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Spiral transitions

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Abstract. Spiral curves are free from singularities and curvature extrema. These are used in path smoothing applications to overcome the abrupt change in curvature and super-elevation of moving object that occurs between tangent and circular curve. Line to circle spiral transition is made of straight line segment and curvature continuous spiral curve. It is extendible to other important types of transitions like line to line and circle to circle. Although the problem of line to circle transition has been addressed by many researchers, there is no comprehensive literature review available. This paper presents state-of-the-art of line to circle spiral transition, applications in different fields, limitations of existing approaches, and recommendations to meet the challenges of compatibility with today's CAD/CAM soft wares, satisfaction of Hermite end conditions, approximation of discrete data for image processing, 3D path smoothness for an unmanned aerial vehicle (UAV), and arc-length parametrization. Whole discussion is concluded with future research directions in various areas of applications.

§1 Introduction

Curvature continuous curves or spirals are used in computer science and engineering to overcome the abrupt change in curvature that occurs between tangent and circular curve [27], necessary to avoid pre-mature wear and tear of moving objects. These curves are called spiral transitions, widely used in path smoothing applications, such as road/rail design, robot trajectories, satellite path design, and computer imaging, etc. A simple spiral transition starts from zero curvature as the tangent to the straight line and ends with the curvature equal that of the connecting arc, briefly called line to circle or J-shape spiral transition [39, 40, 53, 81], having single curvature extremum at the start point. Other transition curves are line to line (U-shape) [32, 82], and circle to circle [29, 34, 35]. Circle to circle transition is further classified in S-shape and C-shape curves. C-shape is again divided into circle to circle with a broken back and circle to circle where one circle completely lies inside the other [34, 70]. Based on Kneser theorem [25], any circle of curvature of spiral encloses all smaller circles of curvature

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and is enclosed by all larger circles of curvature. Hence the transition curve without curvature extrema is possible when one circle completely lies inside the other circle, referred as the *fifth* case [34, 71].

Among all above mentioned cases of spiral transitions, J-shape curve, i.e., line to circle spiral transition is extensively used in aesthetic and engineering applications that require free-form curves and surfaces, either through path fairing or smoothing arc-spline [53]. It is extendable to U-shape (line to circle and circle to line), S-shape (circle to line and line to circle with inflection point), and C-shape with a broken back (circle to line and line to circle with no inflection point). Since a lot of work has been done on straight line to circle spiral transitions, their extension to other types of useful transitions [29, 30], and applications in many areas, there needs a broader literature survey on line to circle spiral transition.

This paper focuses on the review of line to circle transition due to its central importance in path fairing. Next section provides the background of spiral transition curves. Rest of the paper presents state-of-the-art of line to circle spiral transition, practical applications in various important fields, limitations of proposed methodologies, and recommendations to meet the challenges of compatibility with today's CAD/CAM softwares, satisfaction of end conditions, arc-length parametrization, approximation of discrete data for image processing, and 3D path smoothness for an unmanned aerial vehicle (UAV). Finally, whole discussion is concluded with future research directions as per the requirement of practical applications.

§2 Background

A lot of research has been done in the past on spiral transitions. Traditionally, clothoid has been used for many years in path smoothing applications, such as highway or rail design [5, 42]. Clothoid is also referred as the Cornu spiral, the Euler spiral, the American spiral, or simply the transition spiral. It was specially recommended due to the linear relationship between the arc-length and curvature of a clothoid [60, 72, 73]. Clothoid is free from singularities (loops and cusps, and inflection points except for a single inflection point at its beginning) due to the curvature continuity. Since it is neither a polynomial nor a rational curve; it is thus inconvenient to incorporate it into existing Computer-Aided Design (CAD) system. Further, clothoid is transcendental function which is quite inflexible due to the absence of the shape control parameter [36, 41].

Beta-splines, Nu-splines, Bézier curves [10], B-splines, or NURBS (Non Uniform Rational B-Splines), or recently proposed Beta-Bézier curve [13] are used extensively for CAD and computer-aided geometric design (CAGD) applications. These curves are suitable for applications in which fair curves are important [67]. Since they are polynomial, resulting algorithms are convenient for implementation in an interactive computer graphics environment due to the provision of shape control parameters. However, their polynomial nature causes problems in obtaining desirable shapes. They may have unwanted singularities (cusps, loops), inflection points, and curvature extrema. G^2 or even C^2 continuity does not guarantee the absence of unwanted singularities or inflection points [41, 47]. Further, polynomial functions are compatible with unit or chord-length parametrization rather than arc-length [83] which is desired in many applications. Another challenge is to keep the minimum possible number of internal curvature extrema in curve with the given Hermite end conditions [28, 37, 38], i.e., fixing both end points, directions of tangents at end points, and curvatures at end points of the curve. A curve is considered fair (or smooth) if its curvature is monotonically increasing or decreasing, i.e., curvature does not change sign. Regions of monotone curvature are separated by points of extreme curvature. Therefore, the number of curvature extrema of a fair curve should be the smallest and where necessary [18].

