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Clifford分析中一类偏微分方程的求解问题

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摘　要: 该文研究Clifford分析中的一类广义二阶偏微分方程的求解问题, 这类方程在解为实值或向量值时与广义Weinstein方程有着联系. 首先构造Clifford分析中的两个向量微分算子, 研究了这两个算子的性质及相互关系, 得到该偏微分方程$k = 0$时的解. 再根据微分算子的性质迭代给出该偏微分方程一般情况下解的表达式, 并研究了$k = m - 1$时的特殊解.

关键词: Clifford分析; 微分算子; 偏微分方程; 广义Weinstein方程

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§1　引言及预备知识

研究Clifford分析中一类二阶偏微分方程的解, 为求解高维空间中Weinstein型方程的解的表达式打下坚实基础, 进一步丰富了Weinstein型方程的解的表达形式–广义轴对称位势, 使其不再局限于积分表达式, 而是扩展到用广义轴对称位势来给出解的表达式[1-5].

用Clifford分析中的方法和性质解决微分方程是当前研究的热门方向[6-9], 例如: Hestenes, Doran, Lasenby在三维空间中利用Clifford分析方法研究电磁场中Maxwell方程, 将其转换为简单的表达式$DF = J$, 其中D是Clifford代数中的Dirac算子. 也有学者在Clifford分析中研究了偏微分方程与Dirac算子相关的一些二阶偏微分方程. 利用Clifford分析来求解高维空间中偏微分方程是Clifford代数理论中的一类重要应用[10-12], 例如求解广义Weinstein方程和Vekua方程. Weinstein方程是与Laplace方程关系密切的一类偏微分方程, 现如今对Weinstein型方程的研究有[13-14]: 用积分、广义正则函数表示Weinstein方程的解, 用Laplace-Beltrami算子和Riemannian度量求Weinstein方程的解[15], 以及证明其解满足的均值定理, 用微分算子表示Weinstein正则函数等等[16].

本文主要讨论Clifford分析中的一类二阶偏微分方程的解. 首先给出Clifford分析中的一类广义偏微分方程, 根据此方程, 定义两个新的微分算子, 探究此算子的性质, 以及算子与广义偏微分方程之间的关系, 再根据正则函数、共轭正则函数的性质, 得到偏微分方程特殊情况时的解, 最后利用微分算子的性质进行迭代, 可得到Clifford分析中广义偏微分方程的通解表达式.

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设$Cl_{0,n}(\mathbf{R})$是由$(n+1)$维向量空间的基生成的Clifford代数, 该向量空间的标准基为$\{e_0, e_1, e_2, \cdots e_n\}$, 则$Cl_{0,n}(\mathbf{R})$的基为$e_0=1, e_1, e_2, \cdots, e_n; e_1e_2, e_1e_3\cdots, e_1e_n; e_2e_3, \cdots e_2e_n; \cdots; e_1e_2\cdots e_n$. 记$Cl_{0,n}(\mathbf{R})$中的基元素为$e_A=e_{\alpha_1}e_{\alpha_2}\cdots e_{\alpha_h}$, 其中$A=\{\alpha_1,\alpha_2,\cdots\alpha_h\}\subseteq\{1,2,\cdots n\}, 1\le\alpha_1<\alpha_2<\cdots<\alpha_h\le n$并且$e_\varnothing=e_0=1$. 对于任意的$\alpha\in Cl_{0,n}(\mathbf{R})$, 有$\alpha=\sum\limits_A\alpha_Ae_A$, 其中$\alpha_A\in\mathbf{R}$. $Cl_{0,n}(\mathbf{R})$中的元素的乘积是不可交换的, 其乘积规则为

$$e_ie_j+e_je_i=-2\delta_{ij}, i,j\in\{1,2\cdots n\}, \text{其中}\delta_{ij}=\begin{cases}0, & i\neq j,\\ 1, & i=j.\end{cases}$$

