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泛Clifford分析中的k-向量正则函数与相关算子的性质

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摘　要: 该文首先定义了泛Clifford分析的T_i算子并研究了T_i算子与Dirac算子的交换性质, 其次利用Dirac算子和T_i算子构造L算子和R算子, 定义了取值在泛Clifford代数上的k-向量正则函数, 并且研究了k-向量正则函数与其他正则函数类和广义调和函数的关系. 最后给出了泛Clifford代数空间中与R算子和L算子相关的广义的Stokes公式, 它是后续找出k-向量正则函数积分表示的重要基础.

关键词: 泛Clifford代数; L算子; R算子; k-向量正则函数; Stokes公式

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§1　引　言

作为现代分析的重要分支之一, 自1970年以来, 取值在Clifford代数上的函数理论及其应用已被很多学者研究[1-11], 其中很重要的一项工作就是利用Clifford分析中的正则函数来求解高维空间中偏微分方程, 例如Weinstein方程和Vekua方程. 黄沙和李生训给出复Clifford分析中一类二阶偏微分方程于Lie球双曲空间上的Dirichlet问题解的存在性和解的积分表示式[12]. 2006年, Leutwiler和Eriksson给出了k-超正则函数的定义[13]. 2009年, 他们又找出了k-超正则函数的积分表示[14]. 从另一角度来看, k-超正则函数的积分表示就是上半空间$\mathbf{R}_+^{n+1}=\left\{(x_0,x_1,\cdots,x_n)\in\mathbf{R}^{n+1}\mid x_n>0\right\}$上的Weinstein方程的解. 袁洪芬和乔玉英给出了k-超正则函数的开拓定理和唯一性定理, 通过k-超正则函数的P部和Q部满足的两个微分方程, 讨论了此方程与k-超正则函数及其相关函数的关系[15]. 2019年, Doan Cong Dihn在Clifford分析中将h-正则函数与k-超正则函数统一了起来, 引入了(m,h)-正则函数, 并将其与轴对称Helmholtz方程联系起来[16]. 2020年Doan Cong Dihn又在Clifford分析中引入了广义k_i-正则函数, 并将其与广义Weinstein方程联系了起来[17].

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近年来, 取值在泛Clifford代数上的函数论被越来越多的学者所青睐. 2002年, Du Jinyuan, Zhang Zhongxiang给出了泛Clifford代数上特征边界上的正则函数的积分表示[18]. 2011年, Du Jinyuan和Xu Na研究了泛Clifford代数上的Cauchy型积分的边界行为[19]. 2021年, Xu Na, Li Zunfeng, Yang Heju得到了取值在泛Clifford代数上的复正则函数的积分表示[20].

本文首先引入T_i算子并得到其与Dirac算子的交换性质, 进而利用Dirac算子和T_i算子构造了L算子和R算子, 给出取值在泛Clifford代数空间上的k-向量正则函数. 随后研究了LR-k向量正则函数与两个高维偏微分方程组的解的关系, 本文讨论的LR-k向量正则函数是这两个偏微分方程组的解. 本文最后给出与L算子和R算子相关的广义的Stokes公式, 它是后续找出k-向量正则函数积分表示的重要准备工作.

§2 预备知识

设$V_{n,s}$是以$\{e_1,\cdots,e_s,\cdots,e_n\}\,(1\leq s\leq n)$为基的$n$-维实线性空间. 考虑以$\{e_A, A=(h_1,\cdots,h_r)\in\mathcal{P}N, 1\leq h_1<\cdots<h_r\leq n\}$为基的$2^{n-1}$维实向量空间$C(V_{n,s})$, N为集合$\{1,\cdots,n\}$, $\mathcal{P}N$表示N的所有保序子集. $e_{h_1}e_{h_2}\cdots e_{h_r}$记为$e_A$, 这里$A=\{h_1,\cdots,h_r\}\in\mathcal{P}N$. $C(V_{n,s})$中的乘法规则定义为

$$\begin{cases} e_Ae_B=(-1)^{\sharp((A\cap B)\setminus S)}(-1)^{P(A,B)}e_{A\triangle B},\ \ A,B\in\mathcal{P}N, \\ \\ \lambda\mu=\sum\limits_{A\in\mathcal{PN}}\sum\limits_{B\in\mathcal{PN}}\lambda_A\mu_Be_Ae_B, \lambda=\sum\limits_{A\in\mathcal{PN}}\lambda_Ae_A, \mu=\sum\limits_{B\in\mathcal{PN}}\mu_Be_B, \end{cases}\tag{1}$$

其中S为集合$\{1,\cdots,s\}$, $\sharp(A)$为集合A中的元素个数, 并且$P(A,B)=\sum\limits_{j\in B}P(A,j), P(A,j)=\sharp\{i,i\in A,i>j\}$, 对称差集$A\triangle B$也是按上述方式保序的, $\lambda_A,\mu_B\in\mathbf{R}$分别是$e_A$与$e_B$的系数. 根据乘法规则(1), e_1为单位元, 同时有

$$\begin{cases} e_i^2=1, & i=1,2,\cdots,s, \\ e_j^2=-1, & j=s+1,s+2,\cdots,n, \\ e_ie_j=-e_je_i, & 1\leq i<j\leq n. \\ e_{h_1}e_{h_2}\cdots e_{h_r}=e_{h_1h_2\cdots h_r}, & 1\leq h_1<h_2<\cdots<h_r\leq n. \end{cases}\tag{2}$$

显然, $C(V_{n,s})$是一个实线性、可结合但不可以交换的代数, 它被称为$V_{n,s}$上的泛Clifford代数. 为了定义内积与相应的模, 现在定义$C(V_{n,s})$上的对合运算为

$$\begin{cases} \overline{e_A}=(-1)^{\sigma(A)+\sharp(A\cap S)}e_A, & A\in\mathcal{PN}, \\ \\ \bar{\lambda}=\sum\limits_{A\in\mathcal{PN}}\lambda_A\overline{e_A}, & \lambda=\sum\limits_{A\in\mathcal{PN}}\lambda_Ae_A. \end{cases}\tag{3}$$

其中$\sigma(A)=\sharp(A)(\sharp(A)+1)/2$. 根据(3)有

$$\begin{cases} \overline{e_i}=e_i, & i=1,\cdots,s, \\ \overline{e_j}=-e_j, & j=s+1,\cdots,n, \\ \overline{\lambda\mu}=\overline{\mu}\overline{\lambda}, & \lambda,\mu\in C(V_{n,s}). \end{cases}\tag{4}$$

