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一类不连续Sturm-liouville问题的二重特征值

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摘 要: 该文研究具有混合型边界条件的一类不连续Sturm-liouville问题(SLPs)的二重特征值. 文中构造出一个整函数, 找到该问题的特征值及特征值重数与该整函数的关系, 并给出某个实数是该不连续SLPs的二重特征值的一个充分必要条件.

关键词: 不连续Sturm-liouville问题; 特征值; 代数重数; 几何重数

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§1 引 言

本文研究由二阶微分方程(1)和混合型边界条件(2)以及转移边界条件(3)构成的不连续Sturm-liouville问题

$$-y''(x)+q(x)y(x)=\lambda y(x),\quad x\in J=(a,c)\cup(c,b), \tag{1}$$

混合型边界条件

$$Y(b)=\mathrm{e}^{\mathrm{i}\gamma}KY(a), \tag{2}$$

转移边界条件

$$y(c+0)=\delta y(c-0),\quad y'(c+0)=\frac{1}{\delta}y'(c-0), \tag{3}$$

其中$q(x)$是实值函数且满足$q(x)\in L^1(J,\mathbf{R}),\delta\in\mathbf{R}$, $0\le\gamma\le\pi$, $K=\begin{pmatrix}k_{11} & k_{12}\\ k_{21} & k_{22}\end{pmatrix}$是$2*2$实矩阵, 且满足$k_{11}k_{22}-k_{12}k_{21}=1$.

不连续Sturm-liouville问题在物理学、热力学和量子力学中有重要应用. 如衍射问题, 某一点有质量作用的弦振动问题, 声波在水下传播遇到障碍物返回波的传播问题, 以及不同材料重叠形成的薄的叠层板块的热传导问题等, 都可以抽象为带有转移边界条件的SLPs, 即不连续SLPs. 近年来, 关于不连续SLPs的特征值的性质、特征函数振荡性问题、完备性问题及逆谱问题成为众多科研工作者的研究热点[1-8].

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不论是带有分离型边界条件还是混合型边界条件的连续型Sturm-liouville问题, 其特征值的代数重数与几何重数都是相等的[9-10]. 对于带分离型边界条件的不连续Sturm-liouville问题, 特征值的代数重数与几何重数也是相等的[2]. 带有混合型边界条件的不连续Sturm-liouville问题(1)-(3)的特征值的代数重数与几何重数是否相等？还没有相关结论. 本文研究一类带有混合型边界条件的不连续Sturm-liouville问题特征值的代数重数与几何重数的关系, 并给出某个实数是该不连续SLPs的二重特征值的一个充分必要条件.

§2 预备知识

设在(a,c)上, $\varphi_1(x,\lambda),\psi_1(x,\lambda)$是二阶微分方程(1)的分别满足如下初始条件(4)-(5)的解.

$$\varphi_1(a,\lambda)=1,\ \varphi_1'(a,\lambda)=0; \tag{4}$$

$$\psi_1(a,\lambda)=0,\ \psi_1'(a,\lambda)=1. \tag{5}$$

在(c,b)上, $\varphi_2(x,\lambda),\psi_2(x,\lambda)$是二阶微分方程(1)的分别满足如下初始条件(6)-(7)的解.

$$\varphi_2(c+0,\lambda)=\delta\varphi_1(c-0,\lambda),\ \varphi_2'(c+0,\lambda)=\frac{1}{\delta}\varphi_1'(c-0,\lambda); \tag{6}$$

$$\psi_2(c+0,\lambda)=\delta\psi_1(c-0,\lambda),\ \psi_2'(c+0,\lambda)=\frac{1}{\delta}\psi_1'(c-0,\lambda). \tag{7}$$

令

$$\varphi(x,\lambda)=\{\begin{array}{ll}\varphi_1(x,\lambda), & x\in(a,c),\\ \varphi_2(x,\lambda), & x\in(c,b),\end{array} \tag{8}$$

$$\psi(x,\lambda)=\{\begin{array}{ll}\psi_1(x,\lambda), & x\in(a,c),\\ \psi_2(x,\lambda), & x\in(c,b).\end{array} \tag{9}$$

则$\varphi(x,\lambda),\psi(x,\lambda)$是线性无关的, 且二阶微分方程(1)的任意一个解都可以表示为

$$y(x,\lambda)=c_1\varphi(x,\lambda)+c_2\psi(x,\lambda), \tag{10}$$

其中$y(x,\lambda)$满足$y(a,\lambda)=c_1,y'(a,\lambda)=c_2$.