特别地, 若$x\in\mathbf{R}^{n+1}$, 则x记为

$$x=x_0e_0+x_1e_1+x_2e_2+\cdots+x_ne_n,$$

其中$x_i\in\mathbf{R}, (i,j=1,2,\cdots,n)$. 对于任意的$\alpha\in Cl_{0,n}(\mathbf{R})$, 有$\alpha=\sum\limits_A\alpha_Ae_A$, 定义其$\widetilde{\alpha}$为

$$\widetilde{\alpha}=\sum_A(-1)^{|A|}\alpha_Ae_A, \text{其中}|A|=n_A,$$

定义向量微分算子D和$\overline{D}$为

$$D:=e_0\frac{\partial}{\partial x_0}+e_1\frac{\partial}{\partial x_1}+e_2\frac{\partial}{\partial x_2}+\cdots+e_n\frac{\partial}{\partial x_n},$$
$$\overline{D}:=e_0\frac{\partial}{\partial x_0}-e_1\frac{\partial}{\partial x_1}-e_2\frac{\partial}{\partial x_2}-\cdots-e_n\frac{\partial}{\partial x_n}.$$

经过计算可知

$$\overline{D}D=D\overline{D}=\frac{\partial^2}{\partial x_0^2}+\frac{\partial^2}{\partial x_1^2}+\cdots+\frac{\partial^2}{\partial x_n^2}=\triangle.$$

设$\Omega\subset\mathbf{R}^{n+1}$是一个区域, 函数$f(x)=\sum\limits_Af_A(x)e_A$为一个定义于$\Omega$取值于$Cl_{0,n}(\mathbf{R})$的函数, 其中$f_A(x)$为$\Omega\to\mathbf{R}^n$上的函数, 它具有$r$ 阶连续偏导数是指对于所有的$A\subset N$, $f_A(x)\in C^r(\Omega)$. 将具有r阶连续偏导的Clifford函数的集合记为

$$F_\Omega^{(r)}=\left\{f \mid f:\Omega\to Cl_{0,n}(\mathbf{R}), f(x)=\sum_Af_A(x)e_A, f_A(x)\in C^r\right\}.$$

设$f(x)=\sum\limits_Af_A(x)e_A\in F_\Omega^1$, 若$f$满足$Df=0$, 则称$f$为左正则函数, 简称为正则函数, 若满足$fD=0$, 则$f$被称为右正则函数. 设$f(x)=\sum\limits_Af_A(x)e_A\in F_\Omega^2$, 若$Df=0$, 可得$\triangle f=\overline{D}Df=\sum\limits_A\triangle f_Ae_A=0$, 所以$\triangle f_A=0$, 即左正则函数$f(x)=\sum\limits_Af_Ae_A$的所有分量函数$f_A(x)$都满足Laplace方程, 同理, 右正则函数$f(x)=\sum\limits_Af_Ae_A$的所有分量函数$f_A(x)$都满足Laplace方程.

关于Clifford分析中的正则函数的相关理论, 读者可以参考文献[17-21].

§2 Clifford分析中的一类偏微分方程的解

接下来讨论Clifford空间中的一类偏微分方程

$$\triangle f+\frac{2k-2m+3}{x_0}Df-\frac{(m-1)^2-(m-1-k)^2}{x_0^2}f=0. \tag{2.1}$$

其中$m>1, k\ge0$, $f(x)=\sum\limits_Af_Ae_A\subset F_\Omega^2$.

将方程(2.1)的所有解的集合表示为L_k, 其中$f(x)$为Clifford值函数, 并定义两个向量值微分

算子
$$E_k(f)=\frac{x_0^2}{(m-1)^2-(m-1-k)^2}Df,\qquad H_k(f)=\overline{D}f-\frac{2m-2k-1}{x_0}f.$$
定理2.1 设算子$E_k(f)$和$H_k(f)$为上述向量微分算子, 则有

(1) E_k为L_k到L_{k-1}上的算子, 即对于$\forall f\in L_k$, 有$E_k(f)\in L_{k-1}$;

(2) H_k为L_{k-1}到L_k上的算子, 即对于$\forall f\in L_{k-1}$, 有$H_k(f)\in L_k$;

(3) $H_k\circ E_k=Id_{L_k}$, $E_k\circ H_k=Id_{L_{k-1}}$.