$\forall\lambda,\mu\in C(V_{n,s})$, 定义它们的内积为

$$(\lambda,\mu)=\sum_{A,A}\lambda_A\mu_A,\ \text{其中}\ \lambda=\sum_{A\in\mathcal{PN}}\lambda_Ae_A, \mu=\sum_{A\in\mathcal{PN}}\mu_Ae_A.\tag{5}$$

所以$C(V_{n,s})$上的模定义为

$$|\lambda|=\sqrt{(\lambda,\lambda)}=(\sum_{A\in\mathcal{PN}}|\lambda_A|^2)^{\frac{1}{2}}. \tag{6}$$

设Ω是$\mathbf{R}^n$上的一个非空开集. 现在考虑定义在Ω上取值在$C(V_{n,s})$上的函数

$$f:\Omega\longrightarrow C(V_{n,s}),$$

它记为

$$f(x)=\sum_A f_A(x)e_A,\ x=(x_1,x_2,\cdots,x_n)\in\Omega,$$

其中f_A是Ω上的实值函数, 并且称其为函数f的e_A-分量函数. 当所有的函数f_A具有连续性、可微性等性质时, 函数f也都具有这些性质, 即$f\in C^{(r)}(\Omega,C(V_{n,s}))$的充要条件是$f_A\in C^{(r)}(\Omega,C(V_{n,s}))(r\geq 1)$. 函数$f$的共轭表示为

$$\overline{f(x)}=\sum_A f_A(x)\overline{e_A},\ x\in\Omega.$$

引入算子[1]

$$D=\sum_{k=1}^n e_k\frac{\partial}{\partial x_k}:C^{(r)}(\Omega,C(V_{n,s}))\longrightarrow C^{(r-1)}(\Omega,C(V_{n,s})).$$

该算子从左或从右作用于函数要遵循规定

$$D\,[f]=\sum_{k=1}^n\sum_A e_ke_A\frac{\partial f_A}{\partial x_k},\ \ [f]\,D=\sum_{k=1}^n\sum_A e_Ae_k\frac{\partial f_A}{\partial x_k}.$$

由于$e_i(1\leq i\leq s)$是不同于$e_j(s<j\leq n)$的, 所以将算子D分为D_1,D_2两部分, 其中

$$D_1=\sum_{k=1}^s e_k\frac{\partial}{\partial x_k}:C^{(r)}(\Omega,C(V_{n,s}))\longrightarrow C^{(r-1)}(\Omega,C(V_{n,s})),$$

$$D_2=\sum_{k=s+1}^n e_k\frac{\partial}{\partial x_k}:C^{(r)}(\Omega,C(V_{n,s}))\longrightarrow C^{(r-1)}(\Omega,C(V_{n,s})).$$

它们的共轭算子分别为

$$\overline{D_1}=\sum_{k=1}^s\overline{e_k}\frac{\partial}{\partial x_k}=\sum_{k=1}^s e_k\frac{\partial}{\partial x_k}=D_1,\qquad \overline{D_2}=\sum_{k=s+1}^n\overline{e_k}\frac{\partial}{\partial x_k}=-\sum_{k=s+1}^n e_k\frac{\partial}{\partial x_k}=-D_2.$$

将Laplace算子Δ记为$\Delta_1+\Delta_2$, 其中

$$\Delta_1=\sum_{i=1}^s\frac{\partial^2}{\partial x_i^2},\ \ \Delta_2=\sum_{i=s+1}^n\frac{\partial^2}{\partial x_i^2}.$$

则有$D_1\overline{D_1}=\overline{D_1}D_1=\Delta_1,D_2\overline{D_2}=\overline{D_2}D_2=\Delta_2$, 且$\Delta=\Delta_1+\Delta_2=D_1\overline{D_1}+D_2\overline{D_2}$.

§3 k-向量正则函数

对于$f(x)=\sum\limits_A f_A(x)e_A\in C(\Omega,C(V_{n,s}))$, 引入算子

$$T_i(f(x))=\sum_A(-1)^{\delta_{i,A}}f_A(x)e_A,$$

其中$\delta_{i,A}=\begin{cases}1, & 若\quad i\in A,\\ 0. & 若\quad i\notin A.\end{cases}$

定理3.1 对于任意的$m\in\{1,\cdots,n\}$, $i\in\{1,\cdots,s\}$, $j\in\{s+1,\cdots,n\}$, $\alpha\in\mathcal{PN}$, T_i算

子满足如下性质

1) $T_iT_jf=T_jT_if\ (i\neq j)$, $T_m(T_mf)=f$, $T_m(fg)=T_m(f)T_m(g)$,

2) $T_m(e_\alpha f)=(-1)^{\delta_{m,\alpha}}e_\alpha T_mf$,

3) $\overline{D_1}\left[e_iT_i(f)\right]=-e_iT_i(D_1f)$, $\left[T_j(f)e_j\right]\overline{D_2}=T_j(fD_2)e_j$.

证 1) 当$i\neq j$时,

$$T_iT_jf=(-1)^{\delta_{i,A}}\Big[\sum_A(-1)^{\delta_{j,A}}f_A(x)e_A\Big]=\sum_A(-1)^{\delta_{j,A}}(-1)^{\delta_{i,A}}f_A(x)e_A=T_jT_if,$$

当$i=j=m$时有

$$T_m(T_mf)=T_m\Big[\sum_A(-1)^{\delta_{m,A}}f_A(x)e_A\Big]=\sum_A(-1)^{2\delta_{m,A}}f_A(x)e_A=\sum_A f_A(x)e_A=f,$$

对于任意的m有

$$\begin{aligned}T_m(fg)=\ &T_m\Big[\sum_A f_Ae_A\sum_B g_Be_B\Big]=\sum_{A,B}(-1)^{\delta_{m,A+B}}f_Ag_Be_Ae_B=\\&\sum_A(-1)^{\delta_{m,A}}f_Ae_A\sum_B(-1)^{\delta_{m,B}}g_Be_B=T_m(f)T_m(g),\end{aligned}$$

2) 当$m\in\alpha$且$m\in A$时有

$$T_m(e_\alpha f)=T_m\Big[\sum_A f_A(x)e_\alpha e_A\Big]=\sum_A f_A(x)e_\alpha e_A,$$

$$(-1)^{\delta_{m,\alpha}}e_\alpha T_mf=(-1)^{\delta_{m,\alpha}}e_\alpha\Big[\sum_A(-1)^{\delta_{m,A}}f_A(x)e_A\Big]=\sum_A f_A(x)e_\alpha e_A.$$

所以$T_m(e_\alpha f)=(-1)^{\delta_{m,\alpha}}e_\alpha T_mf$.

