x_i)}\ge1. \tag{12}$$

Regarding P_1, it is easy to get that

$$\begin{aligned}&\sum_{(1-\delta)\lambda t\le n\le(1+\delta)\lambda t}P_1\\ \le&\sum_{(1-\delta)\lambda t\le n\le(1+\delta)\lambda t}\ \sum_{1\le j_1,\ldots,j_m,p_1\le n,p_1\ne j_1}\mathrm{P}\left(N(t)=n\middle|X_{1,p_1}>\nu x_1,X_{i,j_i}>\nu x_i,i\in\mathbb{I}\right)\\&\times\mathrm{P}\left(X_{1,p_1}>\nu x_1,X_{i,j_i}>\nu x_i,i\in\mathbb{I}\right)\\ \le&\sum_{1\le j_1,\ldots,j_m,p_1\le(1+\delta)\lambda t,p_1\ne j_1}\mathrm{P}\left(X_{1,p_1}>\nu x_1,X_{i,j_i}>\nu x_i,i\in\mathbb{I}\right).\end{aligned}$$

By Assumption 1, the largest one of $\mathrm{P}\left(X_{1,p_1}>\nu x_1,X_{i,j_i}>\nu x_i,i\in\mathbb{I}\right)$ in the above display is $(g(m))^{(m+1)/2}\bar F_1(\nu x_1)\prod_{i=1}^m\bar F_i(\nu x_i)$. Hence,

$$\sum_{(1-\delta)\lambda t\le n\le(1+\delta)\lambda t}P_1\le(g(m))^{(m+1)/2}((1+\delta)\lambda t)^{m+1}\bar F_1(\nu x_1)\prod_{i=1}^m\bar F_i(\nu x_i),$$

which implies that

$$\lim_{\nu\searrow1}\limsup_{t\to\infty}\sup_{\boldsymbol{x}\ge\boldsymbol{\gamma}t}\frac{\sum_{(1-\delta)\lambda t\le n\le(1+\delta)\lambda t}P_1}{(\lambda t)^m\prod_{i=1}^m\bar F_i(x_i)}=0, \tag{13}$$

since $t\bar F_1(\nu x_1)\le\gamma^{-1}x\bar F_1(\nu x_1)\to0$, by virtue of $\mu_1<\infty$. Similarly, we can show

$$\lim_{\nu\searrow1}\limsup_{t\to\infty}\sup_{\boldsymbol{x}\ge\boldsymbol{\gamma}t}\frac{\sum_{(1-\delta)\lambda t\le n\le(1+\delta)\lambda t}P_s}{(\lambda t)^m\prod_{i=1}^m\bar F_i(\nu x_i)}=0,\ \ s=2,\ldots,m. \tag{14}$$

Hence, combining (12), (13) and (14), the relation (9) is obtained.

In the sequel, we show the asymptotic upper bound. For small $0<\delta<1$, we have

$$\begin{aligned}\mathrm{P}\left(\boldsymbol{S}(t)-\boldsymbol{\mu}\lambda t>\boldsymbol{x}\right)=&\,\mathrm{P}\left(\boldsymbol{S}(t)-\boldsymbol{\mu}\lambda t>\boldsymbol{x},N(t)\le(1+\delta)\lambda t\right)\\&+\mathrm{P}\left(\boldsymbol{S}(t)-\boldsymbol{\mu}\lambda t>\boldsymbol{x},N(t)>(1+\delta)\lambda t\right)\\=:&\,J_1+J_2.\end{aligned}$$

