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The differentiability of solutions for elliptic equations which degenerate on part of the boundary of a convex domain

SONG Jia-xin CAO Yi*

Abstract. In this paper, we study the differentiability of solutions on the boundary for equations of type $L_\lambda u = \frac{\partial^2 u}{\partial x^2} + |x|^{2\lambda} \frac{\partial^2 u}{\partial y^2} = f(x, y)$, where λ is an arbitrary positive number. By introducing a proper metric that is related to the elliptic operator L_λ, we prove the differentiability on the boundary when some well-posed boundary conditions are satisfied. The main difficulty is the construction of new barrier functions in this article.

1 Introduction

It is difficult while important to understand how the geometry of the domain affect solutions of the partial differential equation. Many smoothness results for solutions of the following elliptic equation (1.1) on a convex domain $\Omega' \subset \mathbb{R}^n$ have been obtained.

$$\begin{cases} -a_{ij}(x)\frac{\partial^2 u(x)}{\partial x_i \partial x_j} = f(x) & x \in \Omega', \\ u(x) = g(x) & x \in \partial\Omega'. \end{cases} \tag{1.1}$$

When $g(x) \equiv 0$, Li and Wang [8] have showed that the solution u is differentiable on the boundary through constructing barrier functions. In their later article [9] they generalize the result to the case of nonhomogeneous Dirichlet boundary conditions, that is, $g(x) \not\equiv 0$ when some well-posed boundary conditions are satisfied. In [12] Wang has showed that the solution is of $C^{1,\alpha}$ along the boundary when the boundary of the domain is of $C^{1,\alpha}$, where the convexity of the domain is not needed.

In this paper, we will study the differentiability of the solutions for the following equation

$$\begin{cases} \text{(i) } L_\lambda u(x,y) = \frac{\partial^2 u}{\partial x^2} + |x|^{2\lambda}\frac{\partial^2 u}{\partial y^2} = f(x,y) & (x,y) \in \Omega, \\ \text{(ii) } u(x,y) = g(x,y) & (x,y) \in \partial\Omega, \end{cases} \tag{1.2}$$

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* Corresponding author.

where $\Omega = \{(x, y)|0 \le x \le a, |y| \le b\}$ is a rectangular area.

The study of (1.2(i)) has a long story. When λ is a positive integer, L_λ belongs to a class of operators that are called sum of squares of vector fields. For the case of $\lambda = \frac{1}{2}$, (1.2(i)) is the transonic flow problem:

$$u_{xx} + xu_{yy} = f(x, y)$$

on the elliptic side ($x > 0$), where the flow is subsonic.

The trait of (1.2(i)) is that it has singularities along the y-axis, it is degenerate when x goes to 0. The elliptic partial differential equations of degenerate type has been studied in many papers, such as [1,7,10,11]. In [11], Wang obtained the Hölder estimates for equation $L_\lambda u = f(x, y)$ at the degenerate line by applying the compactness method which is given by Hömander in [4]. In [10] Song and Wang studied the Keldysh's type equation $Lu(x, y) := xa(x, y)u_{xx} + u_{yy} - b(x, y)u_x = f(x, y)$, which has been widely researched in [1,5,6,7], and concluded the Hölder estimates at the degenerate line by constructing the barrier functions which have been widely used in partial differential equations. Especially, Daskalopoulos and Hamilton used it to show the regularity of the degenerate equations (see [2]).

We mainly adopt the idea of Li and Wang [9], but there are some difference since the equation (1.2) is degenerate when x tends to 0. In order to obtain the differentiability, we need to construct new barrier functions because the barrier functions in [9] can not be applied near the degenerate line $\Gamma = \{(0, y) : -b < y < b\}$ and the corner $(0, \pm b)$. This is the main problem we will solve in this paper.

The plan of this paper is as follows. We state our results in section 3 after introducing some preliminaries in section 2. Constructing two barrier functions, we prove the main results by using the Hölder estimate and the Harnack inequality of elliptic equations.

2 Preliminaries

The basic feature of our equation (1.2) is the following scaling structure. If $L_\lambda u = f$ and $v(x, y) = u(rx, r^{1+\lambda}y)$, then

$$L_\lambda v = r^2 f(rx, r^{1+\lambda}y). \tag{2.1}$$

This provides the basic scaling near the singularity and leads us to consider the following intrinsic cubes.

Definition 2.1. If $X = (x, y)$ and $r > 0$, we define $rX = (rx, r^{1+\lambda}y)$. We also define cubes with size r as

$$Q_r = [0, r] \times [-r^{1+\lambda}, r^{1+\lambda}].$$
$$Q_r(X) = X + Q_r.$$

Definition 2.2. Set $X = (x_1, y_1), Y = (x_2, y_2)$, we define an intrinsic distance $d(X, Y)$ as

$$d(X, Y) = |x_1 - x_2| + \frac{|y_1 - y_2|}{|x_1|^\lambda + |x_2|^\lambda + |y_1 - y_2|^{\frac{\lambda}{1+\lambda}}}.$$

The advantage of using $d(X,Y)$ is that it is explicit and it satisfies the scaling property

$$d(rX,rY)=rd(X,Y).$$

In this paper, we assume that $g(x,y)\in C(\partial\Omega)$, $f\in C(\overline{\Omega})$, and then it is convenient for us to use the conception of viscosity solutions. For simplicity, in this paper, the term solutions always indicate viscosity solutions.

3 Differentiability

The main result is

Theorem 3.1. *Suppose that u is a solution of (1.2), g is differentiable at $X\in\partial\Omega$, that is, there exists a function $\sigma:\mathbb{R}_+\to\mathbb{R}_+$ such that $\sigma(r)=o(1)$ and that $|g(Y)-g(X)-Dg(X)(Y-X)|\leq \frac{3r}{2}\sigma(\frac{3r}{2})$ for any $Y\in\partial\Omega\bigcap Q_r(X)$, and $\int_0^{\frac{3}{2}}\frac{\sigma(r)}{r}dr<\infty$, then u is differentiable at X.*

Since (1.2(i)) is elliptic away from Γ and the corners $(0,\pm b)$, then the differentiability of u on those parts can be proved by the method of [9], so we mainly prove the differentiability on Γ and at the corners.

3.1. The differentiability on the boundary Γ

For any $X\in\Gamma$, by shifting the coordinate system, we assume that X is the origin (0,0). It is enough for us to prove the following theorem.