引理2.1(见[11, p293]) 设$\varphi_1(x,\lambda),\varphi_2(x,\lambda),\psi_1(x,\lambda),\psi_2(x,\lambda)$是方程(1)的满足初始条件(4)-(7)的解, 当$x\in(a,c)$时, 令

$$\omega_1(\lambda)=\mathrm{Det}\begin{pmatrix}\varphi_1(x,\lambda) & \psi_1(x,\lambda)\\ \varphi_1'(x,\lambda) & \psi_1'(x,\lambda)\end{pmatrix}; \tag{11}$$

当$x\in(c,b)$时, 令

$$\omega_2(\lambda)=\mathrm{Det}\begin{pmatrix}\varphi_2(x,\lambda) & \psi_2(x,\lambda)\\ \varphi_2'(x,\lambda) & \psi_2'(x,\lambda)\end{pmatrix}. \tag{12}$$

则$\omega_1(\lambda),\omega_2(\lambda)$分别是$\varphi_1(x,\lambda),\psi_1(x,\lambda)$和$\varphi_2(x,\lambda),\psi_2(x,\lambda)$的朗斯基(Wronskian)行列式, 与$x$无关, 且$\omega_1(\lambda)=\omega_2(\lambda)=1$.

引理2.2(见[11, p294]) 设$\varphi_1(x,\lambda),\varphi_2(x,\lambda),\psi_1(x,\lambda),\psi_2(x,\lambda)$是方程(1)的满足初始条件(4)-(7)的解, 令

$$D(\lambda)=k_{11}\psi_2'(b,\lambda)+k_{22}\varphi_2(b,\lambda)-k_{12}\varphi_2'(b,\lambda)-k_{21}\psi_2(b,\lambda), \tag{13}$$

则λ是不连续Sturm-liouville问题(1)-(3)的特征值的充分必要条件是

$$D(\lambda)=2\cos\gamma. \tag{14}$$

. **定义2.1**(见[9, p44]) 不连续Sturm-liouville问题(1)-(3)特征值的代数重数为λ作为(14)的零点的阶数. 不连续Sturm-liouville问题(1)-(3)特征值的几何重数为线性方程组(15)-(16)解空间

的维数.

$$(\varphi_2(b,\lambda)-\mathrm{e}^{\mathrm{i}\gamma}k_{11})c_1+(\psi_2(b,\lambda)-\mathrm{e}^{\mathrm{i}\gamma}k_{12})c_2=0, \tag{15}$$

$$(\varphi_2'(b,\lambda)-\mathrm{e}^{\mathrm{i}\gamma}k_{21})c_1+(\psi_2'(b,\lambda)-\mathrm{e}^{\mathrm{i}\gamma}k_{22})c_2=0. \tag{16}$$

推论2.1

1) 若λ是不连续Sturm-liouville问题(1)-(3)的二重特征值(几何重数), 则$\gamma=0$或π.

2) 若$0<\gamma<\pi$, 则λ是不连续Sturm-liouville问题(1)-(3)的单特征值(几何重数1).

证

1) 若λ是不连续Sturm-liouville问题(1)-(3)的二重特征值(几何重数), 则方程组(15)-(16)解空间的维数是2. 因此方程组(15)-(16)的系数矩阵的秩为零, 即

$$\begin{pmatrix} \varphi_2(b,\lambda)-\mathrm{e}^{\mathrm{i}\gamma}k_{11} & \psi_2(b,\lambda)-\mathrm{e}^{\mathrm{i}\gamma}k_{12} \\ \varphi_2'(b,\lambda)-\mathrm{e}^{\mathrm{i}\gamma}k_{21} & \psi_2'(b,\lambda)-\mathrm{e}^{\mathrm{i}\gamma}k_{22} \end{pmatrix}=0. \tag{17}$$

由于$\varphi_2(b,\lambda),\varphi_2'(b,\lambda),\psi_2(b,\lambda),\psi_2'(b,\lambda)$及$k_{ij}$是实数, 所以$\gamma=0$或$\pi$.

2) 由结论1)可得.

§3 主要结果

首先考虑向量微分方程

$$Y'(x)=(P-\lambda W)Y(x), \tag{18}$$

其中$Y(x)=\begin{pmatrix} y(x) \\ y'(x) \end{pmatrix}$, $P=\begin{pmatrix} 0 & 1 \\ q(x) & 0 \end{pmatrix}$, $W=\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, 若$y(x)$是微分方程(1)的解, 则$Y(x)$是向量微分方程(18)的解.