证 (1) 设$f\in L_k$, 并且$g=E_k(f)$, 则有$g=\dfrac{x_0^2}{(m-1)^2-(m-1-k)^2}Df$, 所以
$$Df=\frac{(m-1)^2-(m-1-k)^2}{x_0^2}g,$$
从而
$$\begin{aligned}\overline{D}g=&\overline{D}\left[\frac{x_0^2}{(m-1)^2-(m-1-k)^2}Df\right]=\\ &\frac{1}{(m-1)^2-(m-1-k)^2}\left[\frac{\partial\left(x_0^2Df\right)}{\partial x_0}-\sum_{i=1}^n e_i\frac{\partial\left(x_0^2Df\right)}{\partial x_i}\right]=\\ &\frac{1}{(m-1)^2-(m-1-k)^2}\left[2x_0Df+x_0^2\frac{\partial Df}{\partial x_0}-x_0^2\sum_{i=1}^n e_i\frac{\partial(Df)}{\partial x_i}\right]=\\ &\frac{1}{(m-1)^2-(m-1-k)^2}\left[2x_0Df+x_0^2\left[\frac{\partial Df}{\partial x_0}-\sum_{i=1}^n e_i\frac{\partial(Df)}{\partial x_i}\right]\right]=\\ &\frac{1}{(m-1)^2-(m-1-k)^2}\left[2x_0Df+x_0^2\overline{D}Df\right]=\\ &\frac{1}{(m-1)^2-(m-1-k)^2}\left[\frac{2\left[(m-1)^2-(m-1-k)^2\right]}{x_0}g+x_0^2\overline{D}Df\right].\end{aligned}$$
又因为$f\in L_k$, 则有
$$\overline{D}Df=\triangle f=\frac{(m-1)^2-(m-1-k)^2}{x_0^2}f-\frac{2k-2m+3}{x_0}Df.$$
所以
$$\begin{aligned}\overline{D}g=&\frac{1}{(m-1)^2-(m-1-k)^2}\left[\frac{2\left[(m-1)^2-(m-1-k)^2\right]}{x_0}g+x_0^2\frac{(m-1)^2-(m-1-k)^2}{x_0^2}f-\right.\\ &\left.x_0^2\frac{(2k-2m+3)}{x_0}\frac{(m-1)^2-(m-1-k)^2}{x_0^2}g\right]=\\ &\frac{2g}{x_0}+f+\frac{2m-2k-3}{x_0}g=\frac{2m-2k-1}{x_0}g+f.\end{aligned}\tag{2.2}$$

那么

$$
\begin{aligned}
D\overline{D}g = & D\left[\frac{2m-2k-1}{x_0}g\right]+Df = \\
& -\frac{2m-2k-1}{x_0^2}g+\frac{2m-2k-1}{x_0}Dg+\frac{(m-1)^2-(m-1-k)^2}{x_0^2}g = \\
& \frac{(m-1)^2-(m-1-k)^2-(2m-2k-1)}{x_0^2}g+\frac{2m-2k-1}{x_0}Dg = \\
& \frac{(m-1)^2-(m-k)^2}{x_0^2}g+\frac{2m-2k-1}{x_0}Dg = \\
& \frac{(m-1)^2-(m-(k-1)-1)^2}{x_0^2}g+\frac{2m-2(k-1)-3}{x_0}Dg.
\end{aligned}
$$

所以

$$
D\overline{D}g+\frac{2(k-1)-2m+3}{x_0}Dg-\frac{(m-1)^2-(m-1-(k-1))^2}{x_0^2}g=0.
$$

即$g=E_k(f)\in L_{k-1}$.

(2) 假设$g\in L_{k-1}$, 且$f=H_k(g)$, 则有$f=\overline{D}g-\dfrac{2m-2k-1}{x_0}g$, 所以

$$
Df=D\overline{D}g+\frac{2m-2k-1}{x_0^2}g-\frac{2m-2k-1}{x_0}Dg.
$$

又因为$g\in L_{k-1}$, 所以有

$$
D\overline{D}g=\overline{D}Dg=-\frac{2(k-1)-2m+3}{x_0}Dg+\frac{(m-1)^2-(m-1-(k-1))^2}{x_0^2}g.
$$

因此

$$
\begin{aligned}
Df= & -\frac{2(k-1)-2m+3}{x_0}Dg+\frac{(m-1)^2-(m-1-(k-1))^2}{x_0^2}g+\frac{2m-2k-1}{x_0^2}g- \\
& \frac{2m-2k-1}{x_0}Dg = \\
& \frac{(m-1)^2-(m-1-k)^2}{x_0^2}g.
\end{aligned}
\tag{2.3}
$$