Theorem 3.2. *Assume that*

(i) $\Omega\cap Q_1=Q_1$;

(ii) $u(x,y)\geq 0$ for $(x,y)\in Q_1$ with $\|u\|_{L^\infty(Q_1)}\leq 1$;

(iii) $f(x,y)\geq 0$ for $(x,y)\in Q_1$ such that $\|f\|_{L^\infty(Q_1)}\leq 1$ and $\int_0^1\frac{\|f\|_{L^\infty(Q_t)}}{t}dt\leq 1$; and

(iv) $0\leq g(x,y)\leq r\sigma(r)$ for $(x,y)\in\partial\Omega$ with $|(x,y)|\leq r$, where σ: $[0,1]\to\mathbb{R}_+$ satisfies $\int_0^1\frac{\sigma(r)}{r}dr\leq 1$.

Then the solution u of (1.2) is differentiable at the origin.

It is clear that u is differentiable in $y-$ direction at the origin since $u(0,y)=g(y)$ and g is differentiable on $\Gamma=\{(0,y):-b<y<b\}$ (see the condition of Theorem 3.1). We will establish Theorem 3.2 by iteration method which is based on the following Lemmas 3.2-3.4. In the following Lemma 3.1, we introduce two barrier functions, which are important for our proof.

Lemma 3.1. *Suppose $0<\delta<1$ and $M\geq 2\sqrt{2}+1$. There exist two second differentiable functions $\Psi_{\delta,M\delta^{\lambda+1}}$ and $\psi_{\delta,M\delta^{\lambda+1}}$ defined on $[0,\delta]\times[-M\delta^{\lambda+1},M\delta^{\lambda+1}]$ which satisfy*

$$\begin{cases} (i)\Psi_{\delta,M\delta^{\lambda+1}} > 1 & if\ x=\delta, |y| \le M\delta^{\lambda+1} \\ (ii)\Psi_{\delta,M\delta^{\lambda+1}} \ge 0 & on\ [0,\delta]\times[-M\delta^{\lambda+1}, M\delta^{\lambda+1}] \\ (iii)\Psi_{\delta,M\delta^{\lambda+1}} \ge 4 & if\ |y| = M\delta^{\lambda+1}, 0\le x\le \delta \\ (iv)\Psi_{\delta,M\delta^{\lambda+1}} \le \frac{5x}{2\delta} & on\ [0,\delta]\times[-\delta^{\lambda+1}, \delta^{\lambda+1}] \\ (v)L_\lambda \Psi \le -1 & in\ (0,\delta)\times(-M\delta^{\lambda+1}, M\delta^{\lambda+1}) \end{cases} \tag{3.1}$$

and

$$\begin{cases} (i)\psi_{\delta,M\delta^{\lambda+1}} < 1 & if\ x=\delta, |y| \le M\delta^{\lambda+1} \\ (ii)\psi_{\delta,M\delta^{\lambda+1}} \le \frac{x}{\delta} & on\ [0,\delta]\times[-M\delta^{\lambda+1}, M\delta^{\lambda+1}] \\ (iii)\psi_{\delta,M\delta^{\lambda+1}} \le 0 & if\ |y| = M\delta^{\lambda+1}, 0\le x\le \delta \\ (iv)\psi_{\delta,M\delta^{\lambda+1}} \ge \frac{x}{4\delta} & on\ [0,\delta]\times[\delta^{\lambda+1}, \delta^{\lambda+1}] \\ (v)L_\lambda \psi \ge 1 & in\ (0,\delta)\times(-M\delta^{\lambda+1}, M\delta^{\lambda+1}) \end{cases} \tag{3.2}$$

respectively.

Proof. Define

$$\Psi_{\delta,M\delta^{\lambda+1}} = \frac{5x}{2\delta} - (\frac{x}{\delta})^2 - \frac{x^2}{2} + \frac{1}{2}((|\frac{y}{\delta^{\lambda+1}}| - 1)^+)^{2+\epsilon}$$

and

$$\psi_{\delta,M\delta^{\lambda+1}} = \frac{1}{4}(\frac{x}{\delta} + (\frac{x}{\delta})^2) + \frac{x^2}{2} - \frac{1}{8}((|\frac{y}{\delta^{\lambda+1}}| - 1)^+)^{2+\epsilon},$$

where $\epsilon > 0$ such that

$$4 - (2+\epsilon)(1+\epsilon)(M-1)^\epsilon \ge 0. \tag{3.3}$$

Clearly, $\Psi_{\delta,M\delta^{\lambda+1}}$ and $\psi_{\delta,M\delta^{\lambda+1}}$ are second differentiable and satisfy (3.1) and (3.2) respectively. □

Lemma 3.2. [3] *Suppose Ω is a bounded domain, and $L_\lambda u \ge 0$ in Ω, $u \le 0$ on $\partial\Omega$, then $u \le 0$ in Ω.*

Lemma 3.3. *There exist positive constants $\delta(<1)$, $\mu(<1)$, N, depending only on λ. If*

$$kx - b \le u(x,y) \le Kx + B \qquad in\ Q_1, \tag{3.4}$$

for some nonnegative constants b, B, k, K, where $B, b \ge \|f\|_{L^\infty(Q_1)}$, then there exist nonnegative constants $\widetilde{k}$ and $\widetilde{K}$, such that

$$\widetilde{k}x - \|f\|_{L^\infty(Q_1)} \le u(x,y) \le \widetilde{K}x + \sigma(1) + \|f\|_{L^\infty(Q_1)} \qquad in\ Q_\delta. \tag{3.5}$$

where either

$$\widetilde{k} = (k - Nb + \mu(K-k))^+, \quad and \quad \widetilde{K} = K + NB; \tag{3.6}$$

or

$$\widetilde{k} = (k - Nb)^+, \quad and \quad \widetilde{K} = K + NB - \mu(K-k). \tag{3.7}$$

Proof. We prove the following claim first.

Claim. There exist positive constant N, δ_1,depending only on λ such that

$$(k - Nb)x \le u(x,y) \le (K+NB)x + \sigma(1) \qquad \text{in } Q_{\delta_1}. \tag{3.8}$$

Proof. Let $\delta_1 = \frac{1}{(2\sqrt{2}+1)^{\frac{1}{\lambda+1}}}$, $(\frac{1}{2\sqrt{2}+1} < \delta_1 < 1)$, set $\Psi = \Psi_{\delta_1,1}$ be defined by Lemma 3.1, we claim that

$$u(x,y)-Kx-B\Psi\le\sigma(1)\qquad\text{on }\partial([0,\delta_1]\times[-1,1]).\tag{3.9}$$

In fact, $\partial([0,\delta_1]\times[-1,1])$ can be separated into three parts: (i) $x=\delta_1,\ |y|\le1$, (ii) $0\le x\le\delta_1,\ |y|=1$, and (iii) $x=0,\ |y|\le1$. On part of (i) and (ii), since $\Psi>1$, we have $u(x,y)-Kx-B\Psi<u(x,y)-Kx-B\le0$ by (3.4). On the part of (iii), since $|(x,y)|\le1$, we have $u(x,y)=g(x,y)\le\sigma(1)$, and then by $Kx+B\Psi\ge0$, it is clear that $u(x,y)-Kx-B\Psi\le\sigma(1)$.

