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The hamiltonicity on the competition graphs of round digraphs

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Abstract. Given a digraph $D = (V, A)$, the competition graph G of D, denoted by $C(D)$, has the same set of vertices as D and an edge between vertices x and y if and only if $N_D^+(x) \cap N_D^+(y) \neq \emptyset$. In this paper, we investigate the competition graphs of round digraphs and give a necessary and sufficient condition for these graphs to be hamiltonian.

§1 Terminology and introduction

Competition graphs arose in connection with an application in ecology (see [2]) and have applications in coding, information transmission in computer and communication networks, channel assignment in communications, and modeling of complex systems arising from study of energy and economic systems, etc. For surveys of the early literature of competition graphs, see [3, 5, 6, 8]. The latest results about competition graphs can be found in [4, 9, 11]. In this paper, we will focus on the connectivity and hamiltonicity of the competition graphs of round digraphs.

It will be assumed that the reader is familiar with the concepts of graphs and digraphs. The other unexplained terms can be found in [1]. In the paper, all graphs are finite, undirected, without loops and multiple edges. All digraphs are finite, directed, without loops and parallel arcs.

For a graph G, $V(G)$ and $E(G)$ denote its vertex and edge sets, respectively. For a vertex set X, we denote by $G[X]$ the subgraph of G induced by X, $G[V(G)-X]$ by $G-X$. In addition, $G - x = G - \{x\}$ for a vertex x of G. A cycle (path) of G is called a *hamiltonian cycle (path)* if it contains all vertices of G. The graph G is said to be *hamiltonian* if it has a hamiltonian cycle.

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Generally, every graph G may be expressed uniquely (up to order) as a disjoint union of connected graphs. These graphs are called the *connected components*, or simply the *components*, of G. The number of components of G is denoted $\omega(G)$. A *cut vertex* of a graph G is a vertex x such that $\omega(G-x) > \omega(G)$. In particular, a cut vertex of a connected graph is a vertex whose deletion results in a disconnected graph.

Let D be a digraph on n vertices, and $V(D)$ and $A(D)$ denote its vertex and arc sets, respectively. If (x, y) is an arc of D, then we say that x *dominates* y and sometimes use the notation $x \to y$ to denote this arc. The *outset* of a vertex $x \in V(D)$ is the set $N_D^+(x) = \{y \mid (x, y) \in A(D)\}$. Similarly, $N_D^-(x) = \{y \mid (y, x) \in A(D)\}$ is the *inset* of x. For a vertex $x \in V(D)$, the *out-degree* (*in-degree*) of x (denoted by $d_D^+(x)$ ($d_D^-(x)$)) is the number of arcs of D with tail (head) x, i.e., $d_D^+(x) = |N_D^+(x)|$ and $d_D^-(x) = |N_D^-(x)|$. We will omit the subscript D if the digraph is known from the context. A subdigraph induced by a subset $X \subseteq V(D)$ is denoted by $D[X]$. In addition, $D - X = D[V(D) - X]$ and $D - x = D - \{x\}$. Similarly as for undirected graphs, a directed cycle of D is called a *hamiltonian cycle* if it contains all vertices of D. The digraph is said to be *hamiltonian* if it has a hamiltonian cycle.

A digraph D is *strong* if every vertex of D is reachable by a directed path from every other vertex of D. For a strong digraph D, a set $S \subseteq V(D)$ is a *separating set* if $D - S$ is not strong. In particular, if S contains exactly one vertex, say s, we call s a *separating vertex*. A digraph D is *k-strong* if $|V(D)| \geq k + 1$ and D has no separating set with less than k vertices. The largest integer k such that D is k-strong is the *vertex-strong connectivity* of D, denoted by $\kappa(D)$. If a digraph D is not strong, we set $\kappa(D) = 0$. Clearly, if $\kappa(D) = k$, then D has a separating set with k vertices. A *strong component* of a digraph D is a maximal induced subdigraph of D which is strong. As we know, the strong components of D can be labelled $D_1, D_2, \ldots, D_t$ (possibly $t = 1$) such that there is no arc from D_j to D_i unless $j < i$. We call such an ordering an *acyclic ordering of strong components* of D. The *underlying graph* of D, denoted by $UG(D)$, is the graph obtained by ignoring the orientations of arcs in D and deleting parallel edges. We say that D is *connected* if its underlying graph is connected.

The *competition graph* $C(D)$ of a digraph D is the (simple undirected) graph G defined by $V(G) = V(D)$ and $E(G) = \{xy \mid x, y \in V(D), x \neq y, N_D^+(x) \cap N_D^+(y) \neq \emptyset\}$.

A digraph D on n vertices is called a *round digraph* if we can label its vertices $v_0, v_1, \ldots, v_{n-1}$ such that for each i, $N_D^+(v_i) = \{v_{i+1}, \ldots, v_{i+d_D^+(v_i)}\}$ and $N_D^-(v_i) = \{v_{i-d_D^-(v_i)}, \ldots, v_{i-1}\}$ if none of them is empty, where the subscripts are taken modulo n. Note that every strong round digraph is hamiltonian. We refer to the ordering $v_0, v_1, \ldots, v_{n-1}$ as a *round labelling* of D. For convenience, in the remainder of this paper, the subscripts of v are taken modulo n. In particular, we use v_i^+ and v_i^- to denote v_{i+1} and v_{i-1} respectively, and $[v_i, v_j]$ to denote the vertex subset $\{v_i, v_{i+1}, \ldots, v_{j-1}, v_j\}$. An example of a round digraph on 10 vertices and its competition graph is shown in Fig. 1.

A digraph D is *semicomplete* if, for every pair x, y of vertices of D, either x dominates y or y dominates x (or both). A digraph D is *locally semicomplete* if for every vertex x, the out-neighbours of x induce a semicomplete digraph and the in-neighbours of x induce a

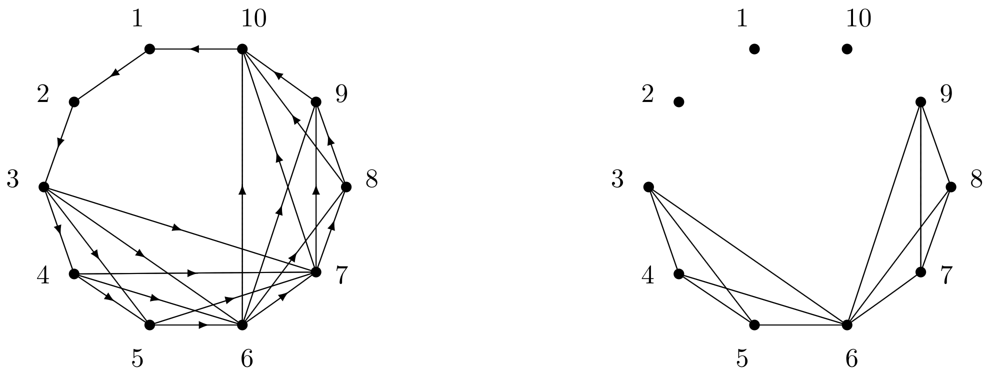

Figure 1: A round digraph on 10 vertices and its competition graph.

semicomplete digraph. In [7], Huang proved the following.