Regarding J_1, it follows from Lemma 3 that

$$\begin{aligned} J_1 \leq & \mathrm{P}\left(\boldsymbol{S}_{\lfloor(1+\delta)\lambda t\rfloor}-\boldsymbol{\mu}\lambda t>\boldsymbol{x}\right) \\ = & \mathrm{P}\left(\boldsymbol{S}_{\lfloor(1+\delta)\lambda t\rfloor}-\boldsymbol{\mu}\lfloor(1+\delta)\lambda t\rfloor>\boldsymbol{x}+\boldsymbol{\mu}\lambda t-\boldsymbol{\mu}\lfloor(1+\delta)\lambda t\rfloor\right) \\ \sim & (\lfloor(1+\delta)\lambda t\rfloor)^m \prod_{i=1}^m \bar{F}_i(x_i+\mu_i\lambda t-\mu_i\lfloor(1+\delta)\lambda t\rfloor) \\ \lesssim & ((1+\delta)\lambda t)^m \prod_{i=1}^m \bar{F}_i((1-\delta\mu_i/\gamma_i)x_i), \end{aligned}$$

for $x_i \geq \gamma_i\lambda t$, $i\in\mathbb{I}$. Using the fact that $F_i\in\mathcal{C}$, $i\in\mathbb{I}$, we have

$$\lim_{\delta\searrow 0}\limsup_{t\to\infty}\sup_{\boldsymbol{x}\geq\boldsymbol{\gamma}t}\frac{J_1}{(\lambda t)^m\prod_{i=1}^m\bar{F}_i(x_i)}\leq 1. \tag{15}$$

Next, we estimate J_2. Let $\{\mathbb{I}^1,\dots,\mathbb{I}^k\}$ be an arbitrary partition of $\mathbb{I}$, $1\leq k\leq m$, that is,

$$\bigcup_{i=1}^k \mathbb{I}^i=\mathbb{I},\ \mathbb{I}^i\cap\mathbb{I}^j=\emptyset,\ i\neq j.$$

Let $\mathscr{I}_k$ be the set of all partitions with k subsets of $\mathbb{I}$, $1\leq k\leq m$. The summation $\sum_{\{\mathbb{I}^1,\dots,\mathbb{I}^k\}\in\mathscr{I}_k}$ is for all partitions $\{\mathbb{I}^1,\dots,\mathbb{I}^k\}$ over the collection $\mathscr{I}_k$. Then,

$$\begin{aligned} & \mathrm{P}(\boldsymbol{S}_n-\boldsymbol{\mu}\lambda t>\boldsymbol{x}, N(t)=n) \\ \leq & \mathrm{P}\left(\bigcup_{j_i=1}^n\left\{X_{i,j_i}>\frac{x_i}{n}\right\}, N(t)=n, i\in\mathbb{I}\right) \\ \leq & \sum_{1\leq j_1,\dots,j_m\leq n}\mathrm{P}\left(X_{i,j_i}>\frac{x_i}{n},\tau_n\leq t, i\in\mathbb{I}\right) \\ = & \sum_{j=1}^n\mathrm{P}\left(X_{ij}>\frac{x_i}{n},\tau_n\leq t, i\in\mathbb{I}\right) \\ & +\sum_{k=2}^m\sum_{\{\mathbb{I}^1,\dots,\mathbb{I}^k\}\in\mathscr{I}_k}\sum_{\substack{1\leq j_s\neq j_l\leq n, s\neq l\\ s,l=1,\dots,k}}\mathrm{P}\left(\bigcap_{i=1}^k\left\{X_{q,j_i}>\frac{x_q}{n}, q\in\mathbb{I}^i\right\},\tau_n\leq t\right), \end{aligned}$$

which yields

$$\begin{aligned} J_2 \leq & \sum_{n>(1+\delta)\lambda t}\left(\sum_{j=1}^n\mathrm{P}\left(X_{i,j}>\frac{x_i}{n},\tau_n\leq t, i\in\mathbb{I}\right)\right. \\ & \left.+\sum_{k=2}^m\sum_{\{\mathbb{I}^1,\dots,\mathbb{I}^k\}\in\mathscr{I}_k}\sum_{\substack{1\leq j_s\neq j_l\leq n, s\neq l\\ s,l=1,\dots,k}}\mathrm{P}\left(\bigcap_{i=1}^k\left\{X_{q,j_i}>\frac{x_q}{n}, q\in\mathbb{I}^i\right\},\tau_n\leq t\right)\right) \\ =: & J_{21}+\sum_{k=2}^m J_{2k}. \end{aligned}$$