Here is a list of conventions used in this review [27].

1. First order geometric continuity: Continuity in position and unit tangent.
2. Second order geometric continuity: Continuity in position, unit tangent, and signed curvature.
3. G^1 Hermite end conditions: If both end points of curve and unit tangents at these end points are given.
4. G^2 Hermite end conditions: If both end points of curve, unit tangents at these end points, and signed curvatures at these end points are given.
5. Curvature extremum: A point on the curve where the derivative of the curvature changes sign.
6. Spiral: A curve with monotone curvature of constant sign or with no curvature extremum.
7. Hodograph: If $\boldsymbol{v} = f(t)$ is the vector equation of the path of the particle, $d\boldsymbol{v}/dt = f'(t)$ is the equation of the hodograph.
8. Pythagorean hodograph: The curve $\boldsymbol{z}(t)\,(= x(t), y(t))$ is Pythagorean hodograph (PH) if $\{x'(t)\}^2 + \{y'(t)\}^2$ can be expressed as the square of a polynomial in t.

2.1 Line to circle spiral transition

Line to circle (or J-shape) spiral transition in normalized form (see Figure 1) can be started from $z(0)$, at the origin $\boldsymbol{O}$, with beginning tangent vector along positive x-axes, and curvature at the point zero. This transition is terminated at point $\boldsymbol{z}(1)$ on circle Ω centered at $\boldsymbol{C}$, with ending tangent vector making an angle θ with x-axes and radius of curvature at the end point r. Here, $\boldsymbol{z}(t)$ is a parametric polynomial with $t(\in [0, 1])$ as the parameter. J-shape transition is extendable to other important types, like C-shape and U-shape spiral transitions. Habib and Sakai [32] discussed in detail the framework and methodology of these transition curves. Their brief overview is as follows.

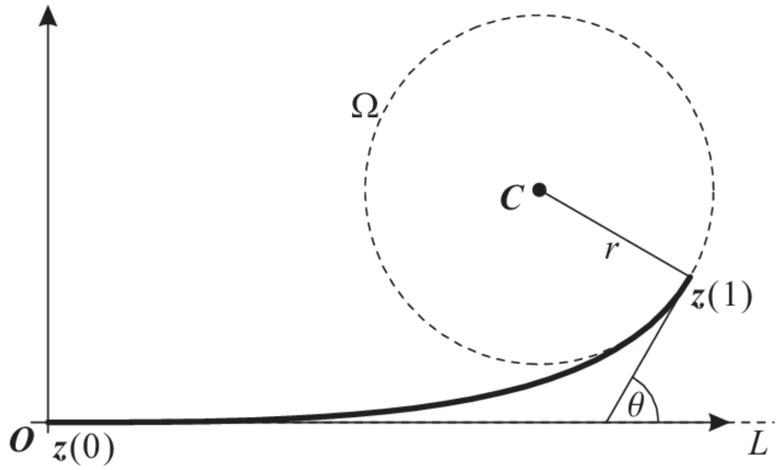
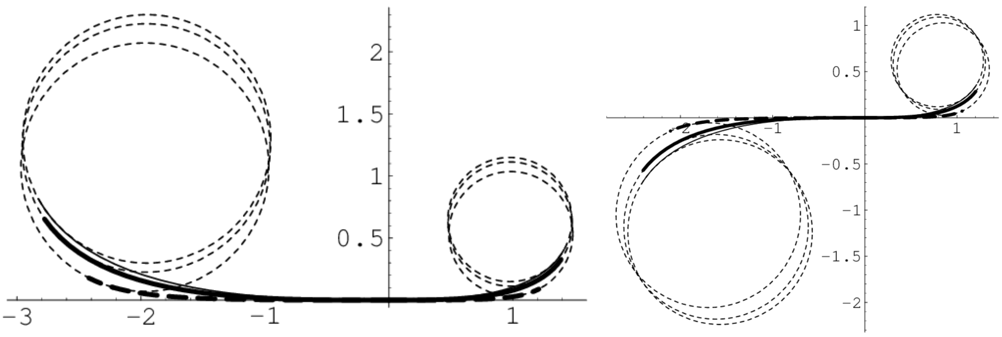


Figure 1: A normalized form of J-shape spiral transition curve [32].

2.1.1 Circle to circle spiral transition

Circle to circle spiral transition is achieved by stitching two J-shape spiral transitions in the form of circle to line and line to circle. Line in the middle is common to both circles. Its setup is shown in Figure 2 for both C-shape and S-shape spiral transitions between two circles. With the change in shape parameter, both circles move. They can be fixed by geometric transformation because the distance between circles remains fixed, however the end points will move along circles, partially violating the Hermite end conditions.