当$m\in\alpha$且$m\notin A$时有

$$T_m(e_\alpha f)=T_m\Big[\sum_A f_A(x)e_\alpha e_A\Big]=-\sum_A f_A(x)e_\alpha e_A,$$

$$(-1)^{\delta_{m,\alpha}}e_\alpha T_mf=(-1)^{\delta_{m,\alpha}}e_\alpha\Big[\sum_A(-1)^{\delta_{m,A}}f_A(x)e_A\Big]=-\sum_A f_A(x)e_\alpha e_A.$$

所以$T_m(e_\alpha f)=(-1)^{\delta_{m,\alpha}}e_\alpha T_mf$.

当$m\notin\alpha$且$m\in A$时有

$$T_m(e_\alpha f)=T_m\Big[\sum_A f_A(x)e_\alpha e_A\Big]=-\sum_A f_A(x)e_\alpha e_A,$$

$$(-1)^{\delta_{m,\alpha}}e_\alpha T_mf=(-1)^{\delta_{m,\alpha}}e_\alpha\Big[\sum_A(-1)^{\delta_{m,A}}f_A(x)e_A\Big]=-\sum_A f_A(x)e_\alpha e_A.$$

所以$T_m(e_\alpha f)=(-1)^{\delta_{m,\alpha}}e_\alpha T_mf$.

当$m\notin\alpha$且$m\notin A$时有

$$T_m(e_\alpha f)=T_m\Big[\sum_A f_A(x)e_\alpha e_A\Big]=\sum_A f_A(x)e_\alpha e_A,$$

$$(-1)^{\delta_{m,\alpha}}e_\alpha T_mf=(-1)^{\delta_{m,\alpha}}e_\alpha\Big[\sum_A(-1)^{\delta_{m,A}}f_A(x)e_A\Big]=\sum_A f_A(x)e_\alpha e_A.$$

所以$T_m(e_\alpha f)=(-1)^{\delta_{m,\alpha}}e_\alpha T_mf$.

3) 由2)可得

$$\overline{D_1}\left[e_iT_i(f)\right]=\sum_{k=1}^{s}e_k\frac{\partial\left[e_iT_i(f)\right]}{\partial x_k}=\sum_{k=i}\frac{\partial T_i(f)}{\partial x_i}+\sum_{k\neq i}e_ke_i\frac{\partial T_i(f)}{\partial x_k};$$

$$-e_iT_i(D_1f)=-e_iT_i(\sum_{k=1}^{s}e_k\frac{\partial f}{\partial x_k})=\sum_{k=i}\frac{\partial T_i(f)}{\partial x_i}-\sum_{k\neq i}e_ie_k\frac{\partial T_i(f)}{\partial x_k};$$

$$[T_j(f)e_j]\,\overline{D_2}=-\sum_{k=s+1}^{n}\frac{\partial\,[T_j(f)e_j]}{\partial x_k}e_k=\sum_{k=j}\frac{\partial T_j(f)}{\partial x_j}-\sum_{k\neq j}\frac{\partial T_j(f)}{\partial x_k}e_je_k;$$

$$T_j(fD_2)e_j=T_j(\sum_{k=s+1}^{n}\frac{\partial f}{\partial x_k}e_k)e_j=\sum_{k=j}\frac{\partial T_j(f)}{\partial x_j}+\sum_{k\neq j}\frac{\partial T_j(f)}{\partial x_k}e_ke_j.$$

又由于$e_ie_k=-e_ke_i, e_je_k=-e_ke_j$, 故有$\overline{D_1}\,[e_iT_i(f)]=-e_iT_i(D_1f)$, $[T_j(f)e_j]\,\overline{D_2}=T_j(fD_2)e_j$.

定义3.2 设$\lambda=\sum\limits_{A\in\mathcal{P}N}\lambda_Ae_A\in C(V_{n,s})$, 其中$\lambda_A\in\mathbf{R}$, 则称$\mathrm{Re}\lambda=\lambda_\emptyset$.

由上述定义可知$\mathrm{Re}(T_i\lambda)=\mathrm{Re}\lambda$.

定理3.3 若存在$i\in\{1,2,\cdots,n\}$使得$T_i\lambda=-\lambda$, 则$\mathrm{Re}\lambda=0$.

证 因为

$$T_i\lambda=T_i\left(\lambda_\emptyset+\sum_{A\in\mathcal{P}N,A\neq\emptyset}\lambda_Ae_A\right)=\lambda_\emptyset+\sum_{A\in\mathcal{P}N,A\neq\emptyset}\lambda_AT_i(e_A).$$

又因为$\mathrm{Re}(T_i\lambda)=\lambda_\emptyset$, $\mathrm{Re}(-\lambda)=-\lambda_\emptyset$. 所以$\lambda_\emptyset=-\lambda_\emptyset$, 也就是$\lambda_\emptyset=0$, 即$\mathrm{Re}\lambda=0$.

对于任意的n维向量$k=(k_1,\cdots,k_n)$ $(k_i\in\mathbf{R})$, 算子L、R和它们的共轭算子表示为

$$\begin{cases}Lf=D\,[f]+\sum\limits_{i=1}^{n}\dfrac{k_ie_i}{x_i}T_i(f),\\[2ex] fR=[f]\,D+\sum\limits_{i=1}^{n}\dfrac{k_i}{x_i}T_i(f)e_i,\end{cases}\tag{7}$$

$$\begin{cases}\overline{L}f=\overline{D}\,[f]+\sum\limits_{i=1}^{n}\dfrac{k_i\overline{e_i}}{x_i}T_i(f),\\[2ex] f\overline{R}=[f]\,\overline{D}+\sum\limits_{i=1}^{n}\dfrac{k_i}{x_i}T_i(f)\overline{e_i}.\end{cases}\tag{8}$$

定义3.4 设$\Omega=\{x=(x_1,\cdots,x_n)|x_i>0,i=1,\cdots,n\}$是$\mathbf{R}^n$中的非空开集. 若函数$f\in C^{(r)}(\Omega,C(V_{n,s}))(r\geq1)$且在$\Omega$中满足$Lf=0(fR=0)$, 则称$f$为$\Omega$上的左(右)$k$-向量正则函数.

