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Asymptotic periodic solutions of some generalized Burgers equations

Smriti Nath Ch. Srinivasa Rao

Abstract. In this paper, we construct asymptotic periodic solutions of some generalized Burgers equations using a perturbative approach. These large time asymptotics (constructed) are compared with relevant numerical solutions obtained by a finite difference scheme.

§1 Introduction

In this article, we study large time asymptotics for periodic solutions of some generalized Burgers equations, namely,

$$\text{GBE1} \qquad u_t + u^2 u_x + \frac{ju}{2t} = \epsilon u_{xx}, \ \ x \in \mathbb{R}, \ \ t > 0, \tag{1}$$

$$\text{GBE2} \qquad u_t + (au + bu^2)u_x = \epsilon u_{xx}, \ \ x \in \mathbb{R}, \ \ t > 0, \tag{2}$$

satisfying the initial-boundary conditions

$$u(x, t_0) = u_0(x), \ x \in \mathbb{R}, \tag{3}$$

$$u_0(0) = u_0(2\pi) = 0, \tag{4}$$

$$u(x, t) = u(x + 2\pi, t), \ x \in \mathbb{R}, \ t > 0, \tag{5}$$

where $\epsilon > 0$ is small and $j \geq 0$, $a > 0$, $b > 0$ are constants. The initial function $u_0(x)$ is continuous, periodic, and bounded on $\mathbb{R}$. We use $t_0 = 1$ for GBE1 (1) and $t_0 = 0$ for GBE2 (2).

In general, generalized Burgers equations are not exactly linearizable via Hopf-Cole like transformations (see Sachdev [18], Nimmo and Crighton [13]).

We follow closely the work of Sachdev *et al.* [21]. Sachdev *et al.* [21] constructed the large time asymptotic periodic solutions of the modified Burgers equation

$$u_t + u^n u_x = \frac{\delta}{2} u_{xx}, \ x \in \mathbb{R}, \ t > 0, \tag{6}$$

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subject to conditions

$$u(x,0) = A\sin x,\ x \in \mathbb{R}, \tag{7}$$

$$u(x,t) = u(x+2\pi,t),\ x \in \mathbb{R},\ t > 0. \tag{8}$$

Let $x_0(t)$ be the zero of the solution u of the initial value problem (6)-(8) and $x_0(0) = 0$. Their numerical study showed that $x_0(t) \equiv 0$ for $n = 3$ and x_0 is a function of t converging to a constant as $t \to \infty$ when $n = 2$. They used a perturbative approach to construct the asymptotic periodic solutions of (6)-(8) for $n = 2, 3$. The idea in this approach is to assume that the large time asymptotic behavior of the periodic solutions of (6)-(8) is described by the relevant solution of "linearized problem" and improve this solution of the linearized problem by incorporating the effect of nonlinear terms. This approach was inspired by the method of dominant balances (see Bender and Orszag [3]). One may refer to Sachdev *et al.* [20], Vaganan and Padmasekaran [26, 27] for the study of large time asymptotic periodic solutions of some other generalized Burgers equations.

Consider the convection-reaction-diffusion problem

$$v_t = v_{xx} + \epsilon(v^m)_x + v^p,\ 0 < x < L,\ t > 0, \tag{9}$$

$$v(0,t) = v(L,t) = 0,\ t \geq 0, \tag{10}$$

$$v(x,0) = v_0(x) \geq 0,\ 0 < x < L, \tag{11}$$

where $\epsilon > 0$, $m \geq p > 1$ and $v_0 \in L^\infty(0,L)$. The analysis of Chen *et al.* [4] showed that there exists $\epsilon_0 > 0$ such that the solutions of (9)-(11) decay exponentially to zero as $t \to \infty$ for all $\epsilon > \epsilon_0$. Levine *et al.* [10] investigated the large time behavior of positive solutions of the problem (9)-(11) with $1 \leq m < p$.

Tersenov [23] studied the initial boundary value problem

$$u_t + g(t,u)u_x + f(t,u) = \epsilon u_{xx},\ -l < x < l,\ 0 < t < T, \tag{12}$$

$$u(x,0) = \phi(x),\ -l \leq x \leq l, \tag{13}$$

$$u(\pm l,t) = 0,\ 0 < t < T, \tag{14}$$

where T and ϵ are positive constants. Assuming certain conditions on f, g, and ϕ, he proved the existence and uniqueness of global "classical" solution of the initial boundary value problem (12)-(14). He also showed that $u \to 0$ as $t \to \infty$. One may refer to Tersenov [24,25] for a related study.

Kato [9] studied the generalized Burgers equation

$$u_t + \left(\frac{b}{2}u^2 + \frac{c}{3}u^3\right)_x = u_{xx},\ x \in \mathbb{R},\ t > 0, \tag{15}$$

subject to the initial condition

$$u(x,0) = u_0(x),\ x \in \mathbb{R}, \tag{16}$$

$b \neq 0$, $c \in \mathbb{R}$. Under suitable conditions on u_0, Kato [9] proved that the optimal rate at which the solution of (15)-(16) converges to the self-similar solution of the Burgers equation is $t^{-1}\log(t)$ in $L^\infty(\mathbb{R})$ as $t \to \infty$. In the present paper, we construct the large time asymptotic periodic solutions of (15).

Oruç *et al.* [14] studied numerically the generalized Burgers equations

$$u_t + u^n u_x = \nu u_{xx},\ a < x < b,\ t > 0,$$
$$u_t + (V + u)u_x = \nu u_{xx},\ x \in \mathbb{R},\ t > 0,$$

satisfying the initial-boundary conditions

$$u(x, 0) = g(x),\ a < x < b,$$
$$u(a, t) = f_1(t),\ u(b, t) = f_2(t),\ t \in [0, T],$$

using Haar wavelet-finite difference method. Here, $\nu > 0$, $V > 0$ are constants and n is a positive integer. They compared their numerical solution with the solution obtained by Sachdev *et al.* [21] (see also Duan *et al.* [6, 7]).

Grundy *et al.* [8] constructed large time asymptotic solutions of the generalized Burgers equation

$$u_t = \delta u_{xx} - \left(u^{\alpha+1}\right)_x - \frac{Ju}{2t},\ J > 0,\ \alpha > 0,$$

with non-negative initial profile, when either $\alpha < 1/(J + 1)$ or $\alpha > 1/(J + 1)$, via balancing arguments. It was assumed that the initial profile is bounded on $(-\infty, \infty)$. Further the initial profile has compact support or it vanishes sufficiently rapidly as $|x| \to \infty$ (for example, like exponential decay). Rao and Satyanarayana [17] constructed large time asymptotic N-wave solutions for $\alpha = 1$ using a balancing argument.

Pocheketa *et al.* [16] investigated classification of Lie symmetries and constructed exact solutions of variable coefficient generalized Burgers equations

$$u_t + u^n u_x + h(t)u = g(t)u_{xx},\ ng \neq 0.$$

Here h and g are smooth functions.

Mishra and Kumar [11] obtained exact solitary wave solutions of the nonlinear convection-reaction-diffusion equations

$$C_t + k(t)CC_x = DC_{xx} + \alpha C - \beta C^2,$$
$$C_t + k(t)C^2 C_x = DC_{xx} + \alpha C - \beta C^4,$$

when (i) k is a constant, (ii) k is a function of t.

The organization of this paper is as follows. In section 2–3, we construct large time asymptotics for periodic solutions of the generalized Burgers equations (1)-(2). In section 4, we validate these asymptotic solutions with the relevant numerical solutions obtained via a finite difference scheme due to Dawson [5]. Section 5 puts forward the conclusions.