由$\overline{D}g=f+\dfrac{2m-2k-1}{x_0}g$知

$$
\begin{aligned}
\overline{D}Df = & \overline{D}\left[\frac{(m-1)^2-(m-1-k)^2}{x_0^2}g\right] = \\
& -2\frac{(m-1)^2-(m-1-k)^2}{x_0^3}g+\frac{(m-1)^2-(m-1-k)^2}{x_0^2}f+ \\
& \frac{(m-1)^2-(m-1-k)^2}{x_0^2}\cdot\frac{2m-2k-1}{x_0}g = \\
& \frac{(2m-2k-3)\left[(m-1)^2-(m-1-k)^2\right]}{x_0^3}g+\frac{(m-1)^2-(m-1-k)^2}{x_0^2}f,
\end{aligned}
$$

又$Df=\dfrac{(m-1)^2-(m-1-k)^2}{x_0^2}g$, 所以$g=\dfrac{x_0^2}{(m-1)^2-(m-1-k)^2}Df$, 因此

$$
\overline{D}Df=\frac{2m-2k-3}{x_0}Df+\frac{(m-1)^2-(m-1-k)^2}{x_0^2}f,
$$

所以
$$\overline{D}Df+\frac{2k-2m+3}{x_0}Df-\frac{(m-1)^2-(m-1-k)^2}{x_0^2}f=0.$$
即$f=H_k(g)\in L_k$.

(3) 假设$f\in L_k$, 且$g=E_k(f), g\in L_{k-1}$, 且由(2.1)可得
$$H_k\circ E_k(f)=H_k(g)=\overline{D}g-\frac{2m-2k-1}{x_0}g=$$
$$\frac{2m-2k-1}{x_0}g+f-\frac{2m-2k-1}{x_0}g=f.$$
所以$H_k\circ E_k(f)=Id_{L_k}$.

假设$g\in L_{k-1}$, 且$f=H_k(g), f\in L_k$, 且由(2.2)可得
$$E_k\circ H_k(g)=E_k(f)=\frac{x_0^2}{(m-1)^2-(m-1-k)^2}Df=$$
$$\frac{x_0^2}{(m-1)^2-(m-1-k)^2}\cdot\frac{(m-1)^2-(m-1-k)^2}{x_0^2}g=g.$$
所以$E_k\circ H_k(g)=Id_{L_{k-1}}$, 从而定理得到证明.

定理2.2　设函数$f\in F_\Omega^{(2)}$, 则$f\in L_0$的充要条件是存在$u(x), h(x)\in F_\Omega^{(2m-1)}$, 使得
$$f=\sum_{i=0}^{2m-3}\frac{(-1)^{i+1}(2m-3)!}{i!}x_0^iD^ih+u. \tag{2.4}$$
其中$Du=0, \overline{D}h=0$.

证　(1) 充分性

对任意的$k\in\mathbf{N}$, 设$h\in F_\Omega^r$有
$$\begin{aligned}D(x_0^kD^kh)=&\frac{\partial}{\partial x_0}(x_0^kD^kh)+\sum_{j=1}^n e_j\frac{\partial(x_0^kD^kh)}{\partial x_j}=\\ &k\cdot x_0^{k-1}D^kh+x_0^k\frac{\partial(D^kh)}{\partial x_0}+\sum_{j=1}^n e_j\cdot x_0^k\frac{\partial(D^kh)}{\partial x_j}=\\ &k\cdot x_0^{k-1}D^kh+x_0^k\left[\frac{\partial(D^kh)}{\partial x_0}+\sum_{j=1}^n e_j\frac{\partial(D^kh)}{\partial x_j}\right]=\\ &k\cdot x_0^{k-1}D^kh+x_0^kD(D^kh)=\\ &k\cdot x_0^{k-1}D^kh+x_0^kD^{k+1}(h),\end{aligned} \tag{2.5}$$
同理有
$$\overline{D}(x_0^kD^kh)=k\cdot x_0^{k-1}D^kh+x_0^k\overline{D}D^k(h), \tag{2.6}$$