注1 当$k=(0,\cdots,0)$时, 定义3.4下的k向量正则函数是Clifford分析中的经典正则函数[1].

注2 当$k=(0,\cdots,0,m)$时, 若f是定义3.4下的k-向量正则函数, 则函数$x_n^mf(x)$是[13]中定义的$2m$-超正则函数.

由于$D=D_1+D_2, \overline{D_1}=D_1, \overline{D_2}=-D_2$, 所以(7)-(8)可以等价写为

$$\begin{cases}L_1f=D_1\,[f]+\sum\limits_{i=1}^{s}\dfrac{k_ie_i}{x_i}T_i(f),\\[2ex] L_2f=D_2\,[f]+\sum\limits_{i=s+1}^{n}\dfrac{k_ie_i}{x_i}T_i(f).\end{cases}\tag{9}$$

$$\begin{cases} fR_1=[f]\,D_1+\sum\limits_{i=1}^{s}\dfrac{k_i}{x_i}T_i(f)e_i, \\ \\ fR_2=[f]\,D_2+\sum\limits_{i=s+1}^{n}\dfrac{k_i}{x_i}T_i(f)e_i, \end{cases} \tag{10}$$

并且$\overline{L_1}f=L_1f$, $f\overline{R_1}=fR_1$, $\overline{L_2}f=-L_2f$, $f\overline{R_2}=-fR_2$.

定义3.5 当函数$f\in C^{(r)}(\Omega,C(V_{n,s}))(r\geq 1)$且在$\Omega$中满足$L_1f=0(fR_1=0)$时, 那么称它为$L_1$左($R_1$右)$k$-向量正则函数. 当其满足$L_2f=0(fR_2=0)$时, 那么称它为$L_2$左($R_2$右)$k$-向量正则函数.

定义3.6 当函数$f\in C^{(r)}(\Omega,C(V_{n,s}))(r\geq 1)$且在$\Omega$中同时满足$L_1f=0$与$fR_2=0$时, 则称函数$f$为$\Omega$上的$LR$-$k$-向量正则函数.

接下来讨论k-向量正则函数与几类偏微分方程的解的关系.

定理3.7 设$f=\sum\limits_{A}f_Ae_A\in C^{(r)}(\Omega,C(V_{n,s}))(r\geq 2)$, 则当$f$是$LR$-$k$向量正则函数时, 有

$$\begin{cases} \varDelta_1f_A-\sum\limits_{i=1}^{s}\dfrac{k_i\left[k_i+(-1)^{\delta_{i,A}}\right]}{x_i^2}f_A=0, \\ \\ \varDelta_2f_A-\sum\limits_{j=s+1}^{n}\dfrac{k_j\left[k_j+(-1)^{\delta_{j,A}}\right]}{x_j^2}f_A=0. \end{cases} \tag{11}$$

证

$$\overline{L_1}L_1f=\overline{D_1}\left[D_1\,[f]+\sum_{j=1}^{s}\frac{k_je_j}{x_j}T_j(f)\right]+\sum_{i=1}^{s}\frac{k_ie_i}{x_i}T_i\left[D_1\,[f]+\sum_{j=1}^{s}\frac{k_je_j}{x_j}T_j(f)\right]=$$
$$\varDelta_1f+\sum_{j=1}^{s}\overline{D_1}\left[\frac{k_je_j}{x_j}T_j(f)\right]+\sum_{i=1}^{s}\frac{k_ie_i}{x_i}T_i(D_1\,[f])+\sum_{i,j=1}^{s}\frac{k_ik_j}{x_ix_j}e_iT_i\,[e_jT_j(f)]=$$
$$\varDelta_1f-\sum_{i=1}^{s}\frac{k_i}{x_i^2}T_i(f)+\sum_{j=1}^{s}\frac{k_j}{x_j}\overline{D_1}\,[e_jT_j(f)]+\sum_{i=1}^{s}\frac{k_i}{x_i}e_iT_i(D_1\,[f])+\sum_{i,j=1}^{s}\frac{k_ik_j}{x_ix_j}e_iT_i\,[e_jT_j(f)]\,.$$

由定理3.1的3)可得

$$\sum_{j=1}^{s}\frac{k_j}{x_j}\overline{D_1}\Big[e_jT_j(f)\Big]=-\sum_{j=1}^{s}\frac{k_j}{x_j}e_jT_j(D_1f)=-\sum_{i=1}^{s}\frac{k_i}{x_i}e_iT_i(D_1f),$$

所以有 $\overline{L_1}L_1f=\varDelta_1f-\sum\limits_{i=1}^{s}\dfrac{k_i}{x_i^2}T_i(f)+\sum\limits_{i,j=1}^{s}\dfrac{k_ik_j}{x_ix_j}e_iT_i\,[e_jT_j(f)]$, 由定理3.1的1)和2)可得

$$\sum_{i,j=1}^{s}\frac{k_ik_j}{x_ix_j}e_iT_i\,[e_jT_j(f)]=\sum_{i,j=1}^{s}\frac{k_ik_j}{x_ix_j}(-1)^{\delta_{i,j}}e_ie_jT_i\,[T_j(f)]=$$
$$\sum_{i,j=1;i\neq j}^{s}\frac{k_ik_j}{x_ix_j}e_ie_jT_iT_jf+\sum_{i=j=1}^{s}\frac{-{k_i}^2}{{x_i}^2}f=-\sum_{i=1}^{s}\frac{{k_i}^2}{{x_i}^2}f,$$

所以有

$$\begin{aligned}\overline{L_1}L_1 f = \quad & \varDelta_1 f-\sum_{i=1}^{s}\frac{k_i}{x_i^2}T_i(f)-\sum_{i=1}^{s}\frac{k_i^2}{x_i^2}f=\varDelta_1 f-\sum_{i=1}^{s}\frac{k_i}{x_i^2}\left[T_i(f)+k_i f\right]=\\ & \sum_A\left[\varDelta_1 f_A-\sum_{i=1}^{s}\frac{k_i\left[k_i+(-1)^{\delta_{i,A}}\right]}{x_i^2}f_A\right]e_A,\end{aligned}$$

因为f是LR-k向量正则函数, 所以$\overline{L_1}L_1 f=0$, 即

$$\sum_A\left[\varDelta_1 f_A-\sum_{i=1}^{s}\frac{k_i\left[k_i+(-1)^{\delta_{i,A}}\right]}{x_i^2}f_A\right]e_A=0,$$