Proposition 1.1 (Huang [7]). *Every round digraph is locally semicomplete.*

In this paper, we investigate the competition graphs of round digraphs. In Section 2, we count the number of connected components of these graphs. In Section 3, we give a necessary and sufficient condition for these graphs to be hamiltonian.

§2 The connectivity of competition graphs of round digraphs

In [10], we obtained the following observation on round digraphs.

Lemma 2.1 (Zhang et al.[10]). *Let D be a round digraph and $v_0, v_1, \ldots, v_{n-1}$ be the round labelling of D. If v_i dominates v_j in D, then v_α dominates v_β for all v_α, v_β satisfying that $v_i, v_\alpha, v_\beta, v_j$ are in the order of the round labelling of D.*

First, we discuss the case when D is a strong round digraph.

Lemma 2.2. *Let D be a strong round digraph with $n \geq 3$ vertices and $v_0, v_1, \ldots, v_{n-1}$ be the round labelling of D. Let G be the competition graph of D. Then for $k \in \{0, 1, \ldots, n-1\}$, the following assertions are equivalent:*

(1) $v_{k-1} \to v_{k+1}$ in D;

(2) v_k is not a separating vertex of D;

(3) $v_{k-1}v_k$ is an edge of G.

Proof. (1)⇒(2). Suppose $v_{k-1} \to v_{k+1}$ in D. Since D is a strong round digraph and $v_0, v_1, \ldots, v_{n-1}$ is the round labelling of D, we see that $v_0v_1 \ldots v_{n-1}v_0$ is a hamiltonian cycle in D. Since $v_{k-1} \to v_{k+1}$ by the hypothesis, $v_0v_1 \ldots v_{k-1}v_{k+1} \ldots v_{n-1}v_0$ is a hamiltonian cycle in $D - v_k$. So v_k is not a separating vertex.

(2)⇒(3). Suppose that v_k is not a separating vertex of D. So v_k cannot be the unique vertex dominated by v_{k-1}. This yields $v_{k-1} \to v_{k+1}$. Combining with $v_k \to v_{k+1}$, we have $v_{k-1}v_k$ is an edge of G.

(3)⇒(1). Since $v_{k-1}v_k$ is an edge of G, there exists a vertex $v_\alpha \in V(D) \setminus \{v_{k-1}, v_k\}$ such that $v_{k-1} \to v_\alpha$ and $v_k \to v_\alpha$. So $v_{k-1} \to v_{k+1}$ by Lemma 2.1. □

Theorem 2.3. *Let D be a strong round digraph with $n \geq 3$ vertices and G be the competition graph of D. Then G is connected if and only if D contains at most one separating vertex.*

Proof. Let $v_0, v_1, \dots, v_{n-1}$ be the round labelling of D.

" ⇒ ". Suppose that D contains two separating vertices, say v_i, v_j. Then $v_{i-1} \nrightarrow v_{i+1}$ and $v_{j-1} \nrightarrow v_{j+1}$ by Lemma 2.2. Let $A = \{v_i, v_{i+1}, \dots, v_{j-1}\}$ and $B = \{v_j, v_{j+1}, \dots, v_{i-1}\}$.

We show that xy is not an edge of G for any $x \in A$ and $y \in B$. If not, then there exists a vertex $z \in V(D) \setminus \{x, y\}$ such that $x \to z$ and $y \to z$ in D for some $x \in A$ and $y \in B$. If $z \in A$, then $y \to z$ implies $v_{i-1} \to v_{i+1}$ according to Lemma 2.1, a contradiction. If $z \in B$, then $x \to z$ implies $v_{j-1} \to v_{j+1}$ according to Lemma 2.1, a contradiction. Thus xy is not an edge of G. As a result, there exists no edge between A and B in G. This means G is not connected, a contradiction.

" ⇐ ". If D contains no separating vertex, then we have $v_{k-1}v_k \in E(G)$ for any $k \in \{0, 1, \dots, n-1\}$ by Lemma 2.2, and hence $v_0v_1 \dots v_{n-1}v_0$ is a hamiltonian cycle of G. Thus G is connected. If D contains exactly one separating vertex v_i, then $v_{k-1}v_k \in E(G)$ for any $k \in \{0, 1, \dots, n-1\} \setminus \{i\}$ by Lemma 2.2. Then $v_iv_{i+1} \dots v_{n-1}v_0 \dots v_{i-2}v_{i-1}$ is a hamiltonian path of G, and hence G is connected. □

Now we consider the case when D is a connected and non-strong round digraph.

Proposition 2.4 (Zhang et al.[10])**.** *Let D be a connected and non-strong round digraph on n vertices. If $D_1, D_2, \dots, D_t$ is an acyclic ordering of strong components of D, then $|V(D_i)| = 1$ for $i = 1, 2, \dots, t$.*

Lemma 2.5. *Let D be a connected and non-strong round digraph on n vertices and let $v_0, v_1, \dots, v_{n-1}$ be an acyclic ordering of strong components of D. Then $v_0, v_1, \dots, v_{n-1}$ is the round labelling of D.*

Proof. Recall that for $0 \leq i < j \leq n-1$, if v_i and v_j are adjacent, we must have $v_i \to v_j$ since $v_0, v_1, \dots, v_{n-1}$ is an acyclic ordering of strong components of D. Also recall that both $N_D^+(v_i)$ and $N_D^-(v_i)$ induce the semicomplete digraph by Proposition 1.1. This means that for any two vertices x, y in $N_D^+(v_i)$ (or $N_D^-(v_i)$), x and y are adjacent.