(a) C-shape (b) S-shape

Figure 2: Family of circle to circle spiral transitions [32].

2.1.2 Line to line spiral transition

Line to line or U-shape spiral transition is achieved by stitching two J-shape spiral transitions in the form of line to circle and circle to line. Circle in the middle is common to both straight lines. Its setup is shown in Figure 3. With the change in shape parameter or the radius of circle, we can generate different curves as per our requirement. All shapes are strictly following Hermite end conditions, see Figure 4.

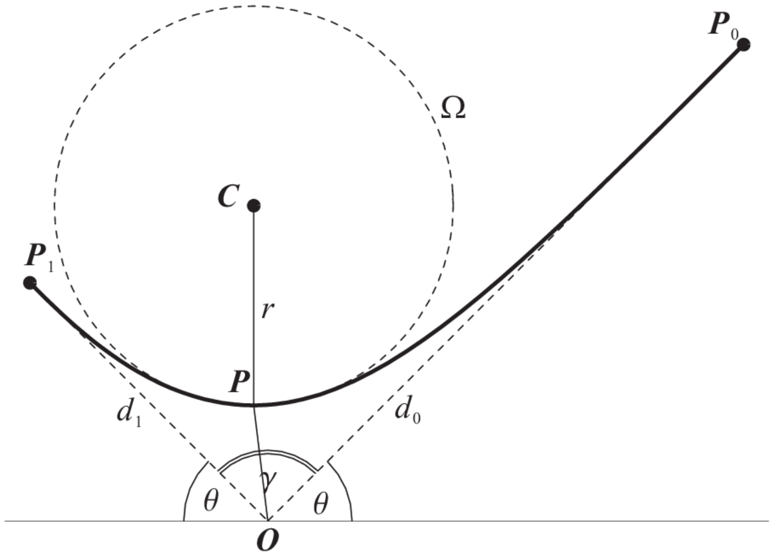


Figure 3: Transition between two straight lines.

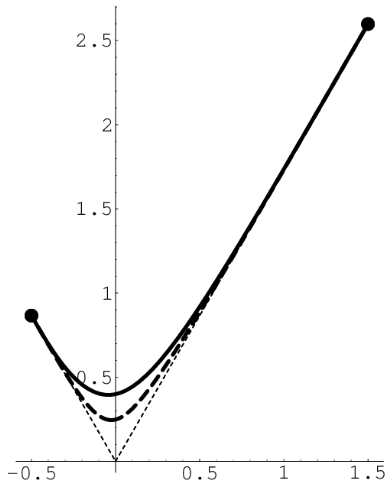


Figure 4: Family of U-shape spiral transitions.

§3 State-of-the-art

Walton and Meek [74, 82] introduced straight line to circle spiral transition. It is generated by a planar cubic Bézier function. A spiral transition generated by the cubic Bézier function is a suitable alternative to the clothoid for many practical applications such as path planning of high speed robots and designing of rail/road. Unlike the clothoid, it is a special case of NURBS (Non-Uniform Rational B-Spline) [24, 68].

Although cubic Bézier function is polynomial and more flexible than clothoid, it has some features which become inconvenient during the curvature analysis and offset of curve, like the following [75].

- Integrand of the arc-length of cubic polynomial has the square root.
- Offset of a cubic polynomial is not a polynomial.
- Offset of a cubic polynomial is not a rational algebraic function of its parameter.

Farouki [21] introduced PH curves. These are special parametric polynomials, do not suffer from the above mentioned undesirable features. The PH quintic spiral described by Walton and Meek [75] is helpful in curve designing because it is polynomial of not very high degree. Even more less degree PH quartic is also useful for the spiral curves [94]. Since straight line segments and circular arcs also have NURBS representations [68], therefore J-shape spiral transitions can be designed with these combinations. Such curves have rational offsets. Furthermore, the arc length of any segment of a curve composed of straight lines, circular arcs, and J-shape transitions can be represented in closed form. Proposed analysis is simplified and extended by Habib and Sakai [32] to achieve more degrees of freedom as the shape control parameter and flexible constraints for easy use in practical applications.

§4 Applications, Research and Issues

Before we discuss main applications of spiral transitions, here is the quick overview of some common in figures [27]. Cross-section of vase in Figure 5 is generated by stitching line to circle spiral transitions in the form of C-shape and S-shape transition curves between two circles. A constrained guided curve is represented in Figure 6, which is composed from J-shape and U-shape spiral transitions. In mechanical engineering, gear is the key component. S-shape transitions are used to design tooth of gear in Figure 7. Another vase profile is given in Figure 8, composed of U-shape spiral transition by using unit and chord-length parametrization. Both are rendered in Figure 9 with shaded surfaces after the revolution of curves in Figure 8 about y-axes. Following is the detail of some major applications of J-shape transitions and their extended shapes.