§2 Large time asymptotics for periodic solutions of GBE1

This section presents the large time asymptotic behavior of periodic solutions of the generalized Burgers equation GBE1 (1) subject to conditions (3)-(5). We assume that the 2π-periodic solution

$$u(x, t) = A_1 t^{-j/2} e^{-\epsilon t} \sin(x - x_0) \tag{17}$$

of the linear partial differential equation

$$u_t + \frac{ju}{2t} = \epsilon u_{xx} \tag{18}$$

describes the large time asymptotic behavior of 2π-periodic solutions of GBE1 (1) satisfying conditions (3)-(5). We are going to improve upon the large time asymptotic solution (17) using a perturbative approach. This idea was quite neatly presented in Bender and Orszag [3] (p. 146) for nonlinear ordinary differential equations and Sachdev [19] (p. 257) for nonlinear partial differential equations.

Our numerical study of GBE1 (1) subject to conditions (3)-(5) showed the movement of the zeros of the solution of GBE1 (1) and their stabilization as $t \to \infty$. This observation is an expected one in view of the study of GBE1 (1) with $j = 0$ presented in Sachdev *et al.* [21]. In view of this discussion, we attempt a solution $u(x,t)$ of GBE1 (1) in the form

$$u(x,t) = A_1 t^{-j/2} e^{-\epsilon t} \sin(x - \tilde{x}_0(t)) + U_1(x,t), \tag{19}$$

$\tilde{x}_0(t) \to x_0$ as $t \to \infty$.
We note that $\tilde{x}_0(t)$ is the zero of the solution u of GBE1 (1) satisfying $\tilde{x}_0(t_0) = 0$. In other words, $\tilde{x}_0(t)$ describes the movement of the zero $x = 0$ of $u(x,t_0)$ as the initial profile $u(x,t_0)$ evolves under GBE1 for time $t > t_0$.

Let

$$\tilde{x}_0(t) = x_0 + x_1(t)e^{-2\epsilon t} + x_2(t)e^{-4\epsilon t} + x_3(t)e^{-6\epsilon t} + \cdots . \tag{20}$$

This form is inspired by the work of Sachdev *et al.* [21]. Using (19) into (1) and ignoring the higher order terms, we have

$$\begin{aligned} U_{1,t} + \frac{j}{2t}U_1 - \epsilon U_{1,xx} \sim & A_1 t^{-j/2} e^{-\epsilon t} \tilde{x}_0'(t) \cos(x - \tilde{x}_0(t)) \\ & - \frac{A_1^3}{4} t^{-3j/2} e^{-3\epsilon t} \left[\cos(x - \tilde{x}_0(t)) - \cos 3(x - \tilde{x}_0(t))\right] \end{aligned} \tag{21}$$

as $t \to \infty$. Equations (20) and (21) imply that

$$\begin{aligned} U_{1,t} + \frac{j}{2t}U_1 - \epsilon U_{1,xx} \sim & e^{-3\epsilon t}\Bigg[\left(A_1(x_1' - 2\epsilon x_1) - \frac{A_1^3}{4} t^{-j}\right) t^{-j/2} \cos(x - \tilde{x}_0(t)) \\ & + \frac{A_1^3}{4} t^{-3j/2} \cos 3(x - \tilde{x}_0(t))\Bigg] \text{ as } t \to \infty. \end{aligned} \tag{22}$$

Motivated by the right hand side of (22), we assume the form for U_1 as

$$U_1(x,t) \sim e^{-3\epsilon t}\left[c_1(t)\cos(x - \tilde{x}_0(t)) + c_2(t)\cos 3(x - \tilde{x}_0(t))\right] \text{ as } t \to \infty. \tag{23}$$

Using (23) in (22) and comparing the coefficients of $\cos(x - \tilde{x}_0(t))$ and $\cos 3(x - \tilde{x}_0(t))$, we have

$$c_1' + \left(\frac{j}{2t} - 2\epsilon\right) c_1 = A_1\left(x_1' - 2\epsilon x_1\right) t^{-j/2} - \frac{A_1^3}{4} t^{-3j/2}, \tag{24}$$

$$c_2' + \left(\frac{j}{2t} + 6\epsilon\right) c_2 = \frac{A_1^3}{4} t^{-3j/2}. \tag{25}$$

Let

$$c_2(t) = t^{-3j/2} \sum_{n=0}^{\infty} a_n t^{-n}. \tag{26}$$

Using (26) in (25) and comparing same powers of t, we obtain

$$a_0 = \frac{A_1^3}{24\epsilon}, \tag{27}$$

$$6\epsilon a_n = (j+n-1)a_{n-1}, n \geq 1. \tag{28}$$

It is important to note that the asymptotic series of form (26) is useful for sufficiently large time and an appropriate number of terms have to be used for getting accurate results. An interesting discussion on an asymptotic series can be seen in Moritz [12].
We require that

$$u(\tilde{x}_0(t), t) = u(\tilde{x}_0(t) + 2\pi, t) = 0,$$

and thus

$$U_1(\tilde{x}_0(t), t) = 0. \tag{29}$$

This, in turn, implies that $c_1(t) + c_2(t) = 0$. Thus

$$c_1(t) = -t^{-3j/2} \sum_{n=0}^{\infty} a_n t^{-n}. \tag{30}$$

Equations (30) and (24) give the differential equation for x_1:

$$x_1' - 2\epsilon x_1 = \frac{8\epsilon}{A_1} t^{-j} \sum_{n=0}^{\infty} a_n t^{-n}. \tag{31}$$

The right hand side of equation (31) suggests a particular solution for (31) of the form

$$x_1(t) = t^{-j} \sum_{n=0}^{\infty} b_n t^{-n}. \tag{32}$$

Substituting expression (32) for x_1 in (31) and comparing the same powers of t, we arrive at

$$b_0 = -\frac{A_1^2}{6\epsilon},$$

$$2\epsilon A_1 b_n = -A_1(n-1+j)b_{n-1} - 8\epsilon a_n, n \geq 1.$$

Thus $\tilde{x}_0(t)$ takes the following form, as $t \to \infty$,

$$\tilde{x}_0(t) = x_0 + e^{-2\epsilon t} t^{-j} \sum_{n=0}^{\infty} b_n t^{-n} + o(e^{-4\epsilon t}). \tag{33}$$

Therefore the large time asymptotic solution $u(x,t)$ of (1) takes the form, as $t \to \infty$,

$$\begin{aligned} u(x,t) = {} & A_1 t^{-j/2} e^{-\epsilon t} \sin(x - \tilde{x}_0(t)) \\ & + e^{-3\epsilon t} \left[c_1(t) \cos(x - \tilde{x}_0(t)) + c_2(t) \cos 3(x - \tilde{x}_0(t)) \right] + \cdots , \end{aligned} \tag{34}$$

where $c_1(t)$, $c_2(t)$, and $\tilde{x}_0(t)$ are given in equations (30), (26), and (33), respectively. Here A_1 and x_0 are unknowns and may be found numerically or otherwise.