In view of (1.2(i)), (3.1(v)) and $B\ge\|f\|_{L^\infty(Q_1)}$, we have

$$L_\lambda(u(x,y)-Kx-B\Psi)\ge0\qquad\text{in }([0,\delta_1]\times[-1,1]).\tag{3.10}$$

According to Lemma 3.2, combining with (3.9) and (3.10), we obtain

$$u(x,y)-Kx-B\Psi\le\sigma(1)\qquad\text{on }[0,\delta_1]\times[-1,1].$$

Therefore by (3.1(iv)) and setting $N=\frac{5}{2\delta_1}$, we obtain the right hand inequality of (3.8). By the same arguments we obtain

$$\begin{cases}L_\lambda(kx-b\Psi-u(x,y))\ge0 & \text{in }[0,\delta_1]\times[-1,1],\\ kx-b\Psi-u(x,y)\le0 & \text{on }\partial([0,\delta_1]\times[-1,1]),\end{cases}$$

then we can obtain

$$u(x,y)\ge(k-Nb)x\qquad\text{in }Q_{\delta_1},$$

we obtain (3.8). □

Let$\delta=\frac{\delta_1}{[2(2\sqrt2+1)]^{\frac{1}{\lambda+1}}}$, $(\frac{1}{2(2\sqrt2+1)^2}<\delta<\delta_1<1)$, and $\Upsilon=\{(x,y)|x=\delta,|y|\le\frac{\delta_1^{\lambda+1}}{2}\}$. Next we will show (3.5) according to two cases:

(i) $u(\delta,0)\ge\frac12(K+k)\delta$,

(ii) $u(\delta,0)\le\frac12(K+k)\delta$.

corresponding to which (3.6) and (3.7) will hold respectively.

Case(i): $u(\delta,0)\ge\frac12(K+k)\delta$. Let

$$v(x,y)=u(x,y)-(k-Nb)x,\tag{3.11}$$

then

$$v(\delta,0)\ge(\frac{K-k}{2}+Nb)\delta.\tag{3.12}$$

Set $\gamma=\frac{\delta_1-\delta}{7}$, and $\widetilde{Q}=[\delta-\gamma,\delta+\gamma]\times[-\delta_1^{\lambda+1},\delta_1^{\lambda+1}]$, we can see $\Upsilon\subset\widetilde{Q}\subset Q_{\delta_1}$, by (3.8) and (3.11) we obtained

$$\begin{cases}L_\lambda v=f & \text{in }\widetilde{Q},\\ v\ge0 & \text{in }\widetilde{Q}.\end{cases}.$$

By the Harnack inequality,

$$\sup_\Upsilon v\le C_1(\inf_\Upsilon v+\|f\|_{L^\infty(Q_1)}),\tag{3.13}$$

where C_1 is a constant depending only on λ. Combining (3.12), (3.13) and $v(x,y)\ge0$, we have

$$\inf_\Upsilon v\ge\{\frac{1}{C_1}(\frac{K-k}{2}+Nb)\delta-\|f\|_{L^\infty(Q_1)}\}^+=:a.$$

Let $\psi=\psi_{\delta,(2\sqrt2+1)\delta^{\lambda+1}}$ be defined by Lemma 3.2,

$$w=\tfrac12\|f\|_{L^\infty(Q_1)}(x-x^2)\quad\text{in }[0,\delta]\times[-(2\sqrt2+1)\delta^{\lambda+1},(2\sqrt2+1)\delta^{\lambda+1}],$$

we claim that

$$\begin{cases} L_\lambda(a\psi - w - v) \geq 0 & \text{in } [0,\delta] \times [-(2\sqrt{2}+1)\delta^{\lambda+1}, (2\sqrt{2}+1)\delta^{\lambda+1}], \\ a\psi - w - v \leq 0 & \text{on } \partial([0,\delta] \times [-(2\sqrt{2}+1)\delta^{\lambda+1}, (2\sqrt{2}+1)\delta^{\lambda+1}]). \end{cases}$$

Indeed, the first inequality is clear since $L_\lambda\psi \geq 1$ and $L_\lambda v = f(x,y)$. For the second inequality, we separate the boundary into three parts: (i) $x = \delta, |y| \leq (2\sqrt{2}+1)\delta^{\lambda+1}$, (ii) $x = 0, |y| \leq (2\sqrt{2}+1)\delta^{\lambda+1}$, (iii) $0 \leq x \leq \delta, |y| = (2\sqrt{2}+1)\delta^{\lambda+1}$. On the part (i), since (3.2(i)) and $w > 0$, we have $a\psi - w - v < 0$; on the parts (ii) and (iii), since $\psi \leq 0,\ w \geq 0$, we have $a\psi - w - v \leq 0$.

By Lemma 3.2, we obtain

$$a\psi - w - v \leq 0 \quad \text{in}[0,\delta] \times [-(2\sqrt{2}+1)\delta^{\lambda+1}, (2\sqrt{2}+1)\delta^{\lambda+1}].$$

Combining with (3.2(iv)), we have

$$\frac{a}{4\delta}x - w - v \leq 0 \quad \text{in } Q_\delta.$$

It follows that

$$\begin{aligned} v(x,y) &\geq \frac{a}{4\delta}x - w \\ &\geq \frac{1}{4\delta}(\frac{1}{C_1}(\frac{K-k}{2} + Nb)\delta - \|f\|_{L^\infty(Q_1)})x + \frac{1}{2}\|f\|_{L^\infty(Q_1)}(x^2 - x) \\ &\geq \frac{K-k}{8C_1}x - \|f\|_{L^\infty(Q_1)}, \end{aligned}$$

i.e.,

$$u(x,y) \geq (K - Nb + \frac{K-k}{8C_1})x - \|f\|_{L^\infty(Q_1)} \qquad \text{in } Q_\delta. \tag{3.14}$$

Let $\mu = \frac{1}{8C_1}$, combining (3.8), (3.14) and $u \geq 0$, we have (3.5) and (3.6) hold.