4.1 Highways and Railways

Spiral transition curve has the particular and traditional importance in the designing of rail/road curves. During their designing we need the special care of desired speed limit, maximum size of vehicles, and collision free route avoiding possible obstacles such as shrine, mountain, or disputed land, etc. Length of designed path also needs to be the shortest for the purposes of land, energy, and time saving.

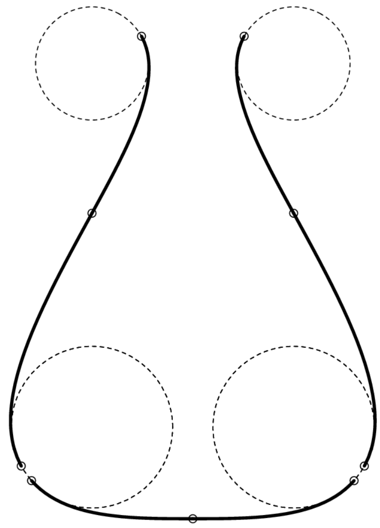


Figure 5: Cross-section of vase.

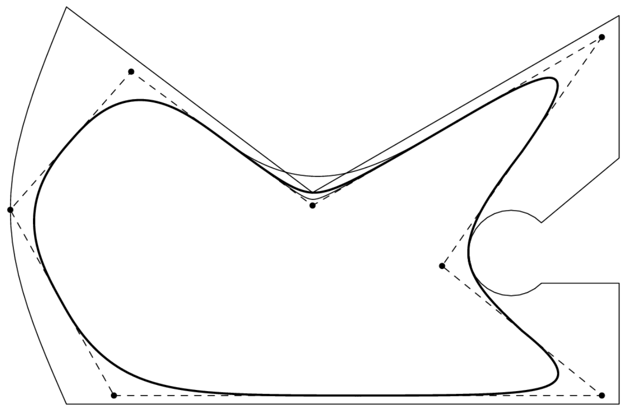
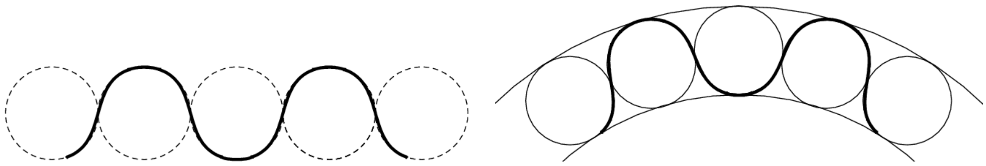


Figure 6: A set of spiral transition curves in closed form.

Standards of geometric design of highways are discussed in detail by Hickerson (1964, p. 17) in American Association of State Highway Officials (AASHO). Hickerson states that "Sudden changes between curves of widely different radii or between long tangents and sharp curves should be avoided by the use of curves of gradually increasing or decreasing radii and, at the same time, introducing an appearance of forced alignment". The importance of this design feature is also highlighted in Gibreel et al. [22] and vehicle accidents are linked to the inconsistency of geometric design of highways (Sarhan and Hassan, 2007).

The clothoid or Cornu spiral (non-polynomial) has been used in highway design for many years [5, 42, 61, 79] due to its good properties particularly the linear relationship between the



(a) Rack (b) Pinion

Figure 7: Design of gear teeth using S-shape spiral transition curves [39].

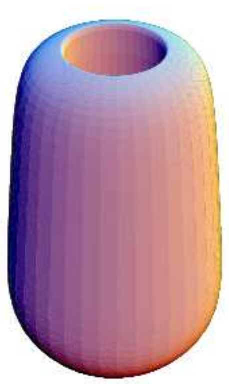
(a) Parametrized by unit-length (b) Parametrized by chord-length

Figure 8: Profile of vase using G^2 cubic Bézier spiral transition curves.

arc-length and curvature. Baass [5] identified five types of transition curves in highway and railway design, namely, straight line to circle (J-shape), circle to circle (C-shape) with a broken back, circle to circle (C-shape) where one circle lies inside the other with a spiral transition, circle to circle (S-shape), and straight line to straight line (U-shape).

However, as discussed in Section 2, the clothoid is unfortunately neither a polynomial nor a rational curve. It is thus inconvenient to incorporate it into existing CAD systems. Parametric polynomials are usually used in CAD/CAM and CAGD (Computer Aided Geometric Design) applications because of their geometric and numerical properties as documented in the literature [18, 19, 31, 33, 34, 70, 76, 77]. Cubic Bézier spiral transition is initially considered as the suitable alternate to the clothoid. However, its PH form does not have the required properties.