Inspired by form (34) of the large time asymptotic solution of GBE1 (1), we make the change of variable $y = x - \tilde{x}_0(t)$. Then the partial differential equation GBE1 (1) in new variable y is given by

$$u_t - \tilde{x}_0'(t) u_y + \frac{ju}{2t} + u^2 u_y = \epsilon u_{yy}. \tag{35}$$

In view of (34), we attempt the large time asymptotic solution of (1) in the form

$$u(y,t) = e^{-\epsilon t} v_0(y,t) + e^{-3\epsilon t} v_1(y,t) + e^{-5\epsilon t} v_2(y,t) + \cdots . \tag{36}$$

Substituting expression (36) for u in the partial differential equation (35) and comparing the coefficients of $e^{-\epsilon t}$, $e^{-3\epsilon t}$, and $e^{-5\epsilon t}$, we get the following partial differential equations for v_0, v_1, and v_2 :

$$
\begin{aligned}
&v_{0,t} - \epsilon v_0 + \frac{j}{2t} v_0 = \epsilon v_{0,yy}, \\
&v_{1,t} - 3\epsilon v_1 - (x_1' - 2\epsilon x_1) v_{0,y} + v_0^2 v_{0,y} + \frac{j}{2t} v_1 = \epsilon v_{1,yy}, \\
&v_{2,t} - 5\epsilon v_2 - [(x_1' - 2\epsilon x_1) v_{1,y} + (x_2' - 4\epsilon x_2) v_{0,y}] + v_0^2 v_{1,y} + 2 v_0 v_1 v_{0,y} + \frac{j}{2t} v_2 = \epsilon v_{2,yy}. \qquad (37)
\end{aligned}
$$

In view of equation (34), we have

$$v_0(y,t) = A_1 t^{-j/2} \sin y, \qquad (38)$$

$$v_1(y,t) = c_1(t) \cos y + c_2(t) \cos 3y, \qquad (39)$$

where $c_1(t)$ and $c_2(t)$ are given by (30) and (26), respectively. Here A_1 is an unknown constant and is referred to as oldage constant. The oldage constant is the constant appearing in the large time (oldage) solution of the initial boundary value problem. Use of forms (38)-(39) for v_0 and v_1 in (37) suggests the following form of solution for v_2:

$$v_2(y,t) = c_3(t) \sin y + c_4(t) \sin 3y + c_5(t) \sin 5y. \qquad (40)$$

Equations (37)-(40) lead to the following differential equations for x_2, c_3, c_4, and c_5:

$$x_2' - 4\epsilon x_2 = 0, \qquad (41)$$

$$c_3' + \left(\frac{j}{2t} - 4\epsilon\right) c_3 = -c_1(x_1' - 2\epsilon x_1) + \frac{A_1^2}{2} t^{-j} c_1, \qquad (42)$$

$$c_4' + \left(\frac{j}{2t} + 4\epsilon\right) c_4 = 3c_1(x_1' - 2\epsilon x_1) - \frac{9}{4} A_1^2 t^{-j} c_1, \qquad (43)$$

$$c_5' + \left(\frac{j}{2t} + 20\epsilon\right) c_5 = \frac{5A_1^2}{4} t^{-j} c_1. \qquad (44)$$

Equation (41) gives that $x_2(t) = 0$, otherwise $x_2(t)$ becomes unbounded.
Equations (30), (32), and (42) imply that

$$c_3' + \left(\frac{j}{2t} - 4\epsilon\right) c_3 = \frac{8\epsilon}{A_1} t^{-5j/2} \left(\sum_{n=0}^{\infty} a_n t^{-n}\right)^2 - \frac{A_1^2}{2} t^{-5j/2} \sum_{n=0}^{\infty} a_n t^{-n}. \qquad (45)$$

The right hand side of (45) suggests the form

$$c_3(t) = t^{-5j/2} \sum_{n=0}^{\infty} B_n t^{-n}. \qquad (46)$$

Substituting expression (46) for c_3 in (45) and comparing the same powers of t, we arrive at the following equations for B_i, $i \geq 0$:

$$4\epsilon B_0 = -\frac{8\epsilon}{A_1} a_0^2 + \frac{A_1^2}{2} a_0, \qquad (47)$$

$$4\epsilon B_n + [2j + n - 1] B_{n-1} = -\frac{8\epsilon}{A_1} \sum_{k=0}^{n} a_k a_{n-k} + \frac{A_1^2}{2} a_n, n \geq 1. \qquad (48)$$

Equations (30), (32), and (43) imply that

$$c_4' + \left(\frac{j}{2t} + 4\epsilon\right) c_4 = -\frac{24\epsilon}{A_1} t^{-5j/2} \left(\sum_{n=0}^{\infty} a_n t^{-n}\right)^2 + \frac{9A_1^2}{4} t^{-5j/2} \sum_{n=0}^{\infty} a_n t^{-n}. \tag{49}$$

Substituting

$$c_4(t) = t^{-5j/2} \sum_{n=0}^{\infty} E_n t^{-n} \tag{50}$$

in (49) and comparing the coefficients of same powers of t, we arrive at the following equations for E_i, $i \geq 0$:

$$4\epsilon E_0 = -\frac{24\epsilon}{A_1} a_0^2 + \frac{9A_1^2}{4} a_0, \tag{51}$$

$$4\epsilon E_n - [2j + n - 1]\, E_{n-1} = -\frac{24\epsilon}{A_1} \sum_{k=0}^{n} a_k a_{n-k} + \frac{9A_1^2}{4} a_n, n \geq 1. \tag{52}$$

Again substituting expression (30) for c_1 and

$$c_5(t) = t^{-5j/2} \sum_{n=0}^{\infty} F_n t^{-n}, \tag{53}$$

in (44) and comparing the coefficients of same powers of t, we obtain the following equations for F_i, $i \geq 0$:

$$F_0 = -\frac{A_1^2}{16\epsilon} a_0, \tag{54}$$

$$20\epsilon F_n - [2j + n - 1]\, F_{n-1} = -\frac{5A_1^2}{4} a_n, n \geq 1. \tag{55}$$

Thus, the large time 2π-periodic asymptotic solution $u(x,t)$ of (1) satisfying (3)-(5) is given by

$$\begin{aligned} u(x,t) =& A_1 t^{-j/2} e^{-\epsilon t} \sin(x - \tilde{x}_0(t)) + e^{-3\epsilon t}[c_1(t)\cos(x - \tilde{x}_0(t)) + c_2(t)\cos 3(x - \tilde{x}_0(t))] \\ &+ e^{-5\epsilon t}[c_3(t)\sin(x - \tilde{x}_0(t)) + c_4(t)\sin 3(x - \tilde{x}_0(t)) + c_5(t)\sin 5(x - \tilde{x}_0(t))] + \cdots, \end{aligned} \tag{56}$$

$$\tilde{x}_0(t) = x_0 + e^{-2\epsilon t} t^{-j} \sum_{n=0}^{\infty} b_n t^{-n} + O(e^{-6\epsilon t})$$

where $c_i(t)$, $i = 1, 2, \ldots, 5$ are given by (30), (26)-(28), (46)-(48), (50)-(52), (53)-(55), respectively.

We present below c_1, c_2, and x_1 in terms of integrals. For $j = 1$, the functions c_1, c_2, and x_1 can be written in terms of a special function called Exponential integral $Ei(x)$.

The functions c_1, c_2, and x_1 in terms of integrals for general j:

The general solution of equation (25) is

$$c_2(t) = \frac{A_1^3}{4} t^{-j/2} e^{-6\epsilon t} \int^t z^{-j} e^{6\epsilon z} dz + \tilde{c} t^{-j/2} e^{-6\epsilon t}, \tag{57}$$

$\tilde{c}$ is an integration constant. One may use integration by parts (see Pinsky [15] p. 350) to show that

$$\int^t z^{-j} e^{6\epsilon z} dz = e^{6\epsilon t} t^{-j} \left[\frac{1}{6\epsilon} + \frac{j}{36\epsilon^2} t^{-1} + O(t^{-2})\right], t \to \infty. \tag{58}$$

Using (58) in (57) and looking for the asymptotic behavior of c_2 as $t \to \infty$, we have

$$c_2(t) \sim t^{-3j/2}\left[\frac{A_1^3}{24\epsilon} + \frac{jA_1^3}{144\epsilon^2}t^{-1} + O(t^{-2})\right]. \tag{59}$$