设$\varphi_1(x,\lambda),\varphi_2(x,\lambda),\psi_1(x,\lambda),\psi_2(x,\lambda)$是方程(1)的满足初始条件(4)-(7)的解, 令

$$\Phi(x,\lambda)=\left\{\begin{array}{ll} \Phi_1(x,\lambda), & x\in(a,c), \\ \Phi_2(x,\lambda), & x\in(c,b). \end{array}\right. \tag{19}$$

其中$\Phi_1(x,\lambda)=\begin{pmatrix} \varphi_1(x,\lambda) & \psi_1(x,\lambda) \\ \varphi_1'(x,\lambda) & \psi_1'(x,\lambda) \end{pmatrix}$, $\Phi_2(x,\lambda)=\begin{pmatrix} \varphi_2(x,\lambda) & \psi_2(x,\lambda) \\ \varphi_2'(x,\lambda) & \psi_2'(x,\lambda) \end{pmatrix}$, 则$\Phi(x,\lambda)$是向量微分方程(18)的解矩阵, 且满足初始条件

$$\Phi_1(a,\lambda)=I,\quad \Phi_2(c+0,\lambda)=\begin{pmatrix} \delta\varphi_1(c-0,\lambda) & \delta\psi_1(c-0,\lambda) \\ \frac{1}{\delta}\varphi_1'(c-0,\lambda) & \frac{1}{\delta}\psi_1'(c-0,\lambda) \end{pmatrix}.$$

定理3.1 设$\varphi_1(x,\lambda),\varphi_2(x,\lambda),\psi_1(x,\lambda),\psi_2(x,\lambda)$是方程(1)的满足初始条件(4)-(7)的解, $\Phi_1(x,\lambda)$, $\Phi_2(x,\lambda),D(\lambda)$定义如(19), (13), 令

$$A(\lambda)=k_{11}\varphi_2'(b,\lambda)-k_{21}\varphi_2(b,\lambda), \tag{20}$$

$$B(\lambda)=k_{11}\psi_2'(b,\lambda)+k_{12}\varphi_2'(b,\lambda)-k_{22}\varphi_2(b,\lambda)-k_{21}\psi_2(b,\lambda), \tag{21}$$

$$B_1(\lambda)=k_{11}\psi_2'(b,\lambda)-k_{21}\psi_2(b,\lambda), \tag{22}$$

$$B_2(\lambda)=k_{22}\varphi_2(b,\lambda)-k_{12}\varphi_2'(b,\lambda), \tag{23}$$

$$C(\lambda)=k_{22}\psi_2(b,\lambda)-k_{12}\psi_2'(b,\lambda). \tag{24}$$

则

$$D'(\lambda)=\int_a^c(A\psi_1^2+(B_2-B_1)\psi_1\varphi_1-C\varphi_1^2)\mathrm{d}t+\int_c^b(A\psi_2^2+(B_2-B_1)\psi_2\varphi_2-C\varphi_2^2)\mathrm{d}t, \tag{25}$$

$$4CD'(\lambda)=-\int_a^c(2C\varphi_1+B\psi_1)^2\mathrm{d}t-\int_c^b(2C\varphi_2+B\psi_2)^2\mathrm{d}t-(4-D^2)\int_a^c\psi_1^2\mathrm{d}t-(4-D^2)\int_c^b\psi_2^2\mathrm{d}t, \tag{26}$$

$$4AD'(\lambda)=\int_a^c(2A\psi_1-B\varphi_1)^2\mathrm{d}t+\int_c^b(2A\psi_2-B\varphi_2)^2\mathrm{d}t+(4-D^2)\int_a^c\varphi_1^2\mathrm{d}t+(4-D^2)\int_c^b\varphi_2^2\mathrm{d}t. \tag{27}$$

证 由(20)-(24)得

$$\Phi_2(b,\lambda)=K\begin{pmatrix} B_2(\lambda) & C(\lambda)\\ A(\lambda) & B_1(\lambda)\end{pmatrix},$$

即

$$K^{-1}\Phi_2(b,\lambda)=\begin{pmatrix} B_2(\lambda) & C(\lambda)\\ A(\lambda) & B_1(\lambda)\end{pmatrix}. \tag{28}$$