Case(ii): $u(\delta,0) \leq \frac{1}{2}(K+k)\delta$. The proof is similar to that of Case(i). Let

$$v(x,y) = (K + NB)x + \sigma(1) - u(x,y) \quad \text{in } Q_{\delta_1}, \tag{3.15}$$

then

$$v(\delta,0) \geq (\frac{K-k}{2} + NB)\delta + \sigma(1). \tag{3.16}$$

By the Harnack inequality,

$$\sup_\Upsilon v \leq C_1(\inf_\Upsilon v + \|f\|_{L^\infty(Q_1)}),$$

combining with (3.16) and $v(x,y) \geq 0$, we have

$$\inf_\Upsilon v \geq \{\frac{1}{C_1}[(\frac{K-k}{2} + NB)\delta + \sigma(1)] - \|f\|_{L^\infty(Q_1)}\}^+ =: a.$$

By the method of Case(i), we obtain

$$\frac{a}{4\delta}x - w - v \leq 0 \quad \text{in } Q_\delta.$$

It follows that

$$v(x,y) \geq \frac{K-k}{8C_1}x - \|f\|_{L^\infty(Q_1)},$$

i.e.,

$$u(x,y) \leq (K + NB - \frac{K-k}{8C_1})x + \|f\|_{L^\infty(Q_1)} + \sigma(1), \tag{3.17}$$

combining (3.8),(3.17) and $u \geq 0$, we have (3.5) and (3.7) hold. □

By $0 \le u \le 1$, scaling and Lemma 3.3, we have

Lemma 3.4. *There exist nonnegative sequences* $\{b_m\}_{m=0}^{\infty}, \{B_m\}_{m=0}^{\infty}, \{k_m\}_{m=0}^{\infty}, \{K_m\}_{m=0}^{\infty}$ *with* $b_0 = B_0 = 1, K_0 = k_0 = 0,$ *and for* $m = 0, 1, 2, \ldots,$

$$b_{m+1} = \delta^{2m}\|f\|_{L^\infty(Q_{\delta^m})},$$
$$B_{m+1} = \delta^{2m}\|f\|_{L^\infty(Q_{\delta^m})} + \delta^m\sigma(\delta^m),$$

and either

$$k_{m+1} = (k_m - N\frac{b_m}{\delta^m} + \mu(K_m - k_m))^+ \quad \textit{and} \quad K_{m+1} = K_m + N\frac{B_m}{\delta^m},$$

or

$$k_{m+1} = (k_m - N\frac{b_m}{\delta^m})^+ \quad \textit{and} \quad K_{m+1} = K_m + N\frac{B_m}{\delta^m} - \mu(K_m - k_m),$$

such that

$$k_m x - b_m \le u(x, y) \le K_m x + B_m \quad \textit{in } Q_{\delta^m}, \tag{3.18}$$

for $m = 0, 1, 2, \cdots.$

Proof. We proof (3.18) by induction. As $m = 0$, it is a direct consequence of assumption that $0 \le u(x, y) \le 1$. Assume that (3.18) hold for $m = l$, that is, set $X = (x_1, y_1) \in Q_{\delta^l}$, we have

$$k_l x_1 - b_l \le u(X) \le K_l x_1 + B_l \quad \text{in } Q_{\delta^l}. \tag{3.19}$$

Set $Y = (x_2, y_2), X = \delta^l Y$ and $Q_1 = \frac{Q_{\delta^l}}{\delta^l}$. Define $\widetilde{u}(Y) = \frac{u(\delta^l Y)}{\delta^l}, \widetilde{f}(Y) = \delta^l f(\delta^l Y)$, then by (2.1) we have

$$L_\lambda \widetilde{u} = \widetilde{f}(Y) \quad \text{in } Q_1. \tag{3.20}$$

By (3.19) we have

$$k_l\delta^l x_2 - b_l \le \widetilde{u}(Y)\cdot\delta^l \le K_l\delta^l x_2 + B_l,$$

that is

$$k_l x_2 - \frac{b_l}{\delta^l} \le \widetilde{u}(Y) \le K_l x_2 + \frac{B_l}{\delta^l} \quad \text{in } Q_1. \tag{3.21}$$

We can see that

$$\begin{aligned}\frac{B_l}{\delta^l} &= \frac{\delta^{2(l-1)}\|f\|_{L^\infty(Q_{\delta^{l-1}})} + \delta^{l-1}\sigma(\delta^{l-1})}{\delta^l}\\ &\ge \frac{\delta^l\|f\|_{L^\infty(Q_{\delta^l})}}{\delta^2} = \frac{\|\widetilde{f}\|_{L^\infty(Q_1)}}{\delta^2}\\ &\ge \|\widetilde{f}\|_{L^\infty(Q_1)},\end{aligned} \tag{3.22}$$

by the same arguments we drive that

$$\frac{b_l}{\delta^l} \ge \|\widetilde{f}\|_{L^\infty(Q_1)}. \tag{3.23}$$

Define $\widetilde{g}(Y) = \frac{g(\delta^l Y)}{\delta^l}, Y \in \partial Q_1 \cap \{x = 0\}$, and then

$$\widetilde{u} = \widetilde{g}, \quad \text{on } \partial Q_1 \cap \{x = 0\}. \tag{3.24}$$

From $g(X) \le \delta^l\sigma(\delta^l), X \in \partial Q_{\delta^l} \cap \{x = 0\}$, it follows that

$$\widetilde{g}(Y) \le \widetilde{\sigma}(1), \tag{3.25}$$

for $Y \in \partial Q_1 \cap \{x = 0\}$, where $\widetilde{\sigma}(1) = \sigma(\delta^l)$. Therefore by (3.21)-(3.25) and Lemma 3.3, we have

$$k_{l+1}x_2 - \|\widetilde{f}\|_{L^\infty(Q_1)} \le \widetilde{u}(Y) \le K_{l+1}x_2 + \widetilde{\sigma}(1) + \|\widetilde{f}\|_{L^\infty(Q_1)} \quad Y \in Q_\delta,$$

that is

$$k_{l+1}x_1 - \delta^{2l}\|f\|_{L^\infty(Q_{\delta^l})} \le u(X) \le K_{l+1}x_1 + \delta^l\sigma(\delta^l) + \delta^{2l}\|f\|_{L^\infty(Q_{\delta^l})} \quad X \in Q_{\delta^{l+1}}.$$

□

Proof of Theorem 3.2. Let $\{b_m\}_{m=0}^\infty, \{B_m\}_{m=0}^\infty, \{k_m\}_{m=0}^\infty, \{K_m\}_{m=0}^\infty$ be defined by Lemma 3.3, and for simplicity, we denote $\|f\|_{L^\infty(Q_{\delta^i})} = f_{\delta^i}, \sigma(\delta^i) = \sigma_{\delta^i}$.

Claim 1. $\lim\limits_{m\to\infty}(K_m - k_m) = 0$.