所以

$$\varDelta_1 f_A-\sum_{i=1}^{s}\frac{k_i\left[k_i+(-1)^{\delta_{i,A}}\right]}{x_i^2}f_A=0.$$

$$\begin{aligned}& fR_2\overline{R_2}=\left[[f]\,D_2+\sum_{j=s+1}^{n}\frac{k_j}{x_j}T_j(f)e_j\right]\overline{D_2}+\sum_{i=s+1}^{n}\frac{k_i}{x_i}T_i\left[[f]\,D_2+\sum_{j=s+1}^{n}\frac{k_j}{x_j}T_j(f)e_j\right]\overline{e_i}=\\ & f\varDelta_2+\left[\sum_{j=s+1}^{n}\frac{k_j}{x_j}T_j(f)e_j\right]\overline{D_2}+\sum_{i=s+1}^{n}\frac{k_i}{x_i}T_i\left[(f)D_2\right]\overline{e_i}+\sum_{i,j=s+1}^{n}\frac{k_ik_j}{x_ix_j}T_i\left[T_j(f)e_j\right]\overline{e_i}=\\ & f\varDelta_2-\sum_{j=s+1}^{n}\frac{k_j}{x_j^2}T_j(f)+\sum_{j=s+1}^{n}\frac{k_j}{x_j}\left[T_j(f)e_j\right]\overline{D_2}+\sum_{i=s+1}^{n}\frac{k_i}{x_i}T_i\left[(f)D_2\right]\overline{e_i}+\\ & \qquad\qquad\qquad\sum_{i,j=s+1}^{n}\frac{k_ik_j}{x_ix_j}T_i\left[T_j(f)e_j\right]\overline{e_i}.\end{aligned}$$

由定理3.1的3), 可得

$$\sum_{j=s+1}^{n}\frac{k_j}{x_j}\Big[T_j(f)e_j\Big]\overline{D_2}=\sum_{j=s+1}^{n}\frac{k_j}{x_j}T_j(fD_2)e_j=-\sum_{i=s+1}^{n}\frac{k_i}{x_i}T_i(fD_2)\overline{e_j},$$

所以有

$$fR_2\overline{R_2}=\quad f\varDelta_2-\sum_{j=s+1}^{n}\frac{k_j}{x_j^2}T_j(f)+\sum_{i,j=s+1}^{n}\frac{k_ik_j}{x_ix_j}T_i\left[T_j(f)e_j\right]\overline{e_i},$$

由定理3.1的1)和2)可得

$$\begin{aligned}\sum_{i,j=s+1}^{n}\frac{k_ik_j}{x_ix_j}T_i\left[T_j(f)e_j\right]\overline{e_i}&=\sum_{i,j=s+1}^{n}\frac{k_ik_j}{x_ix_j}(-1)^{\delta_{i,j}}e_j\overline{e_i}T_iT_j(f)=\\ \sum_{i,j=s+1;i\neq j}^{n}\frac{k_ik_j}{x_ix_j}e_j\overline{e_i}T_iT_j(f)&-\sum_{i=j=s+1}^{n}\frac{{k_i}^2}{{x_i}^2}f=-\sum_{i=j=s+1}^{n}\frac{{k_i}^2}{{x_i}^2}f,\end{aligned}$$

所以有

$$\begin{aligned}& fR_2\overline{R_2}=f\varDelta_2-\sum_{j=s+1}^{n}\frac{k_j}{x_j^2}T_j(f)-\sum_{j=s+1}^{n}\frac{{k_j}^2}{{x_j}^2}f=\\ & f\varDelta_2-\sum_{j=s+1}^{n}\frac{k_j}{x_j^2}\left[T_j(f)+k_jf\right]=\sum_A\left[f_A\varDelta_2-\sum_{i=s+1}^{n}\frac{k_j\left[k_j+(-1)^{\delta_{j,A}}\right]}{x_j^2}f_A\right]e_A.\end{aligned}$$

因为f是LR-k向量正则函数, 所以$fR_2\overline{R_2}=0$, 即

$$\sum_A\left[f_A\varDelta_2-\sum_{j=s+1}^{n}\frac{k_j\left[k_j+(-1)^{\delta_{j,A}}\right]}{x_j^2}f_A\right]e_A=0,$$

那么$f_A\mathit{\Delta}_2-\sum\limits_{j=s+1}^{n}\dfrac{k_j\left[k_j+(-1)^{\delta_{j,A}}\right]}{x_j^2}f_A=0$. 因此(11)得证.

定理3.8 对于任意的$f=\sum\limits_A f_Ae_A\in C^{(r)}(\mathit{\Omega},C(V_{n,s}))(r\geq 2)$, 当$f$是$LR$-$k$向量正则函数, 且$f_A=g_A\prod\limits_{j=1}^{n}x_j^{k_j(-1)^{1+\delta_{j,A}}}$时, 有$g_A$为方程组

$$\begin{cases}\mathit{\Delta}_1 g_A-2\sum\limits_{i\notin\alpha_1}\dfrac{k_i}{x_i}\dfrac{\partial g_A}{\partial x_i}+2\sum\limits_{i\in\alpha_1}\dfrac{k_i}{x_i}\dfrac{\partial g_A}{\partial x_i}=0,\\ \mathit{\Delta}_2 g_A-2\sum\limits_{i\notin\alpha_2}\dfrac{k_i}{x_i}\dfrac{\partial g_A}{\partial x_i}+2\sum\limits_{i\in\alpha_2}\dfrac{k_i}{x_i}\dfrac{\partial g_A}{\partial x_i}=0\end{cases}\tag{12}$$

的解.