由(2.5)可得

$$\begin{aligned} x_0^k D^{(k+1)}(h) =& D({x_0}^k D^k h) - k \cdot x_0^{k-1} D^k(h) = \\ & D(x_0^k D^k h) - k \left[D(x_0^{k-1} D^{k-1}(h)) - (k-1) x_0^{k-2} D^{k-1}(h) \right] = \\ & D(x_0^k D^k h) - kD \left[x_0^{k-1} D^{k-1}(h) \right] + k(k-1) x_0^{k-2} D^{k-1}(h) = \\ & D(x_0^k D^k h) - kD \left[x_0^{k-1} D^{k-1}(h) \right] + k(k-1) D \left[x_0^{k-2} D^{k-2}(h) \right] - \\ & k(k-1)(k-2) x_0^{k-3} D^{k-2}(h) = \\ & D(x_0^k D^k h) - kD \left[x_0^{k-1} D^{k-1}(h) \right] + k(k-1) D \left[x_0^{k-2} D^{k-2}(h) \right] - \\ & \cdots + (-1)^{k-1} k! D \left[x_0 D(h) \right] + (-1)^k k! D \left[D^0(h) \right] = \\ & D \left[\sum_{i=0}^{k} \frac{(-1)^{k-i} k!}{i!} x_0^i D^i(h) \right]. \end{aligned} \tag{2.7}$$

设$f = \displaystyle\sum_{i=0}^{2m-3} \frac{(-1)^{i+1}(2m-3)!}{i!} x_0^i D^i h + u, Du = 0, \overline{D}h = 0$, 则由上式可得

$$\begin{aligned} Df = & D\left[\sum_{i=0}^{2m-3} \frac{(-1)^{i+1}(2m-3)!}{i!} x_0^i D^i h \right] + Du = \\ & D\left[\sum_{i=0}^{2m-3} \frac{(-1)^{2m-3-i}(2m-3)!}{i!} x_0^i D^i h \right] + 0 = \\ & x_0^{2m-3} D^{2m-2} h, \end{aligned}$$

因为$\overline{D}h = 0$, 则由(2.6)可得

$$\begin{aligned} \overline{D}Df = & (2m-3) \cdot x_0^{2m-4} D^{2m-2} h + x_0^{2m-3} \overline{D} D^{2m-2} h = \\ & \frac{2m-3}{x_0} x_0^{2m-3} D^{2m-2} h + x_0^{2m-3} D^{2m-2} \overline{D} h = \\ & \frac{2m-3}{x_0} Df, \end{aligned}$$

即

$$\overline{D}Df - \frac{2m-3}{x_0} Df = 0,$$

所以$f \in L_0$.

(2) 必要性

设函数$f \in F_{\Omega}^{(2)}$, 且$f \in L_0$, 则有

$$\overline{D}Df - \frac{2m-3}{x_0} Df = 0.$$

由(2.6)可知

$$\begin{aligned} \overline{D}(x_0^{-2m+3} DG) =& (-2m+3) x_0^{-2m+2} Df + x_0^{-2m+3} \overline{D} Df = \\ & x_0^{-2m+3} \left[\frac{-2m+3}{x_0} Df + \overline{D} Df \right] = 0. \end{aligned}$$

令$x_0^{-2m+3} Df = g$, 则$\overline{D}g = 0$, 由文献[20]可知存在函数h满足

$$\begin{cases} \overline{D}h = 0, \\ D^{2m-2} h = g. \end{cases}$$

由$x_0^{-2m+3}Df=g$及(2.7)可得

$$Df=x_0^{2m-3}g=x_0^{2m-3}D^{2m-2}h=D\left[\sum_{i=0}^{2m-3}\frac{(-1)^{2m-3-i}(2m-3)!}{i!}x_0^iD^ih\right]=$$

$$D\left[\sum_{i=0}^{2m-3}\frac{(-1)^{i+1}(2m-3)!}{i!}x_0^iD^ih\right].$$

令$u=f-\sum_{i=0}^{2m-3}\frac{(-1)^{i+1}(2m-3)!}{i!}x_0^iD^ih$, 那么

$$Du=Df-D\left[\sum_{i=0}^{2m-3}\frac{(-1)^{i+1}(2m-3)!}{i!}x_0^iD^ih\right]=0.$$

故$f=\sum_{i=0}^{2m-3}\frac{(-1)^{i+1}(2m-3)!}{i!}x_0^iD^ih+u$,其中$Du=0,\overline{D}h=0$.

定理2.3　设$f\in F_\Omega^2$, 则f为方程(2.1)的解的充要条件是存在$u(x),h(x)\in F_\Omega^{(2m-1)}$, 使得$f$有表达式

$$f=\sum_{i=0}^{2m-3-k}\frac{(-1)^{i+k+1}(2m-3)!(2m-3-i)!}{i!(2m-3-i-k)!}x_0^{i-k}D^i+\sum_{i=0}^{k}\frac{(-1)^{i+k}(2k-1-i)!k}{i!(k-i)!}x_0^{i-k}\overline{D}^iu. \tag{2.8}$$

其中$Du=0,\overline{D}h=0$.