再由(22)-(23)得

$$D(\lambda)=B_1(\lambda)+B_2(\lambda)=\mathrm{trace}K^{-1}\Phi_2(b,\lambda),$$

所以

$$D'(\lambda)=\mathrm{trace}[K^{-1}\frac{\partial\Phi_2(b,\lambda)}{\partial\lambda}]. \tag{29}$$

在(c,b)上有

$$\Phi_2'(x)=(P-\lambda W)\Phi_2(x), \tag{30}$$

(30)两边对λ求导得

$$[\frac{\partial\Phi_2(x)}{\partial\lambda}]'=(P-\lambda W)\frac{\partial\Phi_2(x)}{\partial\lambda}-W\Phi_2(x), \tag{31}$$

且

$$\frac{\partial\Phi_2(c+0)}{\partial\lambda}=\begin{pmatrix}\delta & 0\\ 0 & \frac{1}{\delta}\end{pmatrix}\frac{\partial\Phi_1(c-0)}{\partial\lambda}. \tag{32}$$

求解初值问题(31)-(32)可得

$$\frac{\partial\Phi_2(x,\lambda)}{\partial\lambda}=-\int_c^x\Phi_2(x)\Phi_2^{-1}(t)W\Phi_2(t)\mathrm{d}t+\Phi_2(x)\Phi_1^{-1}(c-0)\frac{\partial\Phi_1(c-0)}{\partial\lambda}, \tag{33}$$

所以

$$\frac{\partial\Phi_2(b,\lambda)}{\partial\lambda}=-\int_c^b\Phi_2(b)\Phi_2^{-1}(t)W\Phi_2(t)\mathrm{d}t+\Phi_2(b)\Phi_1^{-1}(c-0)\frac{\partial\Phi_1(c-0)}{\partial\lambda}. \tag{34}$$

于是

$$K^{-1}\frac{\partial\Phi_2(b,\lambda)}{\partial\lambda}=-\int_c^bK^{-1}\Phi_2(b)\Phi_2^{-1}(t)W\Phi_2(t)\mathrm{d}t-\int_a^cK^{-1}\Phi_2(b)\Phi_1^{-1}(t)W\Phi_1(t)\mathrm{d}t,$$

所以

$$\begin{gathered}D'(\lambda)=\mathrm{trace}[K^{-1}\frac{\partial\Phi_2(b,\lambda)}{\partial\lambda}]=\\ -\mathrm{trace}[\int_c^bK^{-1}\Phi_2(b)\Phi_2^{-1}(t)W\Phi_2(t)\mathrm{d}t+\int_a^cK^{-1}\Phi_2(b)\Phi_1^{-1}(t)W\Phi_1(t)\mathrm{d}t]=\\ -\mathrm{trace}[\int_c^b\begin{pmatrix}-B_2\varphi_2\psi_2+C\varphi_1^2 & *\\ * & -A\psi_2^2+B\varphi_2\psi_2\end{pmatrix}\mathrm{d}t+\\ \int_a^c\begin{pmatrix}-B_2\varphi_1\psi_1+C\varphi_1^2 & *\\ * & -A\psi_1^2+B\varphi_1\psi_1\end{pmatrix}\mathrm{d}t]=\\ \int_a^c(A\psi_1^2+(B_2-B_1)\psi_1\varphi_1-C\varphi_1^2)\mathrm{d}t+\int_c^b(A\psi_2^2+(B_2-B_1)\psi_2\varphi_2-C\varphi_2^2)\mathrm{d}t.\end{gathered} \tag{35}$$

即(25)成立.

由(20)-(24)知

$$B_1(\lambda)B_2(\lambda) - A(\lambda)C(\lambda) = 1, \tag{36}$$

$$D(\lambda) = B_1(\lambda) + B_2(\lambda), \tag{37}$$

$$B(\lambda) = B_1(\lambda) - B_2(\lambda), \tag{38}$$

$$4 - D^2(\lambda) = -(4A(\lambda)C(\lambda) + B^2(\lambda)). \tag{39}$$

将(36)-(39)代入(25)可得(26)-(27). 下面给出本文得主要结论.

定理3.2 设$D(\lambda)$定义如(13), λ是不连续Sturm-liouville问题(1)-(3)的特征值. 设边界条件(2)中$\gamma = 0$或$\gamma = \pi$, 则λ是二重特征值(几何重数)的充分必要条件为$D'(\lambda) = 0$.