This is in agreement with the form of solution for c_2 given in (26). Because of the asymptotic behavior (59) as $t \to \infty$, we may write

$$c_2(t) \sim \frac{A_1^3}{4}t^{-j/2}e^{-6\epsilon t}\int^t z^{-j}e^{6\epsilon z}dz.$$

Equation (29) implies that

$$c_1(t) = -c_2(t) \sim -\frac{A_1^3}{4}t^{-j/2}e^{-6\epsilon t}\int^t z^{-j}e^{6\epsilon z}dz. \tag{60}$$

Using (60) in (24) and solving for x_1, we get

$$x_1(t) \sim -\frac{8\epsilon}{A_1}e^{2\epsilon t}\int^t c_1(z)z^{j/2}e^{-2\epsilon z}dz.$$

The functions c_1, c_2, and x_1 in terms of Exponential integrals for $j = 1$:

$$-c_1(t) = c_2(t) \sim \frac{A_1^3}{4\sqrt{t}}e^{-6\epsilon t}Ei(6\epsilon t) \text{ as } t \to \infty \tag{61}$$

and $x_1(t)$ is given by

$$x_1(t) \sim \frac{A_1^2}{4}\left(e^{2\epsilon t}Ei(-2\epsilon t) - e^{-6\epsilon t}Ei(6\epsilon t)\right) \text{ as } t \to \infty.$$

Here

$$Ei(t) := \int_{-\infty}^{t}\frac{e^x}{x}dx,\ t \neq 0$$

is called the Exponential integral and for $t > 0$ the value of the integral is given by the Cauchy principal value (see Andrews [2] p. 103 and Abramowitz and Stegun [1] p. 228).

Thus, we have, for $j = 1$,

$$u(x,t) = A_1t^{-1/2}e^{-\epsilon t}\sin(x - \tilde{x}_0(t)) + e^{-3\epsilon t}[c_1(t)\cos(x - \tilde{x}_0(t)) + c_2(t)\cos 3(x - \tilde{x}_0(t))] + \cdots, \tag{62}$$

$$\tilde{x}_0(t) \sim x_0 + \frac{A_1^2}{4}\left(e^{2\epsilon t}Ei(-2\epsilon t) - e^{-6\epsilon t}Ei(6\epsilon t)\right)e^{-2\epsilon t} + \cdots, \tag{63}$$

where c_1 and c_2 are given by (61).

An attempt was made by Vaganan and Padmasekaran [28] to construct asymptotic periodic solutions of the generalized Burgers equation GBE1. However their (constructed) asymptotic solutions have singularities at a finite time. This discrepancy was due to the choice of c_i's as constants.

§3 Large time behavior of periodic solutions of GBE2

This section presents the large time asymptotics for periodic solutions of the generalized Burgers equation GBE2 (2) satisfying conditions (3)-(5). Again as in Section 2, we improve upon the solution of the corresponding "linearized" partial differential equation of (2) by considering the effect of nonlinear terms in (2). We assume that the large time behavior of solutions of

GBE2 is described by the solution

$$u_{\mathrm{L}}(x,t) = A_1 e^{-\epsilon t}\sin(x - x_0) \tag{64}$$

of the linear partial differential equation

$$u_t = \epsilon u_{xx} \tag{65}$$

satisfying conditions (3)-(5).

Let

$$u(x,t) = A_1 e^{-\epsilon t}\sin(x - \tilde{x}_0(t)) + U_1(x,t), \tag{66}$$

$$\tilde{x}_0(t) = x_0 + x_1(t)e^{-\epsilon t} + x_2(t)e^{-2\epsilon t} + x_3(t)e^{-3\epsilon t} + x_4(t)e^{-4\epsilon t} + \cdots . \tag{67}$$

Our assumption is that

$$U_1(x,t) \ll O(e^{-\epsilon t}) \text{ as } t \to \infty$$

and the form used for $\tilde{x}_0(t)$ is motivated by Sachdev *et al.* [21].
Using (66)-(67) in (2) and ignoring the higher order terms, we have

$$U_{1,t} - \epsilon U_{1,xx} \sim e^{-2\epsilon t}\left[A_1(x_1' - \epsilon x_1)\cos(x - \tilde{x}_0(t)) - \frac{aA_1^2}{2}\sin 2(x - \tilde{x}_0(t))\right] \tag{68}$$

as $t \to \infty$. Because we require that $U_1(\tilde{x}_0(t), t) = 0$, $x_1(t)$ has to be identically equal to zero. Inspired by the right hand side of (68), we attempt U_1 in the form

$$U_1(x,t) \sim e^{-2\epsilon t}c_1(t)\sin 2(x - \tilde{x}_0(t)) \text{ as } t \to \infty. \tag{69}$$

Equations (68)-(69) imply that

$$c_1' + 2\epsilon c_1 \sim -\frac{aA_1^2}{2} \text{ as } t \to \infty.$$

This implies that

$$c_1(t) \sim -\frac{aA_1^2}{4\epsilon} \text{ as } t \to \infty. \tag{70}$$

Thus

$$U_1(x,t) \sim -\frac{aA_1^2}{4\epsilon}e^{-2\epsilon t}\sin 2(x - \tilde{x}_0(t)) \text{ as } t \to \infty$$

and

$$u(x,t) \sim A_1 e^{-\epsilon t}\sin(x - \tilde{x}_0(t)) + e^{-2\epsilon t}c_1(t)\sin 2(x - \tilde{x}_0(t)) \text{ as } t \to \infty. \tag{71}$$

Here c_1 is as in (70).

Let us find the higher order terms as $t \to \infty$. Define

$$y = x - \tilde{x}_0(t).$$

The generalized Burgers equation GBE2 given in (2) transforms to the partial differential equation

$$u_t + \left(-\tilde{x}_0'(t) + au + bu^2\right)u_y = \epsilon u_{yy}. \tag{72}$$

The form for u given in (71) suggests choose $u(y,t)$ in the form:

$$u(y,t) = e^{-\epsilon t}f_1(y,t) + e^{-2\epsilon t}f_2(y,t) + e^{-3\epsilon t}f_3(y,t) + e^{-4\epsilon t}f_4(y,t) + \cdots \tag{73}$$

as $t \to \infty$. Using the form (73) for u in (72) and comparing the coefficients of $e^{-\epsilon t}$, $e^{-2\epsilon t}$, $e^{-3\epsilon t}$, and $e^{-4\epsilon t}$, we get:

$$f_{1,t} - \epsilon f_1 = \epsilon f_{1,yy},$$

$$f_{2,t} - 2\epsilon f_2 - \epsilon f_{2,yy} = (x_1' - \epsilon x_1)\, f_{1,y} - af_1 f_{1,y},$$

$$f_{3,t} - 3\epsilon f_3 - \epsilon f_{3,yy} = (x_1' - \epsilon x_1) f_{2,y} + (x_2' - 2\epsilon x_2) f_{1,y} - a (f_1 f_{2,y} + f_2 f_{1,y}) - b f_1^2 f_{1,y}, \quad (74)$$

$$\begin{aligned} f_{4,t} - 4\epsilon f_4 - \epsilon f_{4,yy} = {} & (x_1' - \epsilon x_1) f_{3,y} + (x_2' - 2\epsilon x_2) f_{2,y} + (x_3' - 3\epsilon x_3) f_{1,y} \\ & - a (f_1 f_{3,y} + f_2 f_{2,y} + f_3 f_{1,y}) - b \left(f_1^2 f_{2,y} + 2 f_1 f_2 f_{1,y}\right). \end{aligned} \quad (75)$$