证　设f为方程(2.1)的解, 即$f\in L_k$, 由定理2.1可得$E_1\circ E_2\circ\cdots\cdots\circ E_kf\in L_0$, 所以由定理2.2可得

$$E_1\circ E_2\circ\cdots\cdots\circ E_kf=g=\sum_{i=0}^{2m-3}\frac{(-1)^{i+1}(2m-3)!}{i!}x_0^iD^ih+u.$$

由定理2.1可得

$$f=H_k\circ H_{k-1}\circ\cdots\cdots\circ H_2\circ H_1(g),$$

可知$f\in L_k$, 则由(2.6)有

$$H_1(g)=\overline{D}g-\frac{2m-3}{x_0}g=$$

$$\sum_{i=0}^{2m-3}\frac{(-1)^{i+1}(2m-3)!i}{i!}x_0^{i-1}D^ih+\sum_{i=0}^{2m-3}\frac{(-1)^{i+1}(2m-3)!}{i!}x_0^iD^i\overline{D}h-$$

$$\sum_{i=0}^{2m-3}\frac{(-1)^{i+1}(2m-3)!(2m-3)}{i!x_0}x_0^iD^ih+\overline{D}u-\frac{2m-3}{x_0}u=$$

$$\sum_{i=0}^{2m-4}\frac{(-1)^{i+2}(2m-3)!(2m-3-i)}{i!}x_0^{i-1}D^ih+\overline{D}u-\frac{2m-3}{x_0}u=$$

$$\sum_{i=0}^{2m-3-1}\frac{(-1)^{i+2}(2m-3)!(2m-3-i)}{i!}x_0^{i-1}D^ih+\sum_{i=0}^{1}\frac{(-1)^{i+1}(1-i)!1}{i!(1-i)!}x_0^{i-1}\overline{D}^iu,$$

所以

$$H_2\circ H_1(g)=\overline{D}(H_1(g))-\frac{2m-5}{x_0}(H_1(g))=$$

$$\sum_{i=0}^{2m-4}\frac{(-1)^{i+2}(2m-3)!(2m-3-i)(i-1)}{i!}x_0^{i-2}D^ih+$$

$$\sum_{i=0}^{2m-4}\frac{(-1)^{i+2}(2m-3)!(2m-3-i)}{i!}x_0^{i-1}\overline{D}D^ih-$$

$$\sum_{i=0}^{2m-4}\frac{(-1)^{i+2}(2m-3)!(2m-3-i)(2m-5)}{i!x_0}x_0^{i-1}D^ih+$$

$$\overline{D}^2u-\frac{2m-5}{x_0}\overline{D}u-\frac{2m-3}{x_0}\overline{D}u+\frac{2m-3}{x_0^2}u+\frac{(2m-5)(2m-3)}{x_0^2}u=$$

$$\sum_{i=0}^{2m-5}\frac{(-1)^{i+3}(2m-3)!(2m-3-i)(2m-i-4)}{i!}x_0^{i-2}D^ih+$$

$$\overline{D}^2u-\frac{4m-8}{x_0}\overline{D}u+\frac{(2m-3)(2m-4)}{x_0^2}u=$$

$$\sum_{i=0}^{2m-3-2}\frac{(-1)^{i+3}(2m-3)!(2m-3-i)(2m-i-4)}{i!}x_0^{i-2}D^ih+$$

$$\sum_{i=0}^{2}\frac{(-1)^{i+2}(3-i)!2}{i!(2-i)!}x_0^{i-2}\overline{D}^iu,$$

依此类推

$$H_k\circ H_{k-1}\cdots H_2\circ H_1(g)=$$

$$\sum_{i=0}^{2m-3-k}\frac{(-1)^{i+k+1}(2m-3)!(2m-3-i)\cdots(2m-3-i-(k-1))}{i!}x_0^{i-k}D^ih+$$

$$\sum_{i=0}^{k}\frac{(-1)^{i+k}(2k-1-i)!k}{i!(k-i)!}x_0^{i-k}\overline{D}^iu=$$

$$\sum_{i=0}^{2m-3-k}\frac{(-1)^{i+k+1}(2m-3)!(2m-3-i)!}{i!(2m-3-i-k)!}x_0^{i-k}D^ih+\sum_{i=0}^{k}\frac{(-1)^{i+k}(2k-1-i)!k}{i!(k-i)!}x_0^{i-k}\overline{D}^iu.$$