证 若λ是二重特征值(几何重数), 由(17)得

$$\Phi_2(b, \lambda) = K,$$

即

$$K^{-1}\Phi_2(b, \lambda) = I. \tag{40}$$

由(28)得

$$B_1(\lambda) = B_2(\lambda) = 1, A(\lambda) = C(\lambda) = 0.$$

由(25)得$D'(\lambda) = 0$.

另一方面, 若$D'(\lambda) = 0$, 设$\gamma = 0$. 因为λ是不连续Sturm-liouville问题(1)-(3)的特征值, 由引理2.2中的(14)得

$$D(\lambda) = 2, \tag{41}$$

再由(26)-(27)得

$$2C\varphi_1(x, \lambda) + B\psi_1(x, \lambda) = 0, \ -B\varphi_1(x, \lambda) + 2A\psi_1(x, \lambda) = 0,$$

$$2C\varphi_2(x, \lambda) + B\psi_2(x, \lambda) = 0, \ -B\varphi_2(x, \lambda) + 2A\psi_2(x, \lambda) = 0.$$

因为$\varphi_1(x, \lambda), \psi_1(x, \lambda)$线性无关, $\varphi_2(x, \lambda), \psi_2(x, \lambda)$线性无关, 所以

$$A(\lambda) = B(\lambda) = C(\lambda) = 0,$$

又

$$B(\lambda) = B_1(\lambda) - B_2(\lambda), D(\lambda) = B_1(\lambda) + B_2(\lambda) = 2,$$

所以

$$B_1(\lambda) = B_2(\lambda) = 1.$$

即(40)成立, 所以λ是二重特征值(几何重数). $\gamma = \pi$时, 同理可证.

定理3.3 $\varphi_1(x, \lambda), \varphi_2(x, \lambda), \psi_1(x, \lambda), \psi_2(x, \lambda)$是方程(1)的满足初始条件(4)-(7)的解, $\Phi_1(x, \lambda)$, $\Phi_2(x, \lambda)$, $D(\lambda)$定义如(19), (13), 设边界条件(2)中$\gamma = 0$或$\gamma = \pi$, 则λ是不连续Sturm-liouville问题(1)-(3)的二重特征值的充分必要条件为$\Phi_2(b, \lambda) = \pm K$. 其中$W = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $K = \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix}$是$2*2$实矩阵, 且满足$k_{11}k_{22} - k_{12}k_{21} = 1$.

证 设λ是不连续Sturm-liouville问题(1)-(3)的特征值, 首先证明λ的代数重数与几何重数相等.

1. 若$0 < \gamma < \pi$, 由推论1知, λ是单特征值(几何重数), 由(14)得$|D(\lambda)| < 2$, 再由(26)得$D'(\lambda) \neq 0$, 所以λ是(14)的一重根, 即λ的代数重数为1, 此时λ的代数重数与几何重数相等.

2. 若$\gamma=0$或$\gamma=\pi$, 若λ是单特征值(几何重数), 由定理1知, $D'(\lambda)\neq 0$, 所以λ是(14)的一重根, 即λ的代数重数为1. 反之若λ的代数重数为1, 即$D'(\lambda)\neq 0$, 由定理1得λ的几何重数为1. 此时λ的代数重数与几何重数相等.

下证若λ_*是二重特征值(几何重数), 则$D'(\lambda_*)=0, D''(\lambda_*)\neq 0$, 即$\lambda$的代数重数为2. 不妨设$\gamma=0$, 即$\Phi_2(b,\lambda_*)=K$. 计算得

$$\Phi_1^{-1}W\Phi_1=\begin{pmatrix}-\varphi_1\psi_1 & -\psi_1^2\\ \varphi_1^2 & \varphi_1\psi_1\end{pmatrix}, \tag{42}$$

$$\Phi_2^{-1}W\Phi_2=\begin{pmatrix}-\varphi_2\psi_2 & -\psi_2^2\\ \varphi_2^2 & \varphi_2\psi_2\end{pmatrix}. \tag{43}$$

将(42)-(43)代入(35)得

$$D'(\lambda)=\operatorname{trace}[K^{-1}\Phi_2(b,\lambda)\int_c^b\begin{pmatrix}\varphi_2\psi_2 & \psi_2^2\\ -\varphi_2^2 & -\varphi_2\psi_2\end{pmatrix}\mathrm{d}t+K^{-1}\Phi_2(b,\lambda)\int_a^c\begin{pmatrix}\varphi_1\psi_1 & \psi_1^2\\ -\varphi_1^2 & -\varphi_1\psi_1\end{pmatrix}\mathrm{d}t], \tag{44}$$