Firstly, we claim that $v_i \to v_{i+1}$ for each $i \in \{0, 1, \dots, n-2\}$. Suppose to the contrary, let i be the minimum index such that $v_i \nrightarrow v_{i+1}$. If $v_i \nrightarrow v_j$ for any $j \in \{i+2, \dots, n-1\}$ (if they exist), then $v_{i-1} \nrightarrow v_j$ since otherwise $v_{i-1} \to v_j$ and $v_{i-1} \to v_i$ implies that $v_i \to v_j$. Continuing this process, we obtain that v_α and v_j are non-adjacent for all $\alpha \in \{0, 1, \dots, i\}$ and $j \in \{i+1, \dots, n-1\}$. This contradicts that D is connected. So assume that $v_i \to v_j$ for some

$j \in \{i+2, \ldots, n-1\}$. Without loss of generality, let j be the minimum index such that $v_i \to v_j$. Then $j \geq i+2$. For convenience, let

$$\alpha \in \{0, 1, \ldots, i\}, \quad \gamma \in \{i+1, \ldots, j-1\}, \quad \beta \in \{j+1, \ldots, n-1\}.$$

Then v_i and v_γ are non-adjacent by the definition of j. Moreover, v_{i-1} and v_γ are also non-adjacent since otherwise $v_{i-1} \to v_\gamma$ and $v_{i-1} \to v_i$ implies $v_i \to v_\gamma$. Continuing this process, we obtain that v_α and v_γ are non-adjacent. Observe that if $v_\gamma \to v_\beta$ then $v_\alpha \nrightarrow v_\beta$ and vise versa. Also if $v_\gamma \to v_\beta$ then $v_j \nrightarrow v_\beta$ and vise versa since otherwise $v_\gamma \to v_j$ and hence $v_i \to v_\gamma$ because $v_i \to v_j$. Thus the component containing $\{v_0, \ldots, v_i, v_j\}$ is not connected with the component containing $\{v_{i+1}, \ldots, v_{j-1}\}$, which contradicts the fact that D is connected. Therefore, $v_i \to v_{i+1}$ for each $i \in \{0, 1, \ldots, n-2\}$.

Now if $v_i \to v_j$ for $0 \leq i < j \leq n-1$, then $v_i \to v_{j-1}$ since $v_{j-1} \to v_j$. Similarly, we obtain that $v_i \to v_k$ for all $k \in \{i+1, \ldots, j\}$. This means that $N_D^+(v_i)$ is a continuous segment in the acyclic ordering $v_0, v_1, \ldots, v_{n-1}$. It is also true for $N_D^-(v_i)$. Thus $v_0, v_1, \ldots, v_{n-1}$ is the round labelling of D. □

Conversely, if $v_0, v_1, \ldots, v_{n-1}$ is the round labelling of D satisfying the indegree of v_0 is zero, then $v_0, v_1, \ldots, v_{n-1}$ is also the acyclic ordering of strong components of D. In the remainder of the paper, if D is a connected and non-strong round digraph, we always assume that the round labelling $v_0, v_1, \ldots, v_{n-1}$ of D satisfies the indegree of v_0 is zero. Let G be the competition graph of D. Since $N_D^+(v_{n-1}) = \emptyset$, we have $v_i v_{n-1} \notin E(G)$ for $i \in \{0, 1, \ldots, n-2\}$. This means v_{n-1} is an isolated vertex in G. So we consider the connectivity of the graph $G - v_{n-1}$. Similarly as for Lemma 2.2, we have the following observation.

Lemma 2.6. *Let D be a connected and non-strong round digraph on n vertices and $v_0, v_1, \ldots, v_{n-1}$ be a round labelling of D. Let G be the competition graph of D. Then for $k \in \{1, 2, \ldots, n-2\}$, the following assertions are equivalent:*

(1) $v_{k-1} \to v_{k+1}$;

(2) v_k is not a cut vertex of $UG(D)$;

(3) $v_{k-1}v_k$ is an edge of G.

Proof. (1)⇒(2). Suppose $v_{k-1} \to v_{k+1}$ in D. Since D is connected, we have $UG(D)$ is connected, i.e., the number of components $\omega(UG(D)) = 1$. Since $v_0, v_1, \ldots, v_{n-1}$ is a round labelling of D and $v_{k-1} \to v_{k+1}$ by the hypothesis, $v_0 v_1 \ldots v_{k-1} v_{k+1} \ldots v_{n-1}$ is a hamiltonian path in $D - v_k$, that is, $v_0 v_1 \ldots v_{k-1} v_{k+1} \ldots v_{n-1}$ is a hamiltonian path in $UG(D) - v_k$. So $\omega(UG(D) - v_k) = 1$. Thus v_k is not a cut vertex of $UG(D)$.

(2)⇒(3). Suppose that v_k is not a cut vertex of $UG(D)$. So there must exist some $v_i \in \{v_0, \ldots, v_{k-1}\}$, $v_j \in \{v_{k+1}, \ldots, v_{n-1}\}$ such that $v_i \to v_j$ in D. By Lemma 2.1, we have $v_{k-1} \to v_{k+1}$. Combining with $v_k \to v_{k+1}$, we obtain $v_{k-1}v_k$ is an edge of G.

(3)⇒(1). Since $v_{k-1}v_k$ is an edge of G, there exists a vertex $v_\alpha \in V(D) \setminus \{v_{k-1}, v_k\}$ such that $v_{k-1} \to v_\alpha$ and $v_k \to v_\alpha$. So $v_{k-1} \to v_{k+1}$ by Lemma 2.1. □

Theorem 2.7. *Let D be a connected and non-strong round digraph on n vertices and $v_0, v_1, \ldots, v_{n-1}$ be the round labelling of D. Let G be the competition graph of D. Then $G - v_{n-1}$ is connected if and only if $v_{k-1} \to v_{k+1}$ for any $k \in \{1, 2, \ldots, n-2\}$.*

Proof. " $\Rightarrow$ ". Suppose there exist some $k \in \{1, 2, \ldots, n-2\}$ satisfying $v_{k-1} \nrightarrow v_{k+1}$. Let $A = \{v_0, v_1, \ldots, v_{k-1}\}$, $B = \{v_k, v_{k+1}, \ldots, v_{n-2}\}$. We show that $v_i v_j$ is not an edge of G for $v_i \in A$ and $v_j \in B$. If not, there exists a vertex $v_\beta \in V(D)$ such that $v_i \to v_\beta$ and $v_j \to v_\beta$. Since $v_0, v_1, \ldots, v_{n-1}$ is also the acyclic ordering of strong components of D, we have $v_\beta \in \{v_{j+1}, v_{j+2}, \ldots, v_{n-1}\}$. Then $v_i \to v_\beta$ implies $v_{k-1} \to v_{k+1}$, a contradiction. So there is no edge between A and B, and $G - v_{n-1}$ is not connected, a contradiction. Thus $v_{k-1} \to v_{k+1}$ for all $k \in \{1, 2, \ldots, n-2\}$.

" $\Leftarrow$ ". Note that $v_{k-1} \to v_{k+1}$ and $v_k \to v_{k+1}$ for $k \in \{1, 2, \ldots, n-2\}$. So $v_{k-1}v_k \in E(G)$ for $k \in \{1, 2, \ldots, n-2\}$. This means $G - v_{n-1}$ is connected. □

In the following, we count the number of the connected components $\omega(G)$, where G is the competition graph of a connected round digraph D. Let $p(D)$ be the number of the separating vertices of D and $q(D)$ be the number of the cut vertices of the underlying graph $UG(D)$.