证 由$f_A=g_A\prod\limits_{j=1}^{n}x_j^{k_j(-1)^{1+\delta_{j,A}}}$可得

$$\begin{aligned}\mathit{\Delta}_1 f_A=&\mathit{\Delta}_1\left(g_A\prod_{j=1}^{n}x_j^{k_j(-1)^{1+\delta_{j,A}}}\right)=\sum_{m=1}^{s}\frac{\partial^2\Big(g_A\prod\limits_{j=1}^{n}x_j^{k_j(-1)^{1+\delta_{j,A}}}\Big)}{\partial x_m^2}=\\&\sum_{m=1}^{s}\frac{\partial^2 g_A}{\partial x_m^2}\prod_{j=1}^{n}x_j^{k_j(-1)^{1+\delta_{j,A}}}+2\sum_{m=1}^{s}\frac{\partial g_A}{\partial x_m}\frac{k_m(-1)^{1+\delta_{j,A}}\prod\limits_{j=1}^{n}x_j^{k_j(-1)^{1+\delta_{j,A}}}}{x_m}+\\&\sum_{m=1}^{s}\left[g_Ak_m(-1)^{1+\delta_{m,A}}\left[k_m(-1)^{1+\delta_{m,A}}-1\right]\frac{\prod\limits_{j=1}^{n}x_j^{k_j(-1)^{1+\delta_{j,A}}}}{x_m^2}\right],\end{aligned}$$

将$f_A=g_A\prod\limits_{j=1}^{n}x_j^{k_j(-1)^{1+\delta_{j,A}}}$代入(11)可得

$$\begin{aligned}&\mathit{\Delta}_1 f_A-\sum_{i=1}^{s}\frac{k_i\left[k_i+(-1)^{\delta_{i,A}}\right]}{x_i^2}f_A=\\&\sum_{m=1}^{s}\frac{\partial^2 g_A}{\partial x_m^2}\prod_{j=1}^{n}x_j^{k_j(-1)^{1+\delta_{j,A}}}+2\sum_{m=1}^{s}\frac{\partial g_A}{\partial x_m}\frac{k_m(-1)^{1+\delta_{j,A}}\prod\limits_{j=1}^{n}x_j^{k_j(-1)^{1+\delta_{j,A}}}}{x_m}+\\&\sum_{m=1}^{s}\left[g_Ak_m(-1)^{1+\delta_{m,A}}\left[k_m(-1)^{1+\delta_{m,A}}-1\right]\frac{\prod\limits_{j=1}^{n}x_j^{k_j(-1)^{1+\delta_{j,A}}}}{x_m^2}\right]-\\&\frac{\sum\limits_{i=1}^{s}k_i\left[k_i+(-1)^{\delta_{i,A}}\right]g_A\prod\limits_{j=1}^{n}x_j^{k_j(-1)^{1+\delta_{j,A}}}}{x_i^2}=\end{aligned}$$

$$\varDelta_1 g_A+2\sum_{m=1}^{s}\frac{\partial g_A}{\partial x_m}\frac{k_m(-1)^{1+\delta_{j,A}}}{x_m}+\sum_{m=1}^{s}\frac{g_A k_m\left[k_m+(-1)^{\delta_{m,A}}\right]}{x_m^2}-\sum_{i=1}^{s}\frac{k_i\left[k_i+(-1)^{\delta_{i,A}}\right]g_A}{x_i^2}=$$
$$\varDelta_1 g_A-2\sum_{i=1}^{s}\frac{\partial g_A}{\partial x_i}\frac{k_i(-1)^{\delta_{i,A}}}{x_i}=0.$$

同理可得$\varDelta_2 g_A-2\sum\limits_{i=s+1}^{n}\frac{\partial g_A}{\partial x_i}\frac{k_i(-1)^{\delta_{i,A}}}{x_i}=0$, 所以有

$$\begin{cases}\varDelta_1 g_A-2\sum\limits_{i=1}^{s}\dfrac{(-1)^{\delta_{i,A}}k_i}{x_i}\dfrac{\partial g_A}{\partial x_i}=0,\\[2ex] \varDelta_2 g_A-2\sum\limits_{i=s+1}^{n}\dfrac{(-1)^{\delta_{i,A}}k_i}{x_i}\dfrac{\partial g_A}{\partial x_i}=0.\end{cases}\tag{13}$$

设$A=\alpha_1+\alpha_2$, 其中$\alpha_1\in\{1,2,\cdots,s\},\alpha_2\in\{s+1,\cdots,n\}$, 则

$$\sum_{i=1}^{s}\frac{(-1)^{\delta_{i,A}}k_i}{x_i}\frac{\partial g_A}{\partial x_i}=\sum_{i\notin\alpha_1}\frac{k_i\partial g_A}{x_i\partial x_i}-\sum_{i\in\alpha_1}\frac{k_i\partial g_A}{x_i\partial x_i},$$
$$\sum_{i=s+1}^{n}\frac{(-1)^{\delta_{i,A}}k_i}{x_i}\frac{\partial g_A}{\partial x_i}=\sum_{i\notin\alpha_2}\frac{k_i\partial g_A}{x_i\partial x_i}-\sum_{i\in\alpha_2}\frac{k_i\partial g_A}{x_i\partial x_i},$$

因此(12)成立.

定理3.9 (广义Stokes公式) 令$\varOmega_1$是$\mathbf{R}^s$中有界域, $\varOmega_2$是$\mathbf{R}^{n-s}$中有界域, $\partial\varOmega_1,\partial\varOmega_2$分别为$\varOmega_1$, $\varOmega_2$的边界且都为实的定向的光滑曲面. 设$f\in C^{(r)}(\varOmega_1\times\varOmega_2,C(V_{n,s}))$, $g\in C^{(r)}(\varOmega_1,C(V_{n,s}))$, $h\in C^{(r)}(\varOmega_2,C(V_{n,s}))$. 则有

$$\mathrm{Re}\left[\int_{\partial\varOmega_1\times\partial\varOmega_2}g(x^S)\mathrm{d}\sigma_1 f(x^S,x^{N\backslash S})\mathrm{d}\sigma_2 h(x^{N\backslash S})\right]=$$
$$\mathrm{Re}\int_{\varOmega_1\times\varOmega_2}\left[(gR_1)(fR_2)h+(gR_1)f(L_2h)+g(L_1(fR_2))h+g(L_1f)(L_2h)\right]\mathrm{d}x,$$

其中$\mathrm{d}\sigma_1=\vec{n_1}\mathrm{d}\theta^S$, $\mathrm{d}\sigma_2=\vec{n_2}\mathrm{d}\theta^{N\backslash S}$, $\vec{n_1}$和$\vec{n_2}$分别是$\partial\varOmega_1$和$\partial\varOmega_2$的外法向量, $\mathrm{d}\theta^S$和$\mathrm{d}\theta^{N\backslash S}$是$\mathbf{R}^S$和$\mathbf{R}^{N\backslash S}$中的面积微元.