Proof. By induction, it is easy to see that $K_m \ge k_m$ for any $m \ge 0$. Therefore, when $m \ge 1$, we have

$$\begin{aligned}
0 \le K_{m+1} - k_{m+1} &\le K_m - k_m + N\frac{B_m + b_m}{\delta^m} - \mu(K_m - k_m)\\
&= (1-\mu)(K_m - k_m) + \frac{N}{\delta^m}(B_m + b_m)\\
&= (1-\mu)(K_m - k_m) + \frac{N}{\delta}(2\delta^{m-1}\|f\|_{L^\infty(Q_{\delta^{m-1}})} + \sigma(\delta^{m-1}))\\
&\le (1-\mu)(K_m - k_m) + \frac{N}{\delta}(2\|f\|_{L^\infty(Q_{\delta^{m-1}})} + \sigma(\delta^{m-1}))\\
&\le (1-\mu)^m(K_1 - k_1) + (1-\mu)^{m-1}\frac{N}{\delta}(2f_{\delta^0} + \sigma_{\delta^0})\\
&\quad + \cdot + (1-\mu)\frac{N}{\delta}(2f_{\delta^{m-2}} + \sigma_{\delta^{m-2}}) + \frac{N}{\delta}(2f_{\delta^{m-1}} + \sigma_{\delta^{m-1}})\\
&\le (1-\mu)^m N(2 + \frac{2}{\delta}\sum_{i=0}^{m-1}\frac{f_{\delta^i}}{(1-\mu)^{1+i}} + \frac{1}{\delta}\sum_{i=0}^{m-1}\frac{\sigma_{\delta^i}}{(1-\mu)^{1+i}}).
\end{aligned}$$

Let $1-\mu = \delta^\alpha$, we calculate that

$$\frac{f_{\delta^i}}{(1-\mu)^{1+i}} = \frac{f_{\delta^i}}{(\delta^\alpha)^{1+i}} = \frac{(\delta^{i-1} - \delta^i)f_{\delta^i}}{(1-\delta)\delta^{2\alpha}\cdot\delta^{(i-1)(\alpha+1)}},$$

then

$$\begin{aligned}
\sum_{i=1}^{m-1}\frac{f_{\delta^i}}{(\delta^\alpha)^{1+i}} &= \frac{1}{(1-\delta)\delta^{2\alpha}}\sum_{i=1}^{m-1}\frac{(\delta^{i-1} - \delta^i)f_{\delta^i}}{\delta^{(i-1)(\alpha+1)}}\\
&\le \frac{1}{(1-\delta)\delta^{2\alpha}}\sum_{i=1}^{m-1}\int_{\delta^i}^{\delta^{i-1}}\frac{f_r}{r^{1+\alpha}}dr \le \frac{1}{(1-\delta)\delta^{2\alpha}}\int_{\delta^{m-1}}^1\frac{f_r}{r^{1+\alpha}}dr,
\end{aligned}$$

and similarly

$$\sum_{i=1}^{m-1}\frac{\sigma_{\delta^i}}{(1-\mu)^{1+i}} = \sum_{i=1}^{m-1}\frac{\sigma_{\delta^i}}{(\delta^\alpha)^{1+i}} \le \frac{1}{(1-\delta)\delta^{2\alpha}}\int_{\delta^{m-1}}^1\frac{\sigma_r}{r^{1+\alpha}}dr.$$

Therefore

$$\begin{aligned}
0 \le K_{m+1} - k_{m+1} &\le \delta^{\alpha m}N(2 + \frac{2}{\delta(1-\delta)\delta^{2\alpha}}\int_{\delta^{m-1}}^1\frac{f_r}{r^{1+\alpha}}dr + \frac{1}{\delta(1-\delta)\delta^{2\alpha}}\int_{\delta^{m-1}}^1\frac{\sigma_r}{r^{1+\alpha}}dr)\\
&\le C_2\delta^{\alpha m}(1 + \int_{\delta^m}^1\frac{f_r + \sigma_r}{r^{1+\alpha}}dr), \qquad (3.26)
\end{aligned}$$

where $C_2 = 2N(1 + \frac{1}{\delta(1-\delta)\delta^{2\alpha}})$.

By L'Hospital's rule, as $\delta^m \to 0$, we have

$$(\delta^m)^\alpha \int_{\delta^m}^1 \frac{f_r + \sigma_r}{r^{1+\alpha}} dr \to 0,$$

combining with (3.26), we obtain

$$\lim_{m\to\infty} (K_m - k_m) = 0. \qquad \square$$

Claim 2. $\{K_m + k_m\}_{m=0}^{\infty}$ is convergent and we set

$$\lim_{m\to\infty} \frac{K_m + k_m}{2} = \theta.$$

Proof. For any $m \geq 2$, we have

$$K_{m+1} + k_{m+1} \leq K_m + k_m + N\frac{B_m}{\delta^m} + \mu(K_m - k_m),$$

$$K_{m+1} + k_{m+1} \geq K_m + k_m + \frac{N}{\delta^m}(B_m - b_m) - \mu(K_m - k_m),$$

therefore

$$\begin{aligned} |(K_{m+1} + k_{m+1}) - (K_m + k_m)| &\leq \mu(K_m - k_m) + \frac{NB_m}{\delta^m} \\ &\leq \mu(K_m - k_m) + \frac{N}{\delta}(f_{\delta^{m-1}} + \sigma_{\delta^{m-1}}). \end{aligned}$$

By (3.26) we have

$$\begin{aligned} &\sum_{j=m}^{\infty} |(K_{j+1} + k_{j+1}) - (K_j + k_j)| \\ &\leq \mu \sum_{j=m}^{\infty} (\delta^{j-1})^\alpha C_2 (1 + \int_{\delta^{j-1}}^1 \frac{f_r + \sigma_r}{r^{1+\alpha}} dr) + \frac{N}{\delta} \sum_{j=m-1}^{\infty} (f_{\delta^j} + \sigma_{\delta^j}). \end{aligned} \tag{3.27}$$