Theorem 2.8. *Let D be a round digraph with $n \geq 3$ vertices and $v_0, v_1, \ldots, v_{n-1}$ be a round labelling of D. Let G be the competition graph of D. Then the following hold:*

(1) if D is strong, then

$$\omega(G) = \begin{cases} 1, & \text{for } p(D) = 0, \\ p(D), & \text{for } p(D) \geq 1; \end{cases}$$

(2) if D is connected and non-strong, then $\omega(G) = q(D) + 2$.

Proof. (1) Let $p(D) = p$. If $p = 0$ or 1, Theorem 2.3 implies that G is connected. Thus $\omega(G) = 1$.

Now we consider the case when $p \geq 2$. Assume that $v_{i_1}, v_{i_2}, \ldots, v_{i_p}$ are the p separating vertices of D with $0 \leq i_1 < i_2 < \ldots < i_p \leq n-1$. Let

$$A_1 = \{v_{i_1}, v_{i_1+1}, \ldots, v_{i_2-1}\}, \quad A_2 = \{v_{i_2}, v_{i_2+1}, \ldots, v_{i_3-1}\}, \quad \ldots, \quad A_p = \{v_{i_p}, v_{i_p+1}, \ldots, v_{i_1-1}\}.$$

For $k \in \{1, 2, \ldots, p\}$, if $A_k \setminus \{v_{i_k}\} = \emptyset$, then $G[A_k]$ is a single vertex; if $A_k \setminus \{v_{i_k}\} \neq \emptyset$, then we have $v_{\alpha-1} \to v_{\alpha+1}$ for any vertex $v_\alpha \in A_k \setminus \{v_{i_k}\}$ by Lemma 2.2, and hence $v_{\alpha-1}v_\alpha \in E(G)$ for $v_\alpha \in A_k \setminus \{v_{i_k}\}$. So $G[A_k]$ is connected in G.

Next, we prove $v_\beta v_\gamma \notin E(G)$ for any $v_\beta \in A_j$, $v_\gamma \in A_k$ for $j \neq k$. Now take $v_\beta \in A_j$ and $v_\gamma \in A_k$ for $j \neq k$. Suppose $v_\beta v_\gamma \in E(G)$ with $\beta < \gamma$. Then there exists a vertex $v_t \in V(D) \setminus \{v_\beta, v_\gamma\}$ such that $v_\beta \to v_t$ and $v_\gamma \to v_t$. For $v_t \in \bigcup_{m=j}^{k-1} A_m$, $v_\gamma \to v_t$ implies $v_{i_j-1} \to v_{i_j+1}$, which contradicts that v_{i_j} is a separating vertex of D. For $v_t \in \bigcup_{\lambda=k}^{j-1} A_\lambda$, $v_\beta \to v_t$ implies $v_{i_k-1} \to v_{i_k+1}$, which contradicts that v_{i_k} is a separating vertex of D. So $v_\beta v_\gamma \notin E(G)$ for any $v_\beta \in A_j$ and $v_\gamma \in A_k$.

Thus we have $\omega(G) = p(D)$ for $p(D) \geq 2$.

(2) Let $q(D) = q$. Note that D is a connected and non-strong round digraph. So $v_0, v_1, \dots$ v_{n-1} is the acyclic ordering of strong components of D. Let $v_{i_1}, v_{i_2}, \dots, v_{i_q}$ be the q cut vertices of $UG(D)$ with $0 \le i_1 < i_2 < \dots < i_q \le n-1$. Clearly, $\{v_{i_1}, v_{i_2}, \dots, v_{i_q}\} \subseteq V(D) \setminus \{v_0, v_{n-1}\}$. Let

$$A_1 = \{v_0, v_1, \dots, v_{i_1-1}\}, \quad A_2 = \{v_{i_1}, v_{i_1+1}, \dots, v_{i_2-1}\}, \quad \dots, \quad A_{q+1} = \{v_{i_q}, v_{i_q+1}, \dots, v_{n-2}\}.$$

For $k \in \{1, 2, \dots, q+1\}$, by Lemma 2.6, we have $v_{\alpha-1} \to v_{\alpha+1}$ for any $v_\alpha \in A_k \setminus \{v_{i_{k-1}}\}$ unless $G[A_k]$ is a single vertex. This implies that $v_{\alpha-1}v_\alpha \in E(G)$ for $v_\alpha \in A_k \setminus \{v_{i_{k-1}}\}$ unless $G[A_k]$ is a single vertex. Hence $G[A_k]$ is connected in G.

Next, we prove $v_\beta v_\gamma \notin E(G)$ for any $v_\beta \in A_j, v_\gamma \in A_k$ for $j \ne k$. Now fix $v_\beta \in A_j$ and $v_\gamma \in A_k$ for $j \ne k$. Suppose $v_\beta v_\gamma \in E(G)$ with $\beta < \gamma$. Then there exists a vertex $v_t \in V(D) \setminus \{v_\beta, v_\gamma\}$ such that $v_\beta \to v_t$ and $v_\gamma \to v_t$. Since $v_0, v_1, \dots, v_{n-1}$ is an acyclic ordering of strong components of D, we have $t > \gamma$. Now $v_\beta \to v_t$ implies $v_{i_j-1} \to v_{i_j+1}$, which contradicts that v_{i_j} is a cut vertex of $UG(D)$.

Recall that v_{n-1} is an isolated vertex of G. Thus we have $\omega(G) = q(D) + 2$. □

§3 The hamiltonicity of competition graphs of round digraphs

Lemma 3.1. *Let D be a connected round digraph on n vertices with $\kappa(D) \le 1$ and $v_0, v_1, \dots,$ v_{n-1} be a round labelling of D. Then there exist the vertices $v_{i_1}, v_{i_2}, \dots, v_{i_t}$ such that*

(1) $\bigcup\limits_{j=1}^{t} N_D^-(v_{i_j}) = \begin{cases} V(D) \setminus \{v_{n-1}\}, & \text{when } \kappa(D) = 0, \\ V(D), & \text{when } \kappa(D) = 1; \end{cases}$

(2) $N_D^-(v_{i_k}) - N_D^-(v_{i_l}) \ne \emptyset$ *for any distinct integers k, l with $1 \le k, l \le t$.*