It is easy to see that, in view of (71),

$$f_1(y,t) = A_1 \sin y, \quad (76)$$

$$f_2(y,t) = c_1 \sin 2y, \; c_1 = -\frac{aA_1^2}{4\epsilon}, \quad (77)$$

$$x_1 = 0. \quad (78)$$

Using (76)-(78) in (74), we have

$$\begin{aligned} f_{3,t} - 3\epsilon f_3 - \epsilon f_{3,yy} = {} & (x_2' - 2\epsilon x_2) A_1 \cos y - a \left(\frac{3A_1 c_1}{2} \sin 3y \right. \\ & \left. - \frac{A_1 c_1}{2} \sin y \right) + \frac{bA_1^3}{4} (\cos 3y - \cos y). \end{aligned} \quad (79)$$

The right hand side of (79) suggests the form:

$$f_3(y,t) = B_1(t) \cos y + B_2(t) \cos 3y + B_3(t) \sin y + B_4(t) \sin 3y. \quad (80)$$

Equations (79)-(80) imply that

$$B_1' - 2\epsilon B_1 = (x_2' - 2\epsilon x_2) A_1 - \frac{bA_1^3}{4}, \quad (81)$$

$$B_2' + 6\epsilon B_2 = \frac{bA_1^3}{4}, \quad (82)$$

$$B_3' - 2\epsilon B_3 = \frac{aA_1 c_1}{2}, \quad (83)$$

$$B_4' + 6\epsilon B_4 = -\frac{3aA_1 c_1}{2}. \quad (84)$$

The requirement $f_3(0,t) = 0$ gives

$$B_2(t) = -B_1(t).$$

Solving (81)-(84),

$$B_2(t) = \frac{bA_1^3}{24\epsilon} = -B_1(t), \quad (85)$$

$$B_3(t) = B_4(t) = -\frac{aA_1 c_1}{4\epsilon}.$$

Using (85) in (81), we obtain an ordinary differential equation for x_2:

$$x_2' - 2\epsilon x_2 = \frac{bA_1^2}{3}. \quad (86)$$

The relevant solution of (86) is

$$x_2(t) = -\frac{bA_1^2}{6\epsilon}. \quad (87)$$

Thus

$$f_3(y,t) = \frac{bA_1^3}{24\epsilon} (-\cos y + \cos 3y) + \frac{a^2 A_1^3}{16\epsilon^2} (\sin y + \sin 3y). \quad (88)$$

Using the expressions for f_i, $i=1,2,3$ in (75), we get

$$f_{4,t}-4\epsilon f_4-\epsilon f_{4,yy}=A_1\left(x_3'-3\epsilon x_3\right)\cos y+\frac{abA_1^4}{6\epsilon}\cos 2y \\ -\frac{abA_1^4}{3\epsilon}\cos 4y-\frac{3a^3A_1^4}{16\epsilon^2}\sin 4y. \tag{89}$$

Let

$$f_4(y,t)=R_1(t)\cos y+R_2(t)\cos 2y+R_3(t)\cos 4y+R_4(t)\sin 4y. \tag{90}$$

Use of (90) in (89) leads to the following equations for R_i, $i=1,2,3,4$.

$$R_1'-3\epsilon R_1=A_1\left(x_3'-3\epsilon x_3\right), \tag{91}$$

$$R_2'=\frac{abA_1^4}{6\epsilon}, \tag{92}$$

$$R_3'+12\epsilon R_3=-\frac{abA_1^4}{3\epsilon}, \tag{93}$$

$$R_4'+12\epsilon R_4=-\frac{3a^3A_1^4}{16\epsilon^2}. \tag{94}$$

Solving equations (92)-(94) for relevant solutions, we have

$$R_2(t)=\frac{abA_1^4}{6\epsilon}t,$$

$$R_3(t)=-\frac{abA_1^4}{36\epsilon^2},$$

$$R_4(t)=-\frac{a^3A_1^4}{64\epsilon^3}.$$

Thus

$$f_4(y,t)=R_1(t)\cos y+\frac{abA_1^4}{6\epsilon}t\cos 2y-\frac{abA_1^4}{36\epsilon^2}\cos 4y-\frac{a^3A_1^4}{64\epsilon^3}\sin 4y. \tag{95}$$

We require that $f_4(0,t)=0$. This condition on f_4, in turn, implies that

$$R_1(t)=-\frac{abA_1^4}{6\epsilon}\left(t-\frac{1}{6\epsilon}\right). \tag{96}$$

Using equations (91) and (96), we get

$$x_3'-3\epsilon x_3=\frac{abA_1^3}{2}\left(t-\frac{1}{2\epsilon}\right). \tag{97}$$

Solving (97) for x_3,

$$x_3(t)=-\frac{abA_1^3}{6\epsilon}\left(t-\frac{1}{6\epsilon}\right). \tag{98}$$

Thus,

$$u(x,t)=e^{-\epsilon t}f_1(y,t)+e^{-2\epsilon t}f_2(y,t)+e^{-3\epsilon t}f_3(y,t)+e^{-4\epsilon t}f_4(y,t)+\cdots, \tag{99}$$

$$\tilde{x}_0(t)=x_0+x_1(t)e^{-\epsilon t}+x_2(t)e^{-2\epsilon t}+x_3(t)e^{-3\epsilon t}+O\left(e^{-4\epsilon t}\right), \tag{100}$$

where f_i, $i=1,2,3,4$ and x_i, $i=1,2,3$ are given by (76)-(78), (87)-(88), (95)-(96), (98), is the large time asymptotic solution for the periodic solutions of GBE2.

§4 Numerical study

In this section, we compare the large time asymptotics obtained for generalized Burgers equations GBE1 and GBE2 (1)-(2) with numerical solutions of these generalized Burgers equations satisfying conditions given in (3)-(5). The relevant numerical solutions are obtained using a finite difference scheme due to Dawson [5]. We choose the spatial domain to be $[0, 2\pi]$ with 2001 spatial points and $\Delta x = 2\pi/2000$, $\Delta t = 10^{-4}$. The constants A_1 and x_0 are found at a time when numerical solution of a generalized Burgers equation and the solution of the relevant linear partial differential equation have maximum error $O(10^{-4})$. We consider two different initial profiles

$$u_0(x) = A_0 \sin x \tag{101}$$

and

$$u_0(x) = \begin{cases} \dfrac{2x}{\pi}, & x \in \left[0, \dfrac{\pi}{2}\right] \\ -\dfrac{2x}{\pi} + 2, & x \in \left[\dfrac{\pi}{2}, \dfrac{3\pi}{2}\right] \\ \dfrac{2x}{\pi} - 4, & x \in \left[\dfrac{3\pi}{2}, 2\pi\right]. \end{cases} \tag{102}$$

It may be noted that we solve GBE1 (1) subject to the initial profile at $t = 1$ and GBE2 (2) with initial profile at $t = 0$.

Figure 1 shows two initial profiles (101) with $A_0 = 1$ and (102). It should be noted that x_0 is the limit of $\tilde{x}_0(t)$ and $\tilde{x}_0(t_0) = 0$.

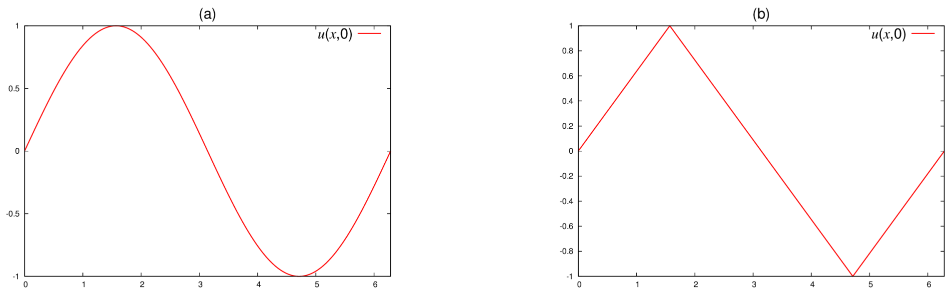

Figure 1: (a) Initial profile (101); (b) Initial profile (102).