Now we estimate the right hand side of (3.27). Let $F_r = \int_r^1 \frac{f_s}{s^{1+\alpha}} ds$, and then

$$\begin{aligned} \sum_{j=m}^{\infty} (\delta^{j-1})^\alpha \int_{\delta^{j-1}}^1 \frac{f_r}{r^{1+\alpha}} dr &= \sum_{j=m-1}^{\infty} (\delta^j)^\alpha F_{\delta^j} \frac{\delta^j - \delta^{j+1}}{\delta^j - \delta^{j+1}} \\ &\leq \sum_{j=m-1}^{\infty} (\delta^{j+1})^{\alpha-1} F_{\delta^j} (\delta^j - \delta^{j+1}) \frac{\delta^j}{(\delta^j - \delta^{j+1})\delta^\alpha} \\ &\leq \frac{1}{(1-\delta)\delta^\alpha} \sum_{j=m-1}^{\infty} \int_{\delta^{j+1}}^{\delta^j} r^{\alpha-1} F_r dr \\ &= \frac{1}{(1-\delta)\delta^\alpha} \int_0^{\delta^{m-1}} r^{\alpha-1} F_r dr \\ &= \frac{1}{(1-\delta)\delta^\alpha} (\int_0^{\delta^{m-1}} \frac{f_s}{s^{1+\alpha}} \int_0^s r^{\alpha-1} dr ds + \int_{\delta^{m-1}}^1 \frac{f_s}{s^{1+\alpha}} \int_0^{\delta^{m-1}} r^{\alpha-1} dr ds) \\ &= \frac{1}{\alpha(1-\delta)\delta^\alpha} (\int_0^{\delta^{m-1}} \frac{f_r}{r} dr + (\delta^{m-1})^\alpha \int_{\delta^{m-1}}^1 \frac{f_r}{r^{1+\alpha}} dr). \end{aligned} \tag{3.28}$$

Similarly,

$$\sum_{j=m}^{\infty} (\delta^{j-1})^\alpha \int_{\delta^{j-1}}^1 \frac{\sigma_r}{r^{1+\alpha}} dr \leq \frac{1}{\alpha(1-\delta)\delta^\alpha} (\int_0^{\delta^{m-1}} \frac{\sigma_r}{r} dr + (\delta^{m-1})^\alpha \int_{\delta^{m-1}}^1 \frac{\sigma_r}{r^{1+\alpha}} dr). \tag{3.29}$$

It is easy to see

$$\begin{aligned}\sum_{j=m-1}^{\infty}(f_{\delta^j}+\sigma_{\delta^j}) &= \sum_{j=m-1}^{\infty}\frac{1}{\delta^{j-1}}(f_{\delta^j}+\sigma_{\delta^j})(\delta^{j-1}-\delta^j)\frac{\delta^{j-1}}{\delta^{j-1}-\delta^j}\\ &\le \frac{1}{1-\delta}\sum_{j=m-1}^{\infty}\int_{\delta^j}^{\delta^{j-1}}\frac{f_r+\sigma_r}{r}dr\\ &= \frac{1}{1-\delta}\int_0^{\delta^{m-2}}\frac{f_r+\sigma_r}{r}dr, \end{aligned} \tag{3.30}$$

and

$$\sum_{j=m}^{\infty}(\delta^{j-1})^{\alpha}=\sum_{j=m-1}^{\infty}(\delta^j)^{\alpha}\le\frac{(\delta^{m-1})^{\alpha}}{1-\delta^{\alpha}}. \tag{3.31}$$

Combining (3.27)-(3.31), for any $m\ge 2$,

$$\begin{aligned}&\sum_{j=m}^{\infty}|(K_{j+1}+k_{j+1})-(K_j+k_j)|\\ &\le C_3\{(\delta^{m-1})^{\alpha}+(\delta^{m-1})^{\alpha}\int_{\delta^{m-1}}^{1}\frac{f_r+\sigma_r}{r^{1+\alpha}}dr+\int_0^{\delta^{m-2}}\frac{f_r+\sigma_r}{r}\},\end{aligned} \tag{3.32}$$

where $C_3=C_2\mu(\frac{1}{1-\delta^{\alpha}}+\frac{1}{\alpha(1-\delta)\delta^{\alpha}})+\frac{N}{\delta(1-\delta)}$.

We conclude that $\{K_m+k_m\}_{m=2}^{\infty}$ is a convergent sequence by the right-hand side of (3.32) tends to 0, as $m\to\infty$. □

Claim 3. Let θ be given by Claim 2, then for each $m=0,1,2,\cdot$, there exists C_m such that $\lim\limits_{m\to\infty}C_m=0$, and that $|u(x,y)-\theta x|\le C_m\delta^m$, for any $(x,y)\in Q_{\delta^m}$.

Proof. For any $m\ge 0$ and any $(x,y)\in Q_{\delta^m}$,

$$|u(x,y)-\theta x|\le|u(x,y)-\frac{K_m+k_m}{2}x|+|(\frac{K_m+k_m}{2}-\theta)x|. \tag{3.33}$$

By Lemma 3.4,

$$-\frac{K_m-k_m}{2}x-b_m\le u(x,y)-\frac{K_m+k_m}{2}x\le\frac{K_m-k_m}{2}x+B_m,$$

Let $\widetilde{C}_m=\frac{K_m-k_m}{2}+\frac{B_m}{\delta^m}$, and then

$$|u(x,y)-\frac{K_m+k_m}{2}x|\le\widetilde{C}_m\delta^m. \tag{3.34}$$

Furthermore, by Claim 1 and $\lim\limits_{m\to\infty}\frac{B_m}{\delta^m}=0$ we have

$$\lim_{m\to\infty}\widetilde{C}_m=0. \tag{3.35}$$

Let $C_m=\widetilde{C}_m+|\frac{K_m+k_m}{2}-\theta|$, and therefore from (3.33) and (3.34), it follows that

$$|u(x,y)-\theta x|\le C_m\delta^m.$$

By Claim 2, and (3.35), $C_m\delta^m\to 0$, as $m\to\infty$. □

From Claim 3, we deuduce that u is differentiable at 0 with derivative θ, the proof of Theorem 3.2 is completed. □

3.2. The differentiability at the corners

Because of the symmetry, we only consider the corner $(0,b)$. By translating the coordinate system, we do assume that (0,0) is the corner. The difference between the corner points and the boundary Γ is that $Q_r \cap \Omega \neq Q_r$, i.e., the assume(i) in Theorem 3.2 is no longer valid.

For simplicity, in the following article, we set

$$\Omega[\delta \times M\delta^{\lambda+1}] = ([0,\delta] \times [-M\delta^{\lambda+1}, M\delta^{\lambda+1}]) \cap \Omega,$$

then $Q_\delta \cap \Omega = \Omega[\delta \times \delta^{\lambda+1}]$.

It is enough for us to prove the following theorem.