Proof. When $\kappa(D) = 0$, D is a connected but non-strong round digraph. Let $v_{i_1} = v_{n-1}$. Since $v_0, v_1, \dots, v_{n-1}$ is the round labelling of D, we have $N_D^-(v_{i_1}) = \{v_{i_1-1}, v_{i_1-2}, \dots, v_{i_1-d_D^-(v_{i_1})}\}$. Let v_{i_2} be the first vertex from v_{i_1-1} to $v_{i_1-d_D^-(v_{i_1})}$ along the sequence in reverse order of the round labelling such that it is dominated by some vertex outside $N_D^-(v_{i_1})$. The vertex v_{i_2} exists since $v_{i_1-d_D^-(v_{i_1})}$ is dominated by $v_{i_1-d_D^-(v_{i_1})-1}$ which is not in $N_D^-(v_{i_1})$. Similarly, $N_D^-(v_{i_2}) = \{v_{i_2-1}, v_{i_2-2}, \dots, v_{i_2-d_D^-(v_{i_2})}\}$. Let v_{i_3} be the first vertex from v_{i_2-1} to $v_{i_2-d_D^-(v_{i_2})}$ such that it is dominated by some vertex outside $N_D^-(v_{i_2})$. Continue in this way until we get a vertex v_{i_t} such that its inset contains v_0. This means we stop when the inset $N_D^-(v_{i_t}) = \{v_{i_t-1}, v_{i_t-2}, \dots, v_0\}$. It is easy to check that $v_{i_1}, v_{i_2}, \dots, v_{i_t}$ satisfy (2). Note that $v_{n-2} \in N_D^-(v_{i_1})$, $v_0 \in N_D^-(v_{i_t})$ and for $j \in \{1, 2, \dots, t-1\}$, $N_D^-(v_{i_j}) \cup N_D^-(v_{i_{j+1}})$ is always a continuous segment in the ordering $v_0, v_1, \dots, v_{n-1}$. Then $\bigcup\limits_{j=1}^{t} N_D^-(v_{i_j}) = V(D) \setminus \{v_{n-1}\}$ and (1) holds. So $v_{i_1}, v_{i_2}, \dots, v_{i_t}$ are the desired vertices. See Fig. 2.

When $\kappa(D) = 1$, D is a strong round digraph and contains at least one separating vertex. We choose arbitrarily a separating vertex as v_{i_1}. Note that $N_D^-(v_{i_1}) = \{v_{i_1-1}, v_{i_1-2}, \dots, v_{i_1-d_D^-(v_{i_1})}\}$. Similarly as in (1), let v_{i_2} be the first vertex from v_{i_1-1} to $v_{i_1-d_D^-(v_{i_1})}$ along the sequence in reverse order of the round labelling such that it is dominated by some vertex outside $N_D^-(v_{i_1})$. Also, $N_D^-(v_{i_2}) = \{v_{i_2-1}, v_{i_2-2}, \dots, v_{i_2-d_D^-(v_{i_2})}\}$. Let v_{i_3} be the first vertex from v_{i_2-1} to $v_{i_2-d_D^-(v_{i_2})}$

such that it is dominated by some vertex outside $N_D^-(v_{i_2})$. Continue in this way until we get a vertex v_{i_t} such that $v_{i_1} \in N_D^-(v_{i_t})$. Such a vertex v_{i_t} exists since v_{i_1} dominates v_{i_1+1}. Also, it is easy to check that $v_{i_1}, v_{i_2}, \ldots, v_{i_t}$ satisfy (2). Moreover, $v_{i_1-1} \in N_D^-(v_{i_1})$, $v_{i_1} \in N_D^-(v_{i_t})$ and for $j \in \{1,2,\ldots,t-1\}$, $N_D^-(v_{i_j}) \cup N_D^-(v_{i_{j+1}})$ is a continuous segment in the ordering $v_0, v_1, \ldots, v_{n-1}$. Then $\bigcup\limits_{j=1}^{t} N_D^-(v_{i_j}) = V(D)$ and (1) holds. So $v_{i_1}, v_{i_2}, \ldots, v_{i_t}$ are the desired vertices. □

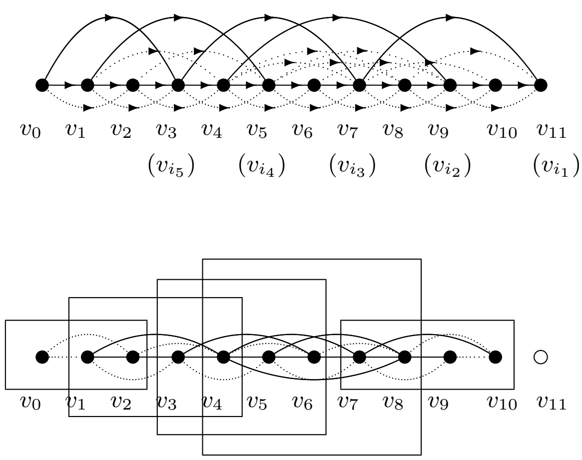

Figure 2: A connected and non-strong round digraph and its round clique cover, where the boxes denote the round intervals $N_D^-(v_{i_1}), N_D^-(v_{i_2}), \ldots, N_D^-(v_{i_5})$.

Lemma 3.2. *Let D be a connected round digraph on n vertices with $\kappa(D) = 1$ and let $v_0, v_1, \ldots, v_{n-1}$ be a round labelling of D. Suppose $v_{i_1}, v_{i_2}, \ldots, v_{i_t}$ are the vertices constructed in the proof of Lemma 3.1. Then*

(1) if v_α is a separating vertex of D, then $v_\alpha \in \{v_{i_1}, v_{i_2}, \ldots, v_{i_t}\}$;

(2) for $t \geq 3$, v_{i_j} is a separating vertex of D if and only if $N_D^-(v_{i_{j-1}}) \cap N_D^-(v_{i_j}) = \emptyset$, where $v_{i_0} = v_{i_t}$.

Proof. (1) Let v_α be a separating vertex of D. Assume $v_\alpha \in N_D^-(v_{i_j})$. By Lemmas 2.1 and 2.2, we have $v_\alpha = v_{i_j - d_D^-(v_{i_j})}$. We claim $v_\alpha = v_{i_{j+1}}$. Suppose not. Then there exists a vertex v_l with $l \leq i_j - d_D^-(v_{i_j}) - 1$ outside $N_D^-(v_{i_j})$ such that $v_{i_{j+1}} (\neq v_\alpha)$ is dominated by v_l. Then, by Lemma 2.1, $v_{i_j - d_D^-(v_{i_j}) - 1} \to v_{i_j - d_D^-(v_{i_j}) + 1}$, which contradicts Lemma 2.2. Thus $v_\alpha = v_{i_{j+1}}$ and (1) holds.