证　应用Stokes公式[1], 有

$$\int_{\partial\varOmega_1\times\partial\varOmega_2}g\mathrm{d}\sigma_1 f\mathrm{d}\sigma_2 h=\int_{\partial\varOmega_1}g\mathrm{d}\sigma_1\left[\int_{\partial\varOmega_2}f\mathrm{d}\sigma_2 h\right]=\int_{\partial\varOmega_1}g\mathrm{d}\sigma_1\left\{\int_{\varOmega_2}([fD_2]\,h+f[D_2h])\mathrm{d}x^{N\backslash S}\right\}=$$
$$\int_{\varOmega_1}[gD_1]\left\{\int_{\varOmega_2}([fD_2]\,h+f[D_2h])\mathrm{d}x^{N\backslash S}\right\}\mathrm{d}x^S+\int_{\varOmega_1}gD_1\left\{\int_{\varOmega_2}([fD_2]\,h+f[D_2h])\mathrm{d}x^{N\backslash S}\right\}\mathrm{d}x^S=$$
$$\int_{\varOmega_1\times\varOmega_2}\left\{[gD_1]\,[fD_2]h+[gD_1]\,f[D_2h]\right\}\mathrm{d}x^{N\backslash S}\wedge\mathrm{d}x^S+$$
$$\int_{\varOmega_1\times\varOmega_2}gD_1\left\{[fD_2]\,h+f[D_2h]\right\}\mathrm{d}x^{N\backslash S}\wedge\mathrm{d}x^S=$$

$$\int\limits_{\Omega_1\times\Omega_2} \{[gD_1]\,[fD_2]h+[gD_1]\,f[D_2h]\}\,\mathrm{d}x^{N\setminus S}\mathrm{d}x^S+$$

$$\int\limits_{\Omega_1\times\Omega_2} g\,\{D_1\,[[fD_2]]\,h+[fD_2]\,[D_1h]+[D_1f]\,[D_2h]+fD_1\,[[D_2h]]\}\,\mathrm{d}x^{N\setminus S}\wedge\mathrm{d}x^S=$$

$$\int\limits_{\Omega_1\times\Omega_2} \{[gD_1]\,[fD_2]h+[gD_1]\,f[D_2h]+gD_1\,[[fD_2]]h+g[D_1f][D_2h]\}\,\mathrm{d}x,$$

因为

$$D_1f=L_1f-\sum_{i=1}^{s}\frac{k_i}{x_i}e_iT_i(f),\ fD_2=fR_2-\sum_{i=s+1}^{n}\frac{k_i}{x_i}T_i(f)e_i,$$

$$gD_1=gR_1-\sum_{i=1}^{s}\frac{k_i}{x_i}T_i(g)e_i,\ D_2h=L_2h-\sum_{i=s+1}^{n}\frac{k_i}{x_i}e_iT_i(H),$$

$$D_1(fD_2)=L_1((f)R_2)-L_1(\sum_{i=s+1}^{n}\frac{k_i}{x_i}T_i(f)e_i)-\sum_{i=1}^{s}\frac{k_i}{x_i}e_iT_i((f)R)+\sum_{i=1}^{s}\frac{k_i}{x_i}e_iT_i(\sum_{i=s+1}^{n}\frac{k_i}{x_i}T_i(f)e_i),$$

则有

$$\int\limits_{\partial\Omega_1\times\partial\Omega_2} g\mathrm{d}\sigma_1f\mathrm{d}\sigma_2h=\int\limits_{\Omega_1\times\Omega_2}\left\{\left[gR_1-\sum_{i=1}^{s}\frac{k_i}{x_i}T_i(g)e_i\right]\cdot\left[fR_2-\sum_{j=s+1}^{n}\frac{k_j}{x_j}T_j(f)e_j\right]\cdot h+\right.$$

$$\left[gR_1-\sum_{i=1}^{s}\frac{k_i}{x_i}T_i(g)e_i\right]\cdot f\cdot\left[L_2h-\sum_{j=s+1}^{n}\frac{k_j}{x_j}e_jT_j(h)\right]+$$

$$g\left[L_1(fR_2)-L_1(\sum_{j=s+1}^{n}\frac{k_j}{x_j}T_j(f)e_j)-\sum_{i=1}^{s}\frac{k_i}{x_i}e_iT_i(fR_2)+\right.$$

$$\left.\sum_{i=1}^{s}\frac{k_i}{x_i}e_iT_i(\sum_{j=s+1}^{n}\frac{k_j}{x_j}T_j(f)e_j)\right]h+$$

$$\left.g\left[L_1f-\sum_{i=1}^{s}\frac{k_i}{x_i}e_iT_i(f)\right]\cdot\left[L_2h-\sum_{j=s+1}^{n}\frac{k_j}{x_j}e_jT_j(h)\right]\right\}\mathrm{d}x=$$

$$\int\limits_{\Omega_1\times\Omega_2}[(gR_1)(fR_2)h+(gR_1)f(L_2h)+g(L_1(fR_2))h+g(L_1f)(L_2h)]\,\mathrm{d}x-$$

$$\int\limits_{\Omega_1\times\Omega_2}(gR_1)\left[\sum_{j=s+1}^{n}\frac{k_j}{x_j}(T_j(f)e_jh+fe_jT_j(h))\right]\mathrm{d}x-$$

$$\int\limits_{\Omega_1\times\Omega_2}\sum_{i=1}^{s}\frac{k_i}{x_i}\,[T_i(g)e_i(fR_2)+ge_iT_i(fR_2)]\,h\cdot\mathrm{d}x-$$

$$\int\limits_{\Omega_1\times\Omega_2}\sum_{i=1}^{s}\frac{k_i}{x_i}\,[T_i(g)e_if+ge_iT_i(f)]\,(L_2h)\mathrm{d}x-$$

$$\int\limits_{\Omega_1\times\Omega_2} g\left[L_1(\sum_{j=s+1}^{n}\frac{k_j}{x_j}T_j(f)e_j)h+(L_1f)\sum_{j=s+1}^{n}\frac{k_j}{x_j}e_jT_j(h)\right]\mathrm{d}x+$$

$$\int\limits_{\Omega_1\times\Omega_2}\left\{\sum_{i=1}^{s}\frac{k_i}{x_i}T_i(g)e_i\sum_{j=s+1}^{n}\frac{k_j}{x_j}T_j(f)e_jh+\sum_{i=1}^{s}\frac{k_i}{x_i}T_i(g)e_if\sum_{j=s+1}^{n}\frac{k_j}{x_j}e_jT_j(h)+\right.$$
$$\left.g\sum_{i=1}^{s}\frac{k_i}{x_i}e_iT_i(\sum_{j=s+1}^{n}\frac{k_j}{x_j}T_j(f)e_j)h+g\sum_{i=1}^{s}\frac{k_i}{x_i}e_iT_i(f)\sum_{j=s+1}^{n}\frac{k_j}{x_j}e_jT_j(h)\right\}\mathrm{d}x=$$
$$\int\limits_{\Omega_1\times\Omega_2}\left[(gR_1)(fR_2)h+(gR_1)f(L_2h)+g(L_1(fR_2))h+g(L_1f)(L_2h)\right]\mathrm{d}x-$$
$$I_1-I_2-I_3-I_4+I_5.$$