Figures 2-4 and Tables 1-3 (pp. 405, 406) present the comparison of asymptotic solutions which are newly constructed, solutions of the linear partial differential equations, and numerical solutions at different times for the initial profiles given above for GBEs (1)-(2).

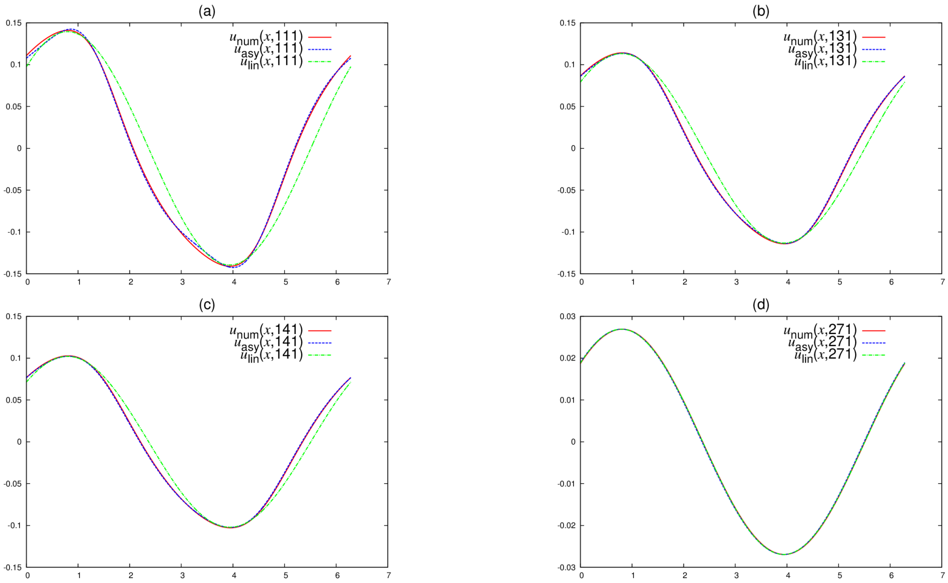

Figure 2: Numerical solution (u_{num}) of the generalized Burgers equation (1) subject to the initial profile $u_0(x) = 2\sin x$, large time asymptotic solution (u_{asy}) given in (56), and the solution (u_{lin}) of the linear partial differential equation (18) given in (17). Here the oldage constant $A_1 = 0.5356$, $x_0 = -0.7747$, $\epsilon = 0.01$, $j = 0.1$, $n = 3$ and (a) $t = 111$ (b) $t = 131$ (c) $t = 141$ (d) $t = 271$.

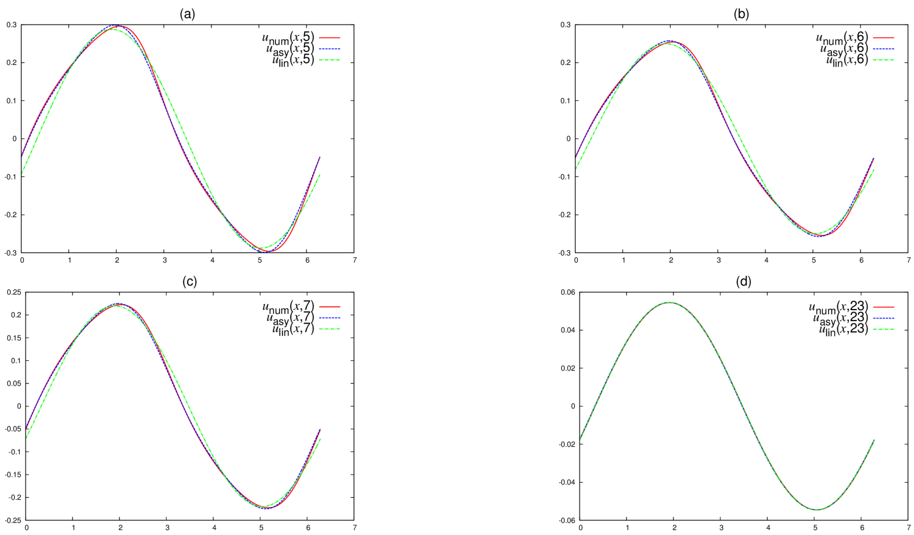

Figure 3: Numerical solution (u_{num}) of the generalized Burgers equation (1) subject to the initial profile (102), large time asymptotic solution (u_{asy}) given in (62), and the solution (u_{lin}) of the linear partial differential equation (18) given in (17). Here the oldage constant $A_1 = 0.8258$, $x_0 = 0.3319$, $\epsilon = 0.05$, $j = 1$ and (a) $t = 5$ (b) $t = 6$ (c) $t = 7$ (d) $t = 23$.

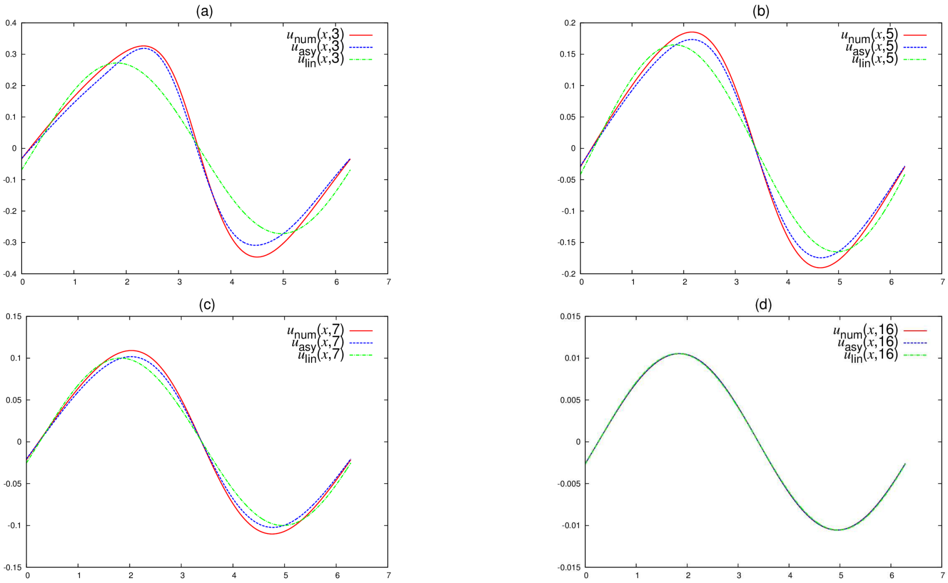

Figure 4: Numerical solution (u_{num}) of the generalized Burgers equation (2) subject to the initial profile (102), large time asymptotic solution (u_{asy}) given in (99), and the solution (u_{lin}) of the linear partial differential equation (65) given in (64) for $a = 1$, $b = 1$. Here the oldage constant $A_1 = 0.5757$, $x_0 = 0.2552$, $\epsilon = 0.25$, and (a) $t = 3$ (b) $t = 5$ (c) $t = 7$ (d) $t = 16$.

Figure 5 shows the movement of the zeros of the solutions of GBE1 and GBE2. We feel that the large time asymptotic solutions constructed in sections 2 and 3 may be useful for more general nonlinear partial differential equations. To illustrate this point, we consider the nonlinear partial differential equation

$$u_t + \sin^2(u)u_x + \frac{ju}{2t} = \epsilon u_{xx}. \tag{103}$$

As $u \to 0$, we may approximate $\sin^2(u)$ by u^2. In other words, GBE1 may be viewed as an approximation to the partial differential equation (103). Our numerical study shows that the asymptotic solution (56) of GBE1 agrees quite nicely with the solutions of GBE1 and also of (103) at different times.