(2) " $\Rightarrow$ ". Let v_{i_j} be a separating vertex of D for some $j \in \{1,2,\ldots,t\}$. Suppose that $N_D^-(v_{i_{j-1}}) \cap N_D^-(v_{i_j}) \neq \emptyset$, say $v_\alpha \in N_D^-(v_{i_{j-1}}) \cap N_D^-(v_{i_j})$. Then $\alpha < i_j$ and $v_\alpha, v_{i_j}, v_{i_{j-1}}$ are in the order of the round labelling of D. Since $v_\alpha \to v_{i_{j-1}}$ and $v_\alpha \to v_{i_j}$, we have $v_{i_j-1} \to v_{i_j+1}$ by Lemma 2.1, a contradiction.

" $\Leftarrow$ ". Suppose that $N_D^-(v_{i_{j-1}}) \cap N_D^-(v_{i_j}) = \emptyset$. Since $v_0, \ldots, v_{n-1}$ is a round labelling of D, $N_D^-(v_{i_{j-1}}) = \{v_{i_{j-1}-1}, v_{i_{j-1}-2}, \ldots, v_{i_{j-1} - d_D^-(v_{i_{j-1}})}\}$. Since $N_D^-(v_{i_{j-1}}) \cap N_D^-(v_{i_j}) = \emptyset$,

$v_{i_j} = v_{i_{j-1}-d_D^-(v_{i_{j-1}})}$ by Lemma 3.1. Suppose to the contrary that v_{i_j} is not a separating vertex of D. Then, by Lemma 2.2, we have $v_{i_j-1} \to v_{i_j+1}$. Then v_{i_j+1} is an in-neighbor of $v_{i_{j-1}}$ having an in-neighbor v_{i_j} not in $N_D^-(v_{i_{j-1}})$. This means that v_{i_j} is not the first vertex from $v_{i_{j-1}}$ to $v_{i_{j-1}-d_D^-(v_{i_{j-1}})}$ along the sequence in reverse order of the round labelling such that it is dominated by some vertices outside $N_D^-(v_{i_{j-1}})$, contradicting the choice of v_{i_j}. □

Suppose $v_{i_1}, v_{i_2}, \ldots, v_{i_t}$ are the vertices constructed in the proof of Lemma 3.1. Note that by the construction, $v_{i_1}, v_{i_2}, \ldots, v_{i_t}$ are unique (up to cyclic permutation) and does not depend on the choice of v_{i_1}. Now we call $N_D^-(v_{i_1}), N_D^-(v_{i_2}), \ldots, N_D^-(v_{i_t})$ a *round clique cover* of D and each clique $G[N_D^-(v_{i_j})]$ (or $N_D^-(v_{i_j})$ for short) a *round interval* of G.

Theorem 3.3. *Let D be a strong round digraph with $n \geq 3$ vertices, $v_0, v_1, \ldots, v_{n-1}$ be a round labelling of D, and $N_D^-(v_{i_1}), N_D^-(v_{i_2}), \ldots, N_D^-(v_{i_t})$ be the round clique cover of D. Let G be the competition graph of D. Then G is hamiltonian if and only if*

(1) D contains no separating vertex or;

(2) D contains exactly one separating vertex v_{i_1} and $|N_D^-(v_{i_{j-1}}) \cap N_D^-(v_{i_j})| \geq 2$ for any $j \in \{2, 3, \ldots, t\}$.

“ $\Rightarrow$ ”. Assume that G is hamiltonian. Then G is connected. Theorem 2.3 implies that D contains at most one separating vertex. If D contains no separating vertex, we are done. Now we consider the case when D contains exactly one separating vertex. Without loss of generality, we may assume v_{i_1} is the separating vertex. It is sufficient to show that $|N_D^-(v_{i_{k-1}}) \cap N_D^-(v_{i_k})| \geq 2$ for all $k \in \{2, 3, \ldots, t\}$. Suppose to the contrary that there exists $\beta \in \{2, 3, \ldots, t\}$ such that $|N_D^-(v_{i_{\beta-1}}) \cap N_D^-(v_{i_\beta})| \leq 1$.

Claim 1. $t \geq 3$.

Suppose $t = 2$. Then G has only two round intervals $N_D^-(v_{i_1})$ and $N_D^-(v_{i_2})$, $v_{i_1} \in N_D^-(v_{i_2})$ and $|N_D^-(v_{i_1}) \cap N_D^-(v_{i_2})| \leq 1$. Suppose $N_D^-(v_{i_1}) \cap N_D^-(v_{i_2}) = \emptyset$. Since $V(D) = N_D^-(v_{i_1}) \cup N_D^-(v_{i_2})$, $v_{i_2+1} \in N_D^-(v_{i_1})$. Since $v_{i_2-1} \in N_D^-(v_{i_2})$ and $N_D^-(v_{i_1}) \cap N_D^-(v_{i_2}) = \emptyset$, $v_{i_2-1} \nrightarrow v_{i_2+1}$. Then v_{i_2} is a separating vertex by Theorem 2.7, which contradicts the assumption that v_{i_1} is the unique separating vertex. Thus $|N_D^-(v_{i_1}) \cap N_D^-(v_{i_2})| = 1$, say $v_\gamma \in N_D^-(v_{i_1}) \cap N_D^-(v_{i_2})$. We claim that x and y are not adjacent in G for any $x \in N_D^-(v_{i_1}) \setminus \{v_\gamma\}$ and $y \in N_D^-(v_{i_2}) \setminus \{v_\gamma\}$. To reach a contradiction, suppose $xy \in E(G)$ for some $x \in N_D^-(v_{i_1}) \setminus \{v_\gamma\}$ and $y \in N_D^-(v_{i_2}) \setminus \{v_\gamma\}$. Take $z \in N_D^+(x) \cap N_D^+(y)$. If $z \in [v_{i_1}^+, x^-]$, then $x \to z$ implies $v_{i_1}^- \to v_{i_1}^+$, i.e., $v_{i_1-1} \to v_{i_1+1}$, which contradicts v_{i_1} is a separating vertex. If $z \in [x^+, v_{i_1}]$, then $y \to z$ implies $y \to v_{i_2}^+$, i.e., $y \to v_{i_2+1}$, which contradicts the choice of v_{i_2}. Hence $xy \notin E(G)$ for any $x \in N_D^-(v_{i_1}) \setminus \{v_\gamma\}$ and $y \in N_D^-(v_{i_2}) \setminus \{v_\gamma\}$. However, this means $G - v_\gamma$ is not connected. It is impossible since G is hamiltonian. Thus $t \geq 3$.