Theorem 3.3. *Assume that*

(i)$u(x,y) \geq 0$ for $(x,y) \in \Omega[1 \times 1]$ with $\|u\|_{L^\infty(\Omega[1\times 1])} \leq 1$;

(ii)$f(x,y) \geq 0$ for $(x,y) \in \Omega[1 \times 1]$ such that $\|f\|_{L^\infty(\Omega[1\times 1])} \leq 1$ and $\int_0^1 \frac{\|f\|_{L^\infty(\Omega[t\times t])}}{t} dt \leq 1$; and

(iii)$0 \leq g(x,y) \leq r\sigma(r)$ for $(x,y) \in \partial\Omega$ with $|(x,y)| \leq r$, where $\sigma : [0,1] \to \mathbb{R}_+$ satisfies $\int_0^1 \frac{\sigma(r)}{r} dr \leq 1$.

Then the solution u of (1.2) is differentiable at 0.

We will establish Theorem 3.3 by the following several lemmas.

Lemma 3.5. *There exist positive constants $\delta(<1)$, $\mu(<1)$, N, depending only on λ. If*

$$kx - b \leq u(x,y) \leq Kx + B \qquad \text{in } \Omega[1 \times 1], \tag{3.36}$$

for some nonnegative constants b, B, k, K,where $B, b \geq \|f\|_{L^\infty(\Omega[1\times 1])}$, then there exist nonnegative constants $\widetilde{k}, \widetilde{K}$ and A, such that

$$\widetilde{k}x - \|f\|_{L^\infty(\Omega[1\times 1])} - A(K+k+b)\kappa(1) \leq u(x,y) \leq \widetilde{K}x + 2\sigma(1) + \|f\|_{L^\infty(\Omega[1\times 1])} \qquad \text{in } \Omega[\delta \times \delta^{\lambda+1}], \tag{3.37}$$

where $\widetilde{k}$ and $\widetilde{K}$ satisfy either (3.6) or (3.7).

Proof. We prove the following claim first.

Claim. There exist positive constant N, δ_1,depending only on λ such that

$$(k - Nb)x - k\delta_1 \leq u(x,y) \leq (K + NB)x + \sigma(1) \qquad \text{in } \Omega[\delta_1 \times \delta_1^{\lambda+1}]. \tag{3.38}$$

Proof. Set δ and Ψ are given by Lemma 3.3.

We claim that

$$u(x,y) - Kx - B\Psi \leq \sigma(1) \qquad \text{on } \partial(\Omega[\delta_1 \times 1]) \qquad (3.39).$$

In fact, $\partial(\Omega[\delta_1 \times 1])$ can be separated into four parts:

$$(i)\ x = \delta_1, -1 \leq y \leq 0, \quad (ii)\ x = 0, -1 \leq y \leq 0,$$
$$(iii)\ 0 \leq x \leq \delta_1, y = -1, \quad (iv)\ 0 \leq x \leq \delta_1, y = 0. \tag{3.40}$$

On the first three parts, we have know that $u(x,y) - Kx - B\Psi \leq \sigma(1)$. On the part (iv), since $\Psi \geq 0, u(x,y) = g(x,y) \leq \sigma(1)$, we have $u(x,y) - Kx - B\Psi \leq \sigma(1)$.

Then we can obtain the right hand inequality of (3.38) by the same arguments in Lemma 3.3.

Moreover,we have

$$L_\lambda(kx - b\Psi - u(x,y)) \geq 0 \qquad \text{in } \Omega[\delta_1 \times 1],$$

and

$$kx - b\Psi - u(x,y) \le k\delta_1 \qquad \text{on } \partial(\Omega[\delta_1 \times 1]).$$

In fact, we separate the boundary into four parts as (3.40), on the first three parts we have $kx - b\Psi - u(x,y) \le 0$, on the part of (iv), we have $kx - b\Psi - u(x,y) \le k\delta_1$.

Then we can obtain

$$u(x,y) \ge (k - Nb)x - k\delta_1 \qquad \text{in } \Omega[\delta_1 \times \delta_1^{\lambda+1}],$$

we obtain the left hand inequality of (3.38). □

Let δ was given by Lemma 3.3 and $\widetilde{\Upsilon} = \{(x,y)|x = \delta, -\frac{\delta_1^{\lambda+1}}{2} \le y \le 0\}$. Next we will show (3.37) according to two cases:

(i) $u(\delta, 0) \ge \frac{1}{2}(K + k)\delta$,

(ii) $u(\delta, 0) \le \frac{1}{2}(K + k)\delta$,

corresponding to which (3.6) and (3.7) will hold respectively.

For case(i), let

$$v(x,y) = u(x,y) - (k - Nb)x + k\delta_1 \qquad \text{in } \Omega[\delta_1 \times \delta_1^{\lambda+1}],$$

then we can obtain

$$\inf_{\widetilde{\Upsilon}} v \ge \{\frac{1}{C_1}[(\frac{K-k}{2} + Nb)\delta + k\delta_1] - \|f\|_{L^\infty(\Omega[1\times 1])}\}^+ =: a.$$

Let $\psi = \psi_{\delta,(2\sqrt{2}+1)\delta^{\lambda+1}}$ be defined by Lemma 3.2,

$$w = \tfrac{1}{2}\|f\|_{L^\infty(\Omega[1\times 1])}(x - x^2) \quad \text{in } \Omega[\delta \times (2\sqrt{2}+1)\delta^{\lambda+1}],$$

we claim that

$$\begin{cases} L_\lambda(a\psi - w - v) \ge 0 & \text{in } \Omega[\delta \times (2\sqrt{2}+1)\delta^{\lambda+1}], \\ a\psi - w - v \le A(K + k + b)\kappa(1) & \text{on } \partial(\Omega[\delta \times (2\sqrt{2}+1)\delta^{\lambda+1}]). \end{cases}.$$

Indeed, the first inequality is clear since $L_\lambda\psi \ge 1$ and $L_\lambda v = f(x,y)$. For the second inequality, we separate the boundary into four parts:

$$\text{(i) } x = \delta, -(2\sqrt{2}+1)\delta^{\lambda+1} \le y \le 0;\ \text{(ii) } x = 0, -(2\sqrt{2}+1)\delta^{\lambda+1} \le y \le 0;$$
$$\text{(iii) } 0 \le x \le \delta, y = -(2\sqrt{2}+1)\delta^{\lambda+1};\ \text{(iv) } 0 \le x \le \delta, y = 0. \tag{3.41}$$

Combining with (3.2(i)) and (3.2(iii)), we have $a\psi - w - v \le 0$ on the first three parts. On the part of (iv), since (3.2(ii)), we have

$$a\psi - w - v \le a\psi \le [\frac{1}{C_1}(K + Nb)\delta + \frac{k\delta_1}{C_1}] \cdot \frac{x}{\delta} \le \frac{(K + k + Nb)\delta_1^2}{C_1\delta}.$$

Set $\kappa(r) = \frac{r}{(2\sqrt{2}+1)^{\frac{1}{\lambda+1}}}$, clearly $\kappa^2(r) \le r\kappa(r)$. Set $A = \max\{\frac{N+1}{C_1\delta}, 1\}$, since $\delta_1 = \frac{1}{(2\sqrt{2}+1)^{\frac{1}{\lambda+1}}} = \kappa(1)$, then we get the second inequality.