Formerly, the fair transition curves had been formed by using two segments, for example, two cubic spiral segments [23], or two PH quintic spiral segments [54, 80]. The use of cubic polynomial is computationally less expensive than PH quintic due to lower degree, therefore



(a) Parametrized by unit-length

(b) Parametrized by chord-length

Figure 9: 3D shaded rendition modal of profiles in Figure 8.

sometimes preferred. Goodman and Meek [23] have discussed S- and C-shape cases by using a pair of cubic Bézier spiral segments to design a fair curve between two fixed end points matching G^2 Hermite conditions. Use of a single segment rather than two has the benefit that designer have fewer entities to be concerned in designing. S- and C-shape cases are also discussed recently by Habib and Sakai [36]. They used a single cubic segment but the transition curve is between two fixed circles instead of two end points, i.e., it does not match G^2 Hermite conditions exactly.

A special case of C-shape transition between two circles is when one circle is completely inside the other circle. Dietz et al. [15] have used a single rational cubic function and it is matching G^2 Hermite conditions. According to them, "Proper care must be taken to use an appropriate measure for optimization and to apply appropriate constraints so that the algorithm is not too susceptible to numerical errors arising from various typical curvature behaviors."

J- and U-shape spiral transitions discussed in Habib and Sakai [41] are also not matching G^2 Hermite conditions exactly. Further, all of the above mentioned schemes are not flexible enough for the use of all types of highway designing due to the boundary conditions on parameters or restricted distance between the centers of given circles.

Finally, all we need is to develop an energy efficient, flexible, and obstacle avoiding fair path planning algorithms, preserving the Hermite conditions, suitable for maximum applications, with minimum possible number of curvature extrema, and with the provision of shape control for easy re-designing and optimization of highways and railways, raising a lot of research issues still need to be solved. Highway or railway re-designing is another important need of developing countries where international standards [8] are usually not observed in road/rail designing. This type of reverse engineering process has not been done before as per our information. Further discussion on reverse engineering of images is included in Section 4.3.

4.2 Robotics

Path planning in robotics is a fundamental capability that a robot must have in order to operate smoothly either it is the movement of arm of humanoid robot [11], or movement of a non-holonomic (car-like) mobile robot in a controlled environment [2, 89], or in an open environment on the ground [17], or under water [3,92] or in space (3D) [1,6]. Path planning with obstacle avoiding in a cluttered environment is critical in many robotic applications particularly unmanned aerial vehicle (UAV) and under water robots. It is the process of finding the shortest, smoothest, and collision free start-to-goal path. Spiral transitions using cubic polynomial in Bézier form is the base of many approaches of smooth path planning. This is an advanced idea which can successfully replace the conventional and existing methods [88, 89]. Conventional methods of path planning for robots are using straight line trajectories, circular arcs or ordinary curves [48]. These methods are either not compatible with today's CAD/CAM softwares due to the use of transcendental functions or may have unwanted singularities and curvature extrema causing jerks in motion or having unnecessary long travelling or consumption of extra time and energy due to slow speed on sharp corners [55, 58, 64]. A lot of research works have addressed collision free and fair path planning for non-holonomic mobile systems in general and car-like robots in particular. Non-holonomic systems are subject to kinematic constraints that restrict their permissible directions of motion. Readers are referred to [45, 88, 89] for the extensive review on this topic.

Similar to highway designing requirement, high speed car-like robots also need the curvature-constrained path planning with given end points, tangent directions at end points, and curvatures at end points. These G^2 Hermite conditions model a trajectory for robots. For example, a car-like robot has a fixed maximum turning radius while the front-wheel steering constrained to move in the direction that the rear wheels are pointing. Therefore the car must follow a curvature-constrained path. Further, due to the spiral transitions, steering of a car needs to be rotated in one direction only while driving along the curve. Path planning of a robot involves the computation of a smooth obstacles avoiding path, possibly optimized with the travelling time or path length [9, 51, 52]. This path planning is divided into local and global constraints. Usually car-like robots come with physical limitations, such as bounds on the curvature, velocity or acceleration. These differential constraints restrict the design of robot path [49,50,65,66]. The path planning problem for UAVs is more challenging since these space vehicles have more complicated dynamics and therefore required very efficient algorithms for fast and real time navigation in 3D space [88].

Main objective of robot path planning is to first find an optimal path satisfying both local (differential) and global (obstacles) constraints and then to smooth (fair) the path as per desire speed of robot. Tradeoff between efficiency and accuracy is another concern again depending on the nature of applications, i.e., the speed of robot, static or moving obstacles, etc. General requirements of path planning and goals are as follows:

- Curve fairing i.e., no curvature extrema, unwanted inflection points and singularities to make movement jerk free.

- Shortest path length to get energy and time efficiency.
- Obstacle avoidance to keep path collision free.
- Matching Hermite end conditions to ensure desired requirement of planar or space curves [86].