Since v_{i_1} is a separating vertex and $t \geq 3$, by Lemma 3.2 (2), $N_D^-(v_{i_t}) \cap N_D^-(v_{i_1}) = \emptyset$. If $|N_D^-(v_{i_{\beta-1}}) \cap N_D^-(v_{i_\beta})| = 0$, i.e., $N_D^-(v_{i_{\beta-1}}) \cap N_D^-(v_{i_\beta}) = \emptyset$, then Lemma 3.2 (2) implies v_{i_β} is also a separating vertex, a contradiction. So $|N_D^-(v_{i_{\beta-1}}) \cap N_D^-(v_{i_\beta})| = 1$, say $v_\gamma \in N_D^-(v_{i_{\beta-1}}) \cap N_D^-(v_{i_\beta})$. Then $v_\gamma = v_{i_\beta}^-$ by Lemma 2.1 and there is no arc from $N_D^-(v_{i_{\beta-1}}) \cup \ldots \cup$

$(N_D^-(v_{i_{\beta+1}}) \setminus \{v_\gamma\})$ to v_{i_β}. We will reach a contradiction by showing $G - v_\gamma$ is not connected. Let

$$V_1 = N_D^-(v_{i_1}) \cup N_D^-(v_{i_2}) \cup \ldots \cup (N_D^-(v_{i_{\beta-1}}) \setminus \{v_\gamma\}),$$
$$V_2 = (N_D^-(v_{i_\beta}) \setminus \{v_\gamma\}) \cup N_D^-(v_{i_{\beta+1}}) \cup \ldots \cup N_D^-(v_{i_t}).$$

Since $t \geq 3$, V_1 or V_2 contains at least two round intervals and we may assume that V_1 does.

Claim 2. If $v_\gamma \in N_D^-(v_{i_j})$, then $j = \beta - 1$ or β.

Suppose $v_\gamma \in N_D^-(v_{i_j})$ for some $j \in \{1, 2, \ldots, t\} \setminus \{\beta, \beta - 1\}$. Note that $v_\gamma = v_{i_{\beta-1} - d_D^-(v_{i_{\beta-1}})}$. Then $v_\gamma \to v_{i_j}$ implies $N_D^-(v_{i_{\beta-1}}) \subseteq N_D^-(v_{i_j})$, which contradicts Lemma 3.1 (2). Thus $j = \beta - 1$ or β if $v_\gamma \in N_D^-(v_{i_j})$.

Claim 3. $V_1 \cap V_2 = \emptyset$.

Suppose $V_1 \cap V_2 \neq \emptyset$. Then there exist $k \in \{1, 2, \ldots, \beta - 1\}$ and $l \in \{\beta, \beta + 1, \ldots, t\}$ such that $(N_D^-(v_{i_k}) \setminus \{v_\gamma\}) \cap (N_D^-(v_{i_l}) \setminus \{v_\gamma\}) \neq \emptyset$, say $x \in (N_D^-(v_{i_k}) \setminus \{v_\gamma\}) \cap (N_D^-(v_{i_l}) \setminus \{v_\gamma\})$. Then $k = \beta - 1$, for otherwise, $x (\in N_D^-(v_{i_l})) \to v_{i_k}$ implies $v_\gamma \to v_{i_k}$ and hence $N_D^-(v_{i_{\beta-1}}) \subseteq N_D^-(v_{i_k})$, a contradiction to Lemma 3.1 (2). Similarly, $l = \beta$, for otherwise $N_D^-(v_{i_k}) \cap N_D^-(v_{i_l}) = \emptyset$, a contradiction. Now we have $x \in (N_D^-(v_{i_{\beta-1}}) \setminus \{v_\gamma\}) \cap (N_D^-(v_{i_\beta}) \setminus \{v_\gamma\})$. However, by the proof of Claim 1, we see that $N_D^-(v_{i_{\beta-1}}) \cap N_D^-(v_{i_\beta}) = \{v_\gamma\}$ and hence $(N_D^-(v_{i_{\beta-1}}) \setminus \{v_\gamma\}) \cap (N_D^-(v_{i_\beta}) \setminus \{v_\gamma\}) = \emptyset$, a contradiction. Thus $V_1 \cap V_2 = \emptyset$.

Claim 4. x and y are not adjacent in G for any $x \in V_1$, $y \in V_2$.

Suppose not. There are $x \in V_1$, $y \in V_2$ with $xy \in E(G)$. Then there is $z \in N_D^+(x) \cap N_D^+(y)$. If $z \in [v_{i_1}^+, x^-]$, then $x \to z$ implies $v_{i_1}^- \to v_{i_1}^+$, i.e., $v_{i_1 - 1} \to v_{i_1 + 1}$, which contradicts v_{i_1} is a separating vertex. Thus $z \in [x^+, v_{i_1}]$. Then, since $x \to z$, $y \to z$, and $v_{i_\beta}^+ \in [y, x]$, we have $y \to v_{i_\beta}^+$ by Lemma 2.1. If $v_{i_\beta}^+ \neq v_{i_{\beta-1}}$, $y \to v_{i_\beta}^+$ and $y \nrightarrow v_{i_{\beta-1}}$ contradict the choice of v_{i_β}. If $v_{i_\beta}^+ = v_{i_{\beta-1}}$, $y \to v_{i_\beta}^+$ implies $N_D^-(v_{i_\beta}) \subseteq N_D^-(v_{i_{\beta-1}})$, a contradiction. Thus $xy \notin E(G)$ for any $x \in V_1$, $y \in V_2$.

We have shown that $G - v_\gamma$ is not connected. The proof of the necessity is completed.

" $\Leftarrow$ ". If D contains no separating vertex, then $v_{k-1} \to v_{k+1}$ for $k \in \{1, 2, \ldots, n\}$ and $v_{k-1}v_k$ is an edge of G for $k \in \{1, 2, \ldots, n\}$ by Lemma 2.2. Thus $v_0 v_1 \ldots v_n v_0$ is a hamiltonian cycle of G.

Now we consider the case when G contains exactly one separating vertex v_{i_1}. By the assumption, we have $|N_D^-(v_{i_{j-1}}) \cap N_D^-(v_{i_j})| \geq 2$ for $j \in \{2, 3, \ldots, t\}$. If $t = 2$, then G has exactly two round intervals $N_D^-(v_{i_1})$ and $N_D^-(v_{i_2})$. Let $v_k, v_l \in N_D^-(v_{i_1}) \cap N_D^-(v_{i_2})$ with $k < l$. Since $v_k, v_l \in N_D^-(v_{i_2})$, $k < l < i_2 < i_1$, and so $v_{i_1}^- \neq v_k, v_l$ and $v_{i_1} \neq v_k, v_l$. Then we have a cycle $C = v_{i_1}^- v_k v_l v_{i_2} v_{i_1}^-$ in G. Note that in the competition graph G of D, both subgraphs induced by $N_D^-(v_{i_1})$ and $N_D^-(v_{i_2})$ are complete graphs. So we can get the hamiltonian cycle C^* in G by inserting each vertex $v_\alpha \in V(G) \setminus V(C)$ to C one by one. In the following we consider the case $t \geq 3$.