于是

$$D''(\lambda)=\operatorname{trace}[K^{-1}\frac{\partial\Phi_2(b,\lambda)}{\partial\lambda}\int_c^b\begin{pmatrix}\varphi_2\psi_2 & \psi_2^2\\ -\varphi_2^2 & -\varphi_2\psi_2\end{pmatrix}\mathrm{d}t+K^{-1}\Phi_2(b,\lambda)\frac{\partial}{\partial\lambda}\int_c^b\begin{pmatrix}\varphi_2\psi_2 & \psi_2^2\\ -\varphi_2^2 & -\varphi_2\psi_2\end{pmatrix}\mathrm{d}t]+$$
$$\operatorname{trace}[K^{-1}\frac{\partial\Phi_2(b,\lambda)}{\partial\lambda}\int_c^b\begin{pmatrix}\varphi_1\psi_1 & \psi_1^2\\ -\varphi_1^2 & -\varphi_1\psi_1\end{pmatrix}\mathrm{d}t+K^{-1}\Phi_2(b,\lambda)\frac{\partial}{\partial\lambda}\int_a^c\begin{pmatrix}\varphi_1\psi_1 & \psi_1^2\\ -\varphi_1^2 & -\varphi_1\psi_1\end{pmatrix}\mathrm{d}t]. \tag{45}$$

把(34)代入(45)得

$$D''(\lambda_*)=\operatorname{trace}[(-\int_c^b\Phi_2^{-1}(t)W\Phi_2(t)\mathrm{d}t+\Phi_1^{-1}(c-0)\frac{\partial\Phi_1(c-0)}{\partial\lambda})\int_c^b\begin{pmatrix}\varphi_2\psi_2 & \psi_2^2-\\ \varphi_2^2 & -\varphi_2\psi_2\end{pmatrix}\mathrm{d}t]+$$
$$\operatorname{trace}[(-\int_c^b\Phi_2^{-1}(t)W\Phi_2(t)\mathrm{d}t+\Phi_1^{-1}(c-0)\frac{\partial\Phi_1(c-0)}{\partial\lambda})\int_a^c\begin{pmatrix}\varphi_1\psi_1 & \psi_1^2-\\ \varphi_1^2 & -\varphi_1\psi_1\end{pmatrix}\mathrm{d}t]= \tag{46}$$
$$\operatorname{trace}[(\int_c^b\begin{pmatrix}\varphi_2\psi_2 & \psi_2^2\\ -\varphi_2^2 & -\varphi_2\psi_2\end{pmatrix}\mathrm{d}t)^2+(\int_a^c\begin{pmatrix}\varphi_1\psi_1 & \psi_1^2-\\ \varphi_1^2 & -\varphi_1\psi_1\end{pmatrix}\mathrm{d}t)^2]= \tag{47}$$
$$2[(\int_c^b\varphi_2\psi_2\mathrm{d}t)^2-\int_c^b\varphi_2^2\mathrm{d}t\int_c^b\psi_2^2\mathrm{d}t]+2[(\int_a^c\varphi_1\psi_1\mathrm{d}t)^2-\int_a^c\varphi_1^2\mathrm{d}t\int_a^c\psi_1^2\mathrm{d}t]. \tag{48}$$

由于φ_1,ψ_1,线性无关, φ_2,ψ_2线性无关知Cauchy-Schwarz不等式严格成立, 所以$D''(\lambda_*)<0$.

反之, 若$D'(\lambda_*)=0, D''(\lambda_*)\neq 0$, 即$\lambda$的代数重数为2, 由定理3.2知$\lambda_*$的几何重数为2.

因此特征值λ的几何重数与代数重数相等. 由定义2.1知λ是不连续Sturm-liouville问题(1)-(3)的二重特征值的充分必要条件为$\Phi_2(b,\lambda)=\pm K$.

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The double eigenvalues of discontinuous Sturm-Liouville problems with coupled boundary conditions

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Abstract: In this paper, the double eigenvalues for discontinuous Sturm-Liouville Problems (SLPs) with periodic boundary condition are investigated. An entire function is constructed, and the relationship between the entire function and eigenvalues and the multiplicity of eigenvalues is found. And the sufficient and necessary condition for a real number to be a double eigenvalue of the discontinuous SLPs is given.

Keywords: discontinuous Sturm-Liouville problems;eigenvalues; algebraic multiplicities; geometric multiplicities

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