Most of the above mentioned objectives can be accomplished by using G^1 or G^2 Hermite cubic Bézier spiral transitions between two straight lines (U-shape) [41, 89]. Usually G^1 Hermite end conditions are considered enough for slow moving robots, however G^2 Hermite conditions are recommended for the high speed robots.

4.3 Computer Imaging

Computer imaging is a rapidly progressive area of computer science covering both digital image processing and computer vision. Recently, computer imaging has also proven to be essential in engineering disciplines, biological sciences, and even in arts programs. Image interpolation is widely used in the presentation process and analysis of digital images for scaling, rotation, self registration, and image re-sampling such as zooming, and warping. Usually C^1 or C^2 continuous spline is used for image interpolation [67], but it may have unwanted singularities. However, if spline segments are replaced with spiral transitions, it can better interpolate images with no risk of loops or cusps.

4.3.1 Approximation of discrete data

Discrete data is obtained from many experiments and required to be approximated (or interpolated) as the pre-process in industrial applications. Since images or video frames are also represented in discrete form and therefore often need to be approximated during image enhancement, restoration, zooming, or to represent accurately the structural changes occurred in image during its illegal tempering (forgery) [4] by detecting intrinsic imbalance through curvature analysis and spline smoothing.

Initially, discrete data was approximated by Meek and Walton [62, 91] by arc-splines made of G^1 curves between straight line segments and circular arcs, i.e., J-shape transitions. Recently, Yun [93] proposed a cumulative averaging method to smoothen the piecewise polynomial interpolation for a given discrete data.

4.3.2 Reverse engineering of images

Reverse engineering of images and fair path planning are two main research and development areas, and very closely related with each other. For example, an optimized re-designing of highways and railways for the maximum possible speed limit along their curves, discussed in Section 4.1, can be planned with pattern recognition of images and spiral transitions of curves.

Reverse engineering of images is a method to create a virtual model of an existing physical part for use in CAD, CAM, and CAE applications. The process of reverse engineering involves measuring an object and then reconstructing it into 2D or 3D model as per need. During the process of reverse engineering, first real objects are digitized through 2D or 3D camera, laser scanner, structured light digitizer, or computed tomography. The measured data is compressed, processed and modelled into required and more useful model. These soft models are reconstructed or redesigned and then transformed into the hard models through 3D printers or conventional machines. At the base of above mentioned methods, preprocessing of digital data plays an important role which includes noise removal, object location, binary segmentation, vectorization, evaluation, and validation.

Various techniques of object extraction from simple planar images have been proposed by different researchers [59, 63]. However their methods are not suitable for the extraction of region of interests (ROI) from aerial images due to the presence of different types of noise like buildings, trees, water, deserts, or un-wanted objects similar in shape with ROI.

It is often required to develop an efficient method and framework for the extraction of ROI from RGB bands of satellite images or aerial color images of high resolution. Thanks to the advances in image processing and pattern recognition techniques, numerous methods have been developed to extract ROI from satellite or aerial images [7,26,69,87]. An image file is generally used to store visual information of various objects. For human observer, an image file proves to be very useful and informative but for any machine it is merely raw data unless some useful information is extracted efficiently and effectively via some algorithms from the image and uses that information in an optimized way. Extracted ROIs are usually converted into vectors. For re-designing of models, these vectors are then used as the input in spline or spiral transition functions.

§5 Research Challenges

Line to circle or J-shape spiral transition is the core geometric tool of many path planning applications and the base for other important transition shapes. Therefore, major research challenges are connected with J-shape spiral transition and are discussed in the following sub sections.

5.1 Hermite end condition and shape control

As discussed above, J- and C-shape transitions are not matching G^2 Hermite conditions exactly. The curve designer is not free to specify the points where the transition curve meets the circles. Further, some of the above mentioned shapes are not flexible enough for the use of all types of design applications due to the boundary conditions on shape control parameters or restricted distance between the centers of given circles. This is opening many research directions specific to applications [27, 85].

5.2 Arc-length Parametrization

Unit and chord-length parametrizations are common and can be easily applied in path planning and surfaces. However, arc-length parametrization is also required in many important applications. For example, it is difficult to regulate the speed and motion control of moving objects in virtual or real environments, like computer animation, simulations, and robotics because the parameter variable and curve length are not, in general, linearly related [83]. Motion of object is easily and accurately controlled if its trajectory is parametrized by arc-length.

Arc-length parametrization is also required in shape recovery of smooth curves or surfaces using the Morphosense. Morphosense is a flexible ribbonlike device, recently developed by the Leti1 department of the CEA2 in Grenoble, France. It incorporates orientation sensors at known distances along its length [14]. Shape reconstruction by morphosense is not just a classical Hermite or spiral transition problem, because the spatial locations of the sensor knots are unknown. Since, only the curvilinear distance or chord-length between successive sensors are known, the shape approximation problem amounts to constructing a spline or transition curve that interpolates a given sequence of way points, with prescribed arc-lengths for each segment between such way points [20].