By Lemma 3.2 and (3.2(iv)), we have

$$\frac{a}{4\delta}x - w - v \le A(K + k + b)\kappa(1) \qquad \text{in } \Omega[\delta \times \delta^{\lambda+1}].$$

It follows that

$$\begin{aligned} v(x,y) &\geq \frac{a}{4\delta}x - w - A(K+k+b)\kappa(1) \\ &\geq \frac{1}{4\delta}\{\frac{1}{C_1}[(\frac{K-k}{2}+Nb)\delta + k\delta_1] - \|f\|_{L^\infty(\Omega[1\times 1])}\}x \\ &\quad + \frac{1}{2}\|f\|_{L^\infty(\Omega[1\times 1])}(x^2 - x) - A(K+k+b)\kappa(1) \\ &\geq \frac{K-k}{8C_1}x - \|f\|_{L^\infty(\Omega[1\times 1])} - A(K+k+b)\kappa(1), \end{aligned}$$

i.e.,

$$u(x,y) \geq (K - Nb + \frac{K-k}{8C_1})x - \|f\|_{L^\infty(\Omega[1\times 1])} - A(K+k+b)\kappa(1) \qquad \text{in } \Omega[\delta\times\delta^{\lambda+1}]. \quad (3.42)$$

Let $\mu = \frac{1}{8C_1}$, combining (3.38),(3.42) and $u \geq 0$, we have (3.37) and (3.6) hold.

For case(ii), the proof is similar to that of Case(i). Let

$$v(x,y) = (K+NB)x + \sigma(1) - u(x,y) \qquad \text{in } \Omega[\delta_1 \times \delta_1^{\lambda+1}],$$

then we obtain

$$\inf_{\widetilde{\Upsilon}} v \geq \{\frac{1}{C_1}[(\frac{K-k}{2}+NB)\delta + \sigma(1)] - \|f\|_{L^\infty(\Omega[1\times 1])}\}^+ =: a.$$

Set ψ and w defined in case (i), we claim that

$$\begin{cases} L_\lambda(a\psi - w - v) \geq 0 & \text{in } \Omega[\delta\times(2\sqrt{2}+1)\delta^{\lambda+1}], \\ a\psi - w - v \leq \sigma(1) & \text{on } \partial(\Omega[\delta\times(2\sqrt{2}+1)\delta^{\lambda+1}]). \end{cases}.$$

Indeed, the first inequality is clear since $L_\lambda\psi \geq 1$. For the second inequality, we also separate the boundary into four parts as (3.41). On the first three parts, by the same arguments to drive $a\psi - w - v \leq 0$. On the last part, by $\psi \leq \frac{x}{\delta}$, we have

$$\begin{aligned} a\psi - w - v &\leq \frac{1}{C_1}[(K+NB)\delta + \sigma(1)]\cdot\frac{x}{\delta} - v \\ &\leq \frac{1}{C_1}(K+NB)x + \frac{1}{C_1}\sigma(1) - [(K+NB)x + \sigma(1) - u(x,y)] \\ &\leq u(x,y) \leq \sigma(1). \end{aligned}$$

Therefore according to the lemma 3.2 and (3.2(iv)), we obtain

$$\frac{a}{4\delta}x - w - v \leq \sigma(1) \qquad \text{in } \Omega[\delta\times\delta^{\lambda+1}].$$

It follows that

$$\begin{aligned} v(x,y) &\geq \frac{a}{4\delta}x - w - \sigma(1) \\ &\geq \frac{1}{4\delta}\{\frac{1}{C_1}[(\frac{K-k}{2}+NB)\delta + \sigma(1)] - \|f\|_{L^\infty(\Omega[1\times 1])}\}x - \frac{1}{2}\|f\|_{L^\infty(\Omega[1\times 1])}(x - x^2) - \sigma(1) \\ &\geq \frac{K-k}{8C_1}x - \|f\|_{L^\infty(\Omega[1\times 1])} - \sigma(1), \end{aligned}$$

i.e.,

$$u(x,y) \leq (K + NB - \frac{K-k}{8C_1})x + \|f\|_{L^\infty(\Omega[1\times 1])} + 2\sigma(1) \qquad \text{in } \Omega[\delta\times\delta^{\lambda+1}]. \quad (3.43)$$

Combining (3.38), (3.43) and $u \geq 0$, we have (3.37) and (3.7) hold. □

As the proof of Lemma 3.4, by $0 \le u \le 1$, scaling and Lemma 3.5, we have

Lemma 3.6. *There exist nonnegative sequences* $\{b_m\}_{m=0}^{\infty}, \{B_m\}_{m=0}^{\infty}, \{k_m\}_{m=0}^{\infty}, \{K_m\}_{m=0}^{\infty}$ *with* $b_0 = B_0 = 1, K_0 = k_0 = 0$*, and for* $m = 0, 1, 2\cdot$,

$$b_{m+1} = \delta^{2m}\|f\|_{L^\infty(\Omega[\delta^m\times\delta^m])} + A(K_m + k_m + b_m)\delta^m\kappa(\delta^m),$$
$$B_{m+1} = \delta^{2m}\|f\|_{L^\infty(\Omega[\delta^m\times\delta^m])} + 2\delta^m\sigma(\delta^m),$$

and either

$$k_{m+1} = (k_m - N\frac{b_m}{\delta^m} + \mu(K_m - k_m))^+ \quad and \quad K_{m+1} = K_m + N\frac{B_m}{\delta^m},$$

or

$$k_{m+1} = (k_m - N\frac{b_m}{\delta^m})^+ \quad and \quad K_{m+1} = K_m + N\frac{B_m}{\delta^m} - \mu(K_m - k_m),$$

such that

$$k_m x - b_m \le u(x, y) \le K_m x + B_m \quad in\ \Omega[\delta^m \times (\delta^m)^{\lambda+1}], \tag{3.44}$$

for $m = 0, 1, 2, \cdot$.

This result is similar to Lemma 3.4, by the similar proving process of Theorem 3.2, we can prove the Theorem 3.3 and get the differentiability at the corners.

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Department of Mathematics and Information Science, Shaanxi Normal University, Xi'an 710119, China.
Email: songjx@snnu.edu.cn(J. Song), caoyi@snnu.edu.cn(Y. Cao)