Figure 6 presents the comparison of the numerical solution of the equation (103) with the numerical solution of GBE1 (1) subject to $u_0(x) = \sin x$, the large time asymptotic solution constructed (56), and the solution (17) of the linear partial differential equation (18) at different times.

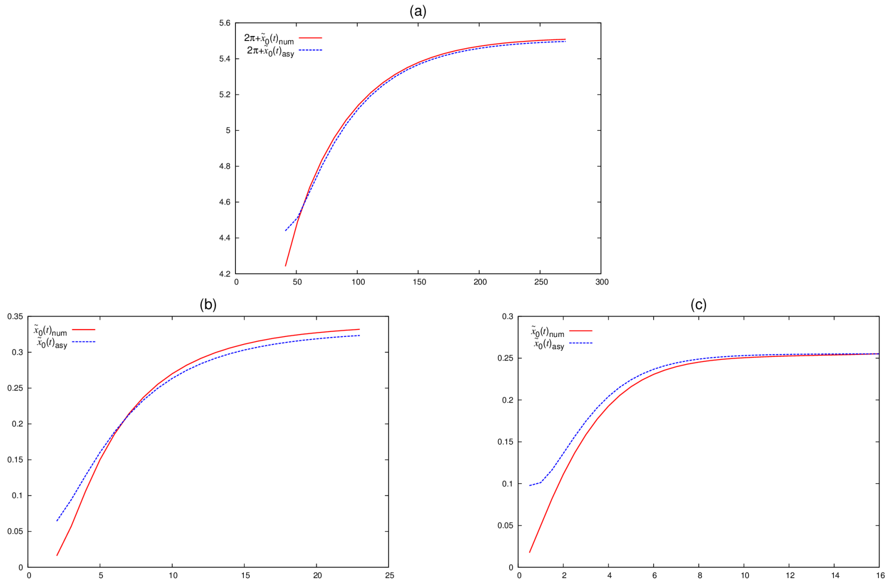

Figure 5: $\tilde{x}_0(t)_{\text{num}}$ is the zero obtained from the numerical solution of GBEs (1) or (2) and $\tilde{x}_0(t)_{\text{asy}}$ is the zero obtained from the formulas (33), (63), or (100) for the initial boundary value problems: (a) GBE1 (1) with $j = 0.1$, $\epsilon = 0.01$ subject to initial profile $u_0(x) = 2\sin x$ (b) GBE1 (1) with $j = 1$, $\epsilon = 0.05$ subject to initial profile (102) (c) GBE2 (2) with $a = 1$, $b = 1$, $\epsilon = 0.25$ subject to initial profile (102).

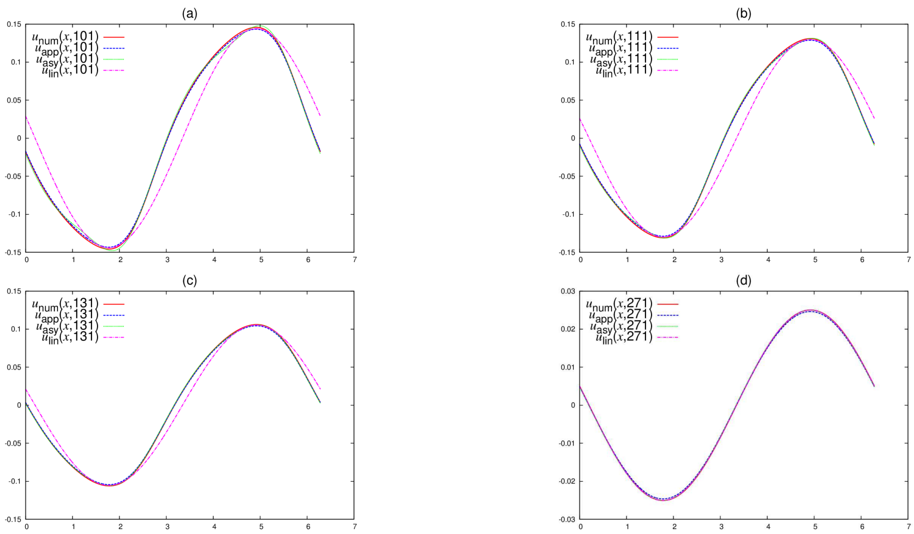

Figure 6: Numerical solution (u_{num}) of the generalized Burgers equation (103), numerical solution (u_{app}) of the generalized Burgers equation (1) subject to the initial profile $u_0(x) = \sin x$, large time asymptotic solution (u_{asy}) given in (56), and the solution (u_{lin}) of the linear partial differential equation (18) given in (17). Here the oldage constant $A_1 = 0.4981$, $x_0 = -2.9396$, $\epsilon = 0.01$, $j = 0.1$ and (a) $t = 101$ (b) $t = 111$ (c) $t = 131$ (d) $t = 271$.

Table 1: Comparison of maximum error between the numerical solution u_{num} of the generalized Burgers equation (1) (subject to the initial profile $u_0(x) = 2\sin x$) with asymptotic solution u_{asy} given in (56) and the solution u_{lin} given in (17) of the linear partial differential equation (18) at different times when $\epsilon = 0.01$, $j = 0.1$.

t	$\|u_{\text{num}} - u_{\text{asy}}\|_\infty$	$\|u_{\text{num}} - u_{\text{lin}}\|_\infty$
61	0.1149	0.2025
71	0.0605	0.1484
81	0.0321	0.1087
91	0.0170	0.0796
101	0.0090	0.0582
111	0.0048	0.0425
121	0.0025	0.0309
131	0.0016	0.0225
141	0.0015	0.0163
151	0.0013	0.0118
161	0.0011	0.0085
171	0.0010	0.0061
181	0.0009	0.0043
191	0.0008	0.0030
201	0.0007	0.0021
211	0.0006	0.0014
221	0.0006	0.0010
231	0.0005	0.0006
241	0.0004	0.0004

Table 2: Comparison of maximum error between the numerical solution u_{num} of the generalized Burgers equation (1) (subject to the initial profile (102)) with asymptotic solution u_{asy} given in (62) and the solution u_{lin} given in (17) of the linear partial differential equation (18) at different times when $\epsilon = 0.05$, $j = 1$.

t	$\|u_{\text{num}} - u_{\text{asy}}\|_\infty$	$\|u_{\text{num}} - u_{\text{lin}}\|_\infty$
2	0.0662	0.1664
3	0.0375	0.1114
4	0.0239	0.0759
5	0.0157	0.0525
6	0.0105	0.0368
7	0.0072	0.0262
8	0.0051	0.0188
9	0.0038	0.0136
10	0.0029	0.0099
11	0.0022	0.0073
12	0.0018	0.0053
13	0.0015	0.0039
14	0.0012	0.0029
15	0.0011	0.0021
16	0.0009	0.0015
17	0.0008	0.0011
18	0.0007	0.0008

§5 Conclusions

In this paper, we have studied the large time asymptotics for the periodic solutions of two generalized Burgers equations (1)-(2) satisfying conditions (3)-(5). We have obtained asymptotic solutions via a perturbative approach. Further the forms of the zeroes $\tilde{x}_0(t)$ of the solutions of GBE1 and GBE2 for large time have been found. The forms for $\tilde{x}_0(t)$ for GBE1 and GBE2 generalize the form used for the zero of the solution of the modified Burgers equation $u_t + u^2u_x = \epsilon u_{xx}$. An interesting observation was that the asymptotic solution for GBE1 with $j = 1$ contained the exponential integral $Ei(t)$. We also have solved numerically the generalized Burgers equations (1)-(2) with initial conditions (101), (102) and boundary conditions (5) using a finite difference scheme due to Dawson [5]. The agreement between the asymptotic and numerical solutions is quite good for the large time we have studied.