Other potential applications are also possible in which precise control over path arc-length may be desired. For example, the layout of carbon-fiber in layered manufacturing processes, or in the manufacturing, sintering, deposition, or curing of composite materials.

Arc-length parametrization of polynomial parametric curves is difficult to compute exactly. It is due to the dependency on way-points, tangents, curvatures, etc, or the interpolation of discrete data, or because of the integral form of total arc-length. Therefore, usually arc-length parametrization has been satisfied approximately through iterative process or numerical methods [46, 85]. To make the curvature based parametric spiral transition curves more useful and accurate in design applications or computer numerical control (CNC) cutting machine system, efficient algorithms are required, which are interesting future research directions.

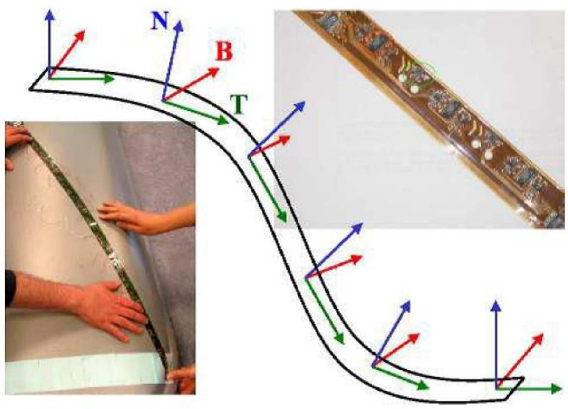


Figure 10: Morphosense ribbon application for shape reconstruction of discrete geometric data [46].

5.2.1 3D Path Planning

Autonomous navigation in known or unknown environments cluttered with static or moving obstacles is a challenging problem since both the obstacles and the robots differential constraints have to be taken into account [90]. It becomes more complicated and difficult when object moves in space, like unmanned aerial vehicle UAV navigation [88], since UAVs have fast and complicated dynamics of real time navigation in space. For an ideal path planning in 3D, there are many important considerations including completeness, robustness, optimality, and computational simplicity. Off course a natural trade off needs to be kept between these requirements [12].

Usually, a path finding algorithm generates a list of collision free way points. Usually unwanted way points are removed through pruning. It generates a piecewise linear path avoiding obstacles. Then J-shape cubic Bézier spiral transitions are used to smooth the path guaranteeing the desired Hermite end conditions, see Figure 11. Since way points are in 3D, the proposed method of Yang et al. [90] is based on the mapping of 3D way points to 2D by using homogeneous coordinate transformation. Then data for 2D fair path is generated and finally this data is mapped back to 3D. Process is somehow completed but the accuracy is compromised due to the reverse mapping of 2D into 3D. Better approach is to solve the problem in 3D space directly. However, this task is complicated and not easy to find the analytical solution.

Reducing the computational complexity is another very important requirement particularly for UAV applications. The UAV has to redesign the path in real time to avoid moving obstacles. In this situation, the computational time of path finding algorithms grows exponentially with the dimension of the configuration space, and hence robust algorithms need to be developed to provide an optimal and robust solution for real time UAV path planning in an unknown natural environment cluttered with obstacles.

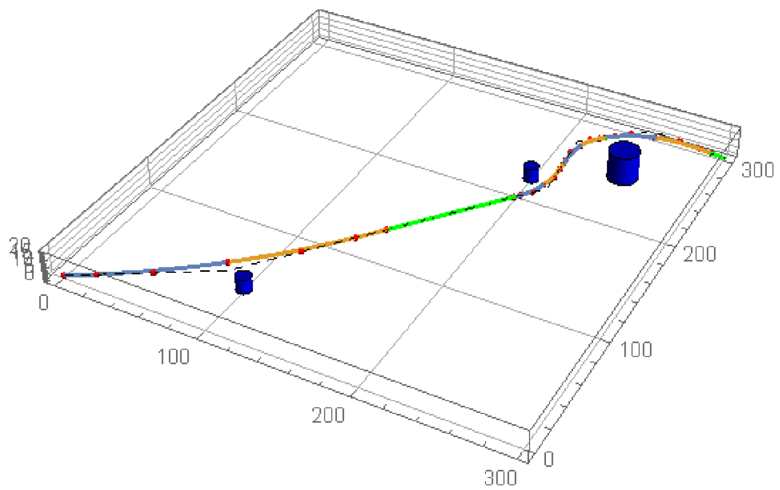


Figure 11: Continuous curvature path smoothing using J-shape spiral transitions in space.