Let $u_{j,j+1}^1$, $u_{j,j+1}^2$ be the first two vertices of $N_D^-(v_{i_j}) \cap N_D^-(v_{i_{j+1}})$ along the inverted round labelling of D. We get a sequence

$$u^1_{1,2}, u^2_{1,2}, u^1_{2,3}, u^2_{2,3}, \ldots, u^1_{j,j+1}, u^2_{j,j+1}, \ldots, u^1_{t-1,t}, u^2_{t-1,t}.$$

Clearly, this sequence is in the order of the inverted round labelling unless two consecutive vertices $u^2_{j,j+1}$ and $u^1_{j+1,j+2}$ are possibly the same for some $j \in \{1, 2, \ldots, t-2\}$. However, it is impossible that three consecutive vertices in the sequence are all the same. This means there must exist two vertices among three consecutive vertices are distinct. We delete one of $u^2_{j,j+1}$ and $u^1_{j+1,j+2}$ if these two vertices are the same. Now we get a new sequence $u_1, u_2, \ldots, u_s$, $t \le s \le 2(t-1)$. Since $t \ge 3$, we have $s \ge 3$.

Claim 5. For any three consecutive vertices u_i, u_{i+1}, u_{i+2} in the sequence $u_1, u_2, \ldots, u_s$, there exits a round interval $N^-_D(v_{i_k})$ such that $u_i, u_{i+1}, u_{i+2} \in N^-(v_{i_k})$.

According to the construction of $u_1, u_2, \ldots, u_s$, we consider the following cases.

- If $u_i = u^1_{j,j+1}$, $u_{i+1} = u^2_{j,j+1}$, $u_{i+2} = u^1_{j+1,j+2}$, then $u_i, u_{i+1}, u_{i+2} \in N^-_D(v_{i_{j+1}})$;
- If $u_i = u^1_{j,j+1}$, $u_{i+1} = u^2_{j,j+1} = u^1_{j+1,j+2}$, $u_{i+2} = u^2_{j+1,j+2}$, then $u_i, u_{i+1}, u_{i+2} \in N^-_D(v_{i_{j+1}})$;
- If $u_i = u^2_{j,j+1}$, $u_{i+1} = u^1_{j+1,j+2}$, $u_{i+2} = u^2_{j+1,j+2}$, then $u_i, u_{i+1}, u_{i+2} \in N^-_D(v_{i_{j+1}})$;
- If $u_i = u^2_{j,j+1} = u^1_{j+1,j+2}$, $u_{i+1} = u^2_{j+1,j+2}$, $u_{i+2} = u^1_{j+2,j+3}$, then $u_i, u_{i+1}, u_{i+2} \in N^-_D(v_{i_{j+2}})$;
- If $u_i = u^2_{j,j+1} = u^1_{j+1,j+2}$, $u_{i+1} = u^2_{j+1,j+2} = u^1_{j+2,j+3}$, $u_{i+2} = u^2_{j+2,j+3}$, then u_i, u_{i+1}, $u_{i+2} \in N^-_D(v_{i_{j+2}})$. □

In the following, we will construct a hamiltonian cycle in G.

First, we construct a cycle in G containing the vertices $u_1, u_2, \ldots, u_s$ and $v^-_{i_1}$. Clearly, $u_1 = u^1_{1,2}$ and $u_2 = u^2_{1,2}$. Moreover, $v^-_{i_1} \ne u_1, u_2$ and $v^-_{i_1}, u_1, u_2 \in N^-_D(v_{i_1})$. If $s = 2k$ is an even integer, set $C = v^-_{i_1} u_1 u_3 \ldots u_{2k-1} u_{2k} u_{2k-2} \ldots u_2 v^-_{i_1}$; If $s = 2k+1$ is an odd integer, set $C = v^-_{i_1} u_2 u_4 \ldots u_{2k} u_{2k+1} u_{2k-1} \ldots u_3 u_1 v^-_{i_1}$. Then C is the desired cycle.

Next, we construct the hamiltonian cycle C^* in G by inserting each vertex $v_\alpha \in V(G) \backslash V(C)$ to C one by one. This can be done since $v_\alpha \in N^-_D(v_{i_j})$ and each $G[N^-_D(v_{i_j})]$ is a complete subgraph of G for $j \in \{1, 2, \ldots, t\}$.

The proof of the sufficiency is completed. □

Theorem 3.4. *Let D be a connected but non-strong round digraph on n vertices and $v_0, v_1, \ldots, v_{n-1}$ be the round labelling of D. Let G be the competition graph of D and $N^-_D(v_{i_1}), N^-_D(v_{i_2}), \ldots, N^-_D(v_{i_t})$ be the round clique cover of D. Then $G - v_{n-1}$ is hamiltonian if and only if $|N^-_D(v_{i_{j-1}}) \cap N^-_D(v_{i_j})| \ge 2$ for any $j \in \{2, 3, \ldots, t\}$.*

Proof. We construct a digraph D' from D by deleting v_{n-1} and adding all arcs from each vertex of $N^-_D(v_{n-1})$ to v_0. Clearly, D' is also a round digraph and $v_0, v_1, \ldots, v_{n-2}$ is the round labelling of D'. Also, $N^-_{D'}(v_i) = N^-_D(v_i)$ for $i \in \{1, 2, \ldots, n-2\}$ and $N^-_{D'}(v_0) = N^-_D(v_{n-1})$. Let G' be the competition graph of D'. It is easy to check that $G - v_{n-1} = G'$.

Now $G-v_{n-1}$ is hamiltonian if and only if G' is hamiltonian. Note that D' is strong and has exactly one separating vertex v_0. Recall that $N_D^-(v_{i_1}) = N_D^-(v_{n-1})$ and $N_{D'}^-(v_{i_1}) = N_{D'}^-(v_0)$. Since $N_D^-(v_{i_1}), N_D^-(v_{i_2}), \ldots, N_D^-(v_{i_t})$ is the round clique cover of D, we have $N_{D'}^-(v_{i_1}), N_{D'}^-(v_{i_2}), \ldots, N_{D'}^-(v_{i_t})$ is the round clique cover of D'. By Theorem 3.3, G' is hamiltonian if and only if $|N_{D'}^-(v_{i_{j-1}}) \cap N_{D'}^-(v_{i_j})| \geq 2$ for any $j \in \{2, 3, \ldots, t\}$. In other words, G' is hamiltonian if and only if $|N_D^-(v_{i_{j-1}}) \cap N_D^-(v_{i_j})| \geq 2$ for any $j \in \{2, 3, \ldots, t\}$. Thus $G - v_{n-1}$ is hamiltonian if and only if $|N_D^-(v_{i_{j-1}}) \cap N_D^-(v_{i_j})| \geq 2$ for any $j \in \{2, 3, \ldots, t\}$. The Theorem holds. □

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