Our study may help the construction or analysis of the large time asymptotics of periodic solutions of the more general nonlinear partial differential equations of the form

$$u_t + f(u)u_x + g(t, u) = \epsilon u_{xx},$$

subject to the initial-boundary conditions (3)-(5). To illustrate this point, let us consider the following nonlinear partial differential equations:

$$u_t + (e^u - 1)u_x = \epsilon u_{xx}, \tag{104}$$

$$u_t + \sin^2(u)u_x = \epsilon u_{xx}, \tag{105}$$

$$u_t + \sin^2(u)u_x + \frac{ju}{2t} = \epsilon u_{xx}. \tag{106}$$

As $u \to 0$, equations (104) and (105)-(106) may be approximated by the generalized Burgers equations GBE2 with $a = 1, b = 1/2$ and GBE1, respectively. Our numerical study shows that the asymptotic solution (99) with $a = 1$, $b = 1/2$ of GBE2, the asymptotic solution (56) with $j = 0$ of GBE1, the asymptotic solution (56) of GBE1, respectively, agree quite well with the solutions of (104)-(106) subject to the initial condition $u_0(x) = \sin x$ for large time. In this paper, we have presented only the results for the equation (106) when $j = 0.1, \epsilon = 0.01$. One may refer to Smriti [22] for a detailed numerical study of equations (104)-(105) and comparison of the numerical solutions of the partial differential equations (104)-(105) with the corresponding large time asymptotic solutions.

Table 3: Comparison of maximum error between the numerical solution u_{num} of the generalized Burgers equation (2) (subject to the initial profile (102)) with asymptotic solution u_{asy} given in (99) and the solution u_{lin} given in (64) of the linear partial differential equation (65) at different times when $a = 1$, $b = 1$, $\epsilon = 0.25$.

t	$\|u_{\text{num}} - u_{\text{asy}}\|_\infty$	$\|u_{\text{num}} - u_{\text{lin}}\|_\infty$
1	0.1163	0.2905
2	0.0638	0.2093
3	0.0381	0.1302
4	0.0240	0.0790
5	0.0161	0.0481
6	0.0112	0.0298
7	0.0080	0.0187
8	0.0058	0.0119
9	0.0042	0.0077
10	0.0030	0.0050
11	0.0022	0.0033
12	0.0015	0.0021
13	0.0010	0.0013
14	0.0005	0.0008
15	0.0002	0.0004

References

[1] M Abramowitz, I A Stegun (eds.). *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, Dover, New York, 1972.

[2] L C Andrews. *Special functions for engineers and applied mathematicians*, Macmillan Publishing Company, New York, 1985.

[3] C M Bender, S A Orszag. *Advanced mathematical methods for scientists and engineers*, McGraw-Hill Book Company, Singapore, 1978.

[4] T F Chen, H A Levine, P E Sacks. *Analysis of a convective reaction–diffusion equation*, Nonlinear Anal TMA, 1988, 12(12): 1349–1370.

[5] C N Dawson. *Godunov-mixed methods for advective flow problems in one space dimension*, SIAM J Numer Anal, 1991, 28(5): 1282–1309.

[6] Y Duan, L Kong, R Zhang, *A lattice Boltzmann model for the generalized Burgers–Huxley equation*, Physica A, 2012, 391: 625–632.

[7] Y Duan, R Liu, Y Jiang. *Lattice Boltzmann model for the modified Burgers' equation*, Appl Math Comput, 2008, 202: 489–497.

[8] R E Grundy, P L Sachdev, C N Dawson. *Large time solution of an initial value problem for a generalized Burgers equation*, In: P L Sachdev and R E Grundy (eds.), *Nonlinear Diffusion Phenomenon*, Narosa Publishing House, New Delhi, 1994, 68–83.

[9] M Kato. *Large time behavior of solutions to the generalized Burgers equations*, Osaka J Math, 2007, 44: 923–943.

[10] H A Levine, L E Payne, P E Sacks, B Straughan. *Analysis of a convective reaction–diffusion equation II*, SIAM J Math Anal, 1989, 20(1): 133–147.

[11] A Mishra, R Kumar. *Exact solutions of variable coefficient nonlinear diffusion–reaction equations with a nonlinear convective term*, Phys Lett A, 2010, 374: 2921–2924.

[12] H Moritz. *The strange behavior of asymptotic series in mathematics, celestial mechanics and physical geodesy*, In: E W Grafarend, F W Krumm and V S Schwarze (eds.), *Geodesy - The challenge of the 3rd millennium*, Springer-Verlag Berlin Heidelberg, 2003, 371–377.

[13] J J C Nimmo, D G Crighton. *Bäcklund transformations for nonlinear parabolic equations: the general results*, Proc R Soc Lond A, 1982, 384: 381–401.

[14] Ö Oruç, F Bulut, A Esen. *A Haar wavelet-finite difference hybrid method for the numerical solution of the modified Burgers' equation*, J Math Chem, 2015, 53: 1592–1607.

[15] M A Pinsky. *Partial differential equations and boundary-value problems with applications*, American Mathematical Society, Providence, Rhode Island, 2011.

[16] O A Pocheketa, R O Popovych, O O Vaneeva. *Group classification and exact solutions of variable-coefficient generalized Burgers equations with linear damping*, Appl Math Comput, 2014, 243: 232–244.

[17] Ch S Rao, E Satyanarayana. *Asymptotic N-wave solutions of the nonplanar Burgers equation*, Stud Appl Math, 2008, 121: 199–221.

[18] P L Sachdev. *A generalised Cole–Hopf transformation for nonlinear parabolic and hyperbolic equations*, J Appl Math Phys, 1978, 29(6): 963–970.

[19] P L Sachdev. *Self-similarity and beyond. Exact solutions of nonlinear problems*, Chapman & Hall/CRC, Boca Raton, Florida, 2000.

[20] P L Sachdev, B O Enflo, Ch S Rao, B M Vaganan, P Goyal. *Large-time asymptotics for periodic solutions of some generalized Burgers equations*, Stud Appl Math, 2003, 110: 181–204.

[21] P L Sachdev, Ch S Rao, B O Enflo. *Large-time asymptotics for periodic solutions of the modified Burgers equation*, Stud Appl Math, 2005, 114: 307–323.

[22] Smriti Nath. *Large time asymptotics to solutions of some generalized Burgers equations*, Ph. D thesis, IIT Madras, India, 2016.

[23] A S Tersenov. *On the generalized Burgers equation*, Nonlinear Differ Equ Appl, 2010, 17: 437–452.

[24] A S Tersenov. *On the first boundary value problem for quasilinear parabolic equations with two independent variables*, Arch Rational Mech Anal, 2000, 152: 81–92.

[25] Ar S Tersenov. *On solvability of some boundary value problems for a class of quasilinear parabolic equations*, Sib Math J, 1999, 40(5): 972–980.

[26] B M Vaganan, S Padmasekaran. *Large-time asymptotics for periodic solutions of nonplanar Burgers equation with linear damping*, Int J Pure Appl Math, 2007, 41(3): 301–316.

[27] B M Vaganan, S Padmasekaran. *Large time asymptotic behaviors for periodic solutions of generalized Burgers equations with spherical symmetry or linear damping*, Stud Appl Math, 2009, 124: 1–18.

[28] B M Vaganan, S Padmasekaran. *Large-time asymptotics for periodic solutions of modified nonplanar and modified nonplanar damped Burgers equations*, In: B M Vaganan (ed.), *Nonlinear Waves and Diffusion Processes*, Narosa Publishing House, New Delhi, 2006, 50–59.

Department of Mathematics, Indian Institute of Technology Madras, Chennai-600036, India.
Email: chsrao@iitm.ac.in