§6 Conclusion

The main advantage of cubic Bézier polynomial is the additional degrees of freedom, i.e., a total of seven versus five for a clothoid segment. So the clothoid is less flexible in matching Hermite end conditions of the transition curve. For example, when J-shape transition is required from the fixed beginning point on the straight line and a fixed ending point on the circle with the flexibility of adjusting the circle while keeping the ending point fixed. It is assumed that the change in tangent vector angle will not exceed 90 degree. On same pattern, transition curve through clothoid is proved unique in [82], hence it does not have enough flexibility for adjusting the curvature and tangent vector angle at the ending point, whereas it might be possible through cubic Bézier polynomial. Another example is the C- or S-shape transitions between two circles. The resulting number of degrees of freedom is 10 minus the number of continuity conditions at their joint, which is not sufficient to fix the points, tangent directions, and curvatures at the endpoints of the transition curve. B-spline and its variation NURBS are also very popular spline tools. However, they are complicated due to the involvement of knot vectors and the prescribed upper limit of curvature. Further, offset of a B-spline is a non-rational curve, as discussed in [44, 53]. However, cubic Bézier polynomial being the special case of B-spline is easy to fair and control the desired path.

Another significant benefit of using cubic Bézier spiral transitions is computational cost. The cubic Bézier spiral is a polynomial of lower degree, whereas approximation through Fresnel integrals is required when drawing spiral segments through clothoid function. Further, as mentioned above, the cubic Bézier spiral segment is polynomial so it is easily cast as a NURBS which can be conveniently incorporated when using popular CAD/CAM packages for curve or surface design, whereas the clothoid spiral is nonrational and incompatible with today's CAD/CAM softwares.

Most of the research issues are related with the minimization of number of curvature extrema in a transition curve. Descartes rule of sign ([43], pp. 439-443) can be used for the solution of such kind of problems as it already proved to be very useful for the partial solution of some other problems discussed in [27]. We may also need the use of Sylvester's resultant [84] which is another powerful tool for the solution of above mentioned complicated research problems.

Derivation of J- and C-shape transitions matching G^2 Hermite conditions is also a challenging task. U-shape transition is usually composed with the combination of two J-shape transitions, one from straight line to circle and other from circle to straight line. The use of a pair of cubic Bézier segments to design C- and S-shape transitions with given G^2 Hermite data was presented by Walton et al. [82]. Since each Bézier segment is a transition between straight line and circle, their scheme can be modified for J- and U-shape transitions.

Line to circle spiral transition is useful in many applications due to its extension in U-shape and many types of C-shape and S-shape transitions, as discussed above. However few specific types of C-shape and S-shape are not covered, e.g., when distance between two circles is less than or equal to the sum of radii of both circles and transition is required with no curvature extremum. These cases are sometimes used in rail/road designing, satellite path planning, or in the making of cam cross-section, etc. [15, 27, 34, 36, 37, 70, 78].

New multimedia technologies are based on digital and being presented in discrete form, like images and video frames. J-shape transition has great potential to approximate or interpolate discrete data [62, 93] as discussed in Section sDiscreteData.

§7 Future research directions

There are several interesting research directions in the domain of spiral transitions. Many research issues are highlighted during the discussion on applications of spiral transitions in Section 4 followed by research challenges in Section 5, and conclusion in previous section. Above discussion is continued here with the highlights on important research directions from the latest published research work in [20, 85].

Wu and Yang [85] proposed techniques of interpolation of intrinsically defined planar curves from G^1 and G^2 Hermite data. Since the function used is non Bézier, depriving from many attractive properties, one can use Bézier polynomial to construct G^2 spiral transition curves with arc length constraint to make the curvature based transition curves more useful in many practical applications.

The PH form of a quintic curve has some useful properties: a) Arc length of a PH curve is a polynomial of its parameter, b) Offset of PH curve is rational. Further, a quintic is the lowest degree PH curve that may have an inflection point [27, 33, 34, 80]. Although PH curves are more complicated than their simple counter parts, they are providing more flexible conditions, i.e., more wider range of free shape control parameters. Arc-length parametrization, smoothing of arc spline, and designing of machine gears can also be tried with the PH quintic transition curves [16].

Recently, Farouki [20] addressed the problem of constructing a planar polynomial curve with given G^1 Hermite end conditions and a specified arc length by exploiting the complex PH quintic curve representation. Proposed method avoids the need of any type of iterative numerical process [56, 57]. The present results of arc-length parametrization may possibly be extended but not limited to: (i) interpolation of local data by imposing G^2 Hermite end conditions; (ii) constructing a J-shape spiral transition; and (iii) interpolation of spatial data.

The proposed technique for spur gear design in [39] can be extended to many types of gear typologies, e.g., bevel, helical, hypoid, and crown, etc. Future work is required for the development of geometric design of machine gears for the purpose of minimum possible friction among gear teeth.

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