By (2), for every $p>\max\{J^+_{F_i}, i\in\mathbb{I}\}$, there is a constant C such that $\mathrm{P}(X_i>x_i/n)\leq$

$Cn^p\bar{F}_i(x_i), i\in\mathbb{I}$. Hence, for J_{21}, an upper bound can be constructs as

$$\begin{aligned}J_{21}\leq&\sum_{n>(1+\delta)\lambda t}\sum_{j=1}^{n}\mathrm{P}\left(X_{i,j}>\frac{x_i}{n},\sum_{1\leq z\neq j\leq n}\theta_z\leq t, i\in\mathbb{I}\right)\\=&\sum_{n>(1+\delta)\lambda t}n\mathrm{P}\left(X_{i,j}>\frac{x_i}{n}, i\in\mathbb{I}\right)\mathrm{P}(\tau_{n-1}\leq t)\\\leq&C^m g(m)\prod_{i=1}^{m}\bar{F}_i(x_i)\sum_{n>(1+\delta)\lambda t}n^{mp+1}\mathrm{P}(\tau_{n-1}\leq t),\end{aligned}$$

where, in the last step, we also used the condition (3). Now by Lemma 2, we get

$$J_{21}=o(t)\prod_{i=1}^{m}\bar{F}_i(x_i). \tag{16}$$

As for J_{2k}, $2\leq k\leq m$, again by (2) and condition (3), we have

$$\begin{aligned}J_{2k}\leq&\sum_{n>(1+\delta)\lambda t}\sum_{\{\mathbb{I}^1,\ldots,\mathbb{I}^k\}\in\mathscr{I}_k}\sum_{\substack{1\leq j_s\neq j_l\leq n, s\neq l\\ s,l=1,\ldots,k}}\mathrm{P}\left(\bigcap_{i=1}^{k}\left\{X_{q,j_i}>\frac{x_q}{n}, q\in\mathbb{I}^i\right\},\sum_{1\leq z\neq j_1,\ldots,j_k\leq n}\theta_z\leq t\right)\\=&\sum_{n>(1+\delta)\lambda t}\sum_{\{\mathbb{I}^1,\ldots,\mathbb{I}^k\}\in\mathscr{I}_k}n(n-1)\cdots(n-k+1)\prod_{i=1}^{k}\hat{C}_{\mathbb{I}^i}(\bar{F}(x_q/n), q\in\mathbb{I}^i)\mathrm{P}(\tau_{n-k}\leq t)\\\leq&\sum_{n>(1+\delta)\lambda t}\sum_{\{\mathbb{I}^1,\ldots,\mathbb{I}^k\}\in\mathscr{I}_k}n(n-1)\cdots(n-k+1)\prod_{i=1}^{k}\left[g(m)\prod_{q\in\mathbb{I}^i}(Cn^p\bar{F}(x_q))\right]\mathrm{P}(\tau_{n-k}\leq t),\end{aligned}$$

which, together with Lemma 2, implies

$$\begin{aligned}J_{2k}\leq&C^m(g(m))^k\prod_{i=1}^{m}\bar{F}_i(x_i)\sum_{\{\mathbb{I}^1,\ldots,\mathbb{I}^k\}\in\mathscr{I}_k}\sum_{n>(1+\delta)\lambda t}n^{mp+k}\mathrm{P}(\tau_{n-k}\leq t)\\=&o(t)\prod_{i=1}^{m}\bar{F}_i(x_i),\ 2\leq k\leq m.\end{aligned}\tag{17}$$

Thus, (15), (16), and (17) guarantee (10) hold. The proof of Theorem 1 is completed. □

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