即

$$f=\sum_{i=0}^{2m-3-k}\frac{(-1)^{i+k+1}(2m-3)!(2m-3-i)!}{i!(2m-3-i-k)!}x_0^{i-k}D^ih+\sum_{i=0}^{k}\frac{(-1)^{i+k}(2k-1-i)!k}{i!(k-i)!}x_0^{i-k}\overline{D}^iu.$$

定理2.4 当方程(2.1)中的$k=m-1$时, 可相应变化为

$$\Delta f+\frac{1}{x_0}Df-\frac{(m-1)^2}{x_0^2}f=0. \tag{2.9}$$

令$g=\sum_{i=0}^{2m-3}\frac{(-1)^{i+1}(2m-3)!}{i!}x_0^iD^ih+u$, 则由定理2.3可知, f为方程(2.9)的解的充要条件为存在$u(x),h(x)\in F_{\Omega}^{(2m-1)}$, 使得

$$f=\sum_{i=0}^{m-1}\frac{(-1)^{m+i-1}(2m-3-i)!}{i!(m-1-i)!}x_0^{i-m+1}\left[(m-1)\overline{D}^iu-(m-1-i)D^ih\right]. \tag{2.10}$$

其中$Du=0,\overline{D}h=0$.

证　当$k=m-1$时,

$$f=\sum_{i=0}^{2m-3-(m-1)}\frac{(-1)^{m+i}(2m-3)!(2m-3-i)!}{i!(m-2-i)!}x_0^{i-m+1}D^ih^*+$$

$$\sum_{i=0}^{m-1}\frac{(-1)^{i+m-1}(2m-3-i)!(m-1)}{i!(m-1-i)!}x_0^{i-m+1}\overline{D}^iu=$$

$$\sum_{i=0}^{m-2}\frac{(-1)^{m+i}(2m-3)!(2m-3-i)!}{i!(m-2-i)!}x_0^{i-m+1}D^ih^*+$$

$$\sum_{i=0}^{m-1}\frac{(-1)^{i+m-1}(2m-3-i)!(m-1)}{i!(m-1-i)!}x_0^{i-m+1}\overline{D}^iu,$$

其中$\overline{D}h^*=0,Du=0$, 令$h=(2m-3)!h^*$, 则$\overline{D}h=0$, 那么存在h,u使得

$$f=\sum_{i=0}^{m-2}\frac{(-1)^{m+i}(2m-3-i)!}{i!(m-2-i)!}x_0^{i-m+1}D^ih+$$

$$\sum_{i=0}^{m-1}\frac{(-1)^{i+m-1}(2m-3-i)!(m-1)}{i!(m-1-i)!}x_0^{i-m+1}\overline{D}^iu=$$

$$\sum_{i=0}^{m-1}\frac{(-1)^{m+i}(2m-3-i)!(m-1-i)}{i!(m-1-i)!}x_0^{i-m+1}D^ih+$$

$$\sum_{i=0}^{m-1}\frac{(-1)^{i+m-1}(2m-3-i)!(m-1)}{i!(m-1-i)!}x_0^{i-m+1}\overline{D}^iu=$$

$$\sum_{i=0}^{m-1}\frac{(-1)^{i+m-1}(2m-3-i)!}{i!(m-1-i)!}x_0^{i-m+1}\left[(m-1)\overline{D}^iu-(m-1-i)D^ih\right]$$

是方程(2.9)的解, 其中$Du=0,\overline{D}h=0$.

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The problem of solving a class of partial differential equations in Clifford analysis

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Abstract: This paper investigates the problem of solving a class of generalized second-order partial differential equations in Clifford analysis, which are related to generalized Weinstein equations when the solutions are real or vector-valued. Firstly, two vector differential operators in Clifford analysis are constructed, their properties and relations are studied, and the solution of the partial differential equation $k=0$ is obtained. Then, according to the properties of differential operators, the expression of the general solution of the partial differential equation is given iteratively, and the special solution for $k=m-1$ is studied.

Keywords: Clifford analysis; differential operator; partial differential equation; generalized Weinstein equation

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