对于I_1有

$$I_1=\int\limits_{\Omega_1\times\Omega_2}(gR_1)\left[\sum_{j=s+1}^{n}\frac{k_j}{x_j}(T_j(f)e_jh+fe_jT_j(h))\right]\mathrm{d}x,$$
$$\mathrm{Re}I_1=\int\limits_{\Omega_1\times\Omega_2}\sum_{j=s+1}^{n}\mathrm{Re}\left[gR_1\frac{k_j}{x_j}(T_j(f)e_jh+fe_jT_j(h))\right]\mathrm{d}x,$$

对于$\forall s+1\leq j\leq n$, 由定理3.1, 有

$$T_j\left[gR_1\frac{k_j}{x_j}(T_j(f)e_jh+fe_jT_j(h))\right]=gR_1\frac{k_j}{x_j}\left[T_j\left[T_j(f)e_jh\right]+T_j\left[fe_jT_j(h)\right]\right]=$$
$$gR_1\frac{k_j}{x_j}\left[f(-e_j)T_j(h)+T_j(f)(-e_j)h\right]=$$
$$-\left[gR_1\frac{k_j}{x_j}(T_j(f)e_jh+fe_jT_j(h))\right],$$

由定理3.3可得$\mathrm{Re}\left[gR_1\frac{k_j}{x_j}(T_j(f)e_jh+fe_jT_j(h))\right]=0$, 因此$\mathrm{Re}I_1=0$.

对于I_2有

$$I_2=\int\limits_{\Omega_1\times\Omega_2}\sum_{i=1}^{s}\frac{k_i}{x_i}\left[T_i(g)e_i(fR_2)+ge_iT_i(fR_2)\right]h\cdot\mathrm{d}x,$$
$$\mathrm{Re}I_2=\int\limits_{\Omega_1\times\Omega_2}\sum_{i=1}^{s}\mathrm{Re}\left[\frac{k_i}{x_i}\left[T_i(g)e_i(fR_2)+ge_iT_i(fR_2)\right]h\right]\mathrm{d}x,$$

对于$\forall 1\leq i\leq s$, 由定理3.1有

$$T_i\left[\frac{k_i}{x_i}\left[T_i(g)e_i(fR_2)+ge_iT_i(fR_2)\right]h\right]=\frac{k_i}{x_i}\Big[T_i\left[T_i(g)e_i(fR_2)\right]+T_i\left[ge_iT_i(fR_2)\right]\Big]h=$$
$$\frac{k_i}{x_i}\left[g(-e_i)T_i(fR_2)+T_i(g)(-e_i)(fR_2)\right]h=$$
$$-\left[\frac{k_i}{x_i}\left[T_i(g)e_i(fR_2)+ge_iT_i(fR_2)\right]h\right],$$

由定理3.3可得$\mathrm{Re}\left[\frac{k_i}{x_i}\left[T_i(g)e_i(fR_2)+ge_iT_i(fR_2)\right]h\right]=0$, 因此$\mathrm{Re}I_2=0$.

对于I_3有

$$I_3=\int\limits_{\Omega_1\times\Omega_2}\sum_{i=1}^{s}\frac{k_i}{x_i}\left[T_i(g)e_if+ge_iT_i(f)\right](L_2h)\mathrm{d}x,$$

$$\mathrm{Re}I_3=\int\limits_{\Omega_1\times\Omega_2}\sum_{i=1}^{s}\mathrm{Re}\Big[\frac{k_i}{x_i}\left[T_i(g)e_if+ge_iT_i(f)\right](L_2h)\Big]\mathrm{d}x,$$

对于$\forall 1\le i\le s$, 由定理3.1, 有

$$\begin{aligned}&T_i\Big[\frac{k_i}{x_i}\left[T_i(g)e_if+ge_iT_i(f)\right](L_2h)\Big]=\frac{k_i}{x_i}\Big[T_i\left[T_i(g)e_if\right]+T_i\left[ge_iT_i(f)\right]\Big](L_2h)=\\&\frac{k_i}{x_i}\Big[g(-e_i)T_i(f)+T_i(g)(-e_i)f\Big](L_2h)=\\&-\Big[\frac{k_i}{x_i}\left[T_i(g)e_if+ge_iT_i(f)\right](L_2h)\Big],\end{aligned}$$

由定理3.3可得$\mathrm{Re}\Big[\frac{k_i}{x_i}\left[T_i(g)e_if+ge_iT_i(f)\right](L_2h)\Big]=0$, 因此$\mathrm{Re}I_3=0$.

对于I_4有

$$I_4=\int\limits_{\Omega_1\times\Omega_2}g\left[L_1(\sum_{j=s+1}^{n}\frac{k_j}{x_j}T_j(f)e_j)h+(L_1f)\sum_{j=s+1}^{n}\frac{k_j}{x_j}e_jT_j(h)\right]\mathrm{d}x,$$

$$\mathrm{Re}I_4=\int\limits_{\Omega_1\times\Omega_2}\sum_{j=s+1}^{n}\frac{k_j}{x_j}\mathrm{Re}\Big[g\left[L_1(T_j(f)e_j)h+g(L_1f)e_jT_j(h)\right]\Big]\mathrm{d}x,$$

对于$\forall s+1\le j\le n$, 由定理3.1有

$$T_j\Big[g\left[L_1(T_j(f)e_j)h+g(L_1f)e_jT_j(h)\right]\Big]=\quad g\left[T_jL_1(T_j(f))(-e_j)T_j(h)+T_j(L_1f)(-e_j)h\right],$$

因为$T_j(L_1f)=L_1(T_j(f))$, 有

$$\begin{aligned}&g\left[T_jL_1(T_j(f))(-e_j)T_j(h)+T_j(L_1f)(-e_j)h\right]=\\&-g\Big[T_j(T_j(L_1f))e_jT_j(h)+L_1(T_j(f))e_jh\Big]=\\&-g\Big[(L_1f)e_jT_j(h)+L_1(T_j(f))e_jh\Big],\end{aligned}$$

由定理3.3可得$\mathrm{Re}\Big[g\left[L_1(T_j(f)e_j)h+g(L_1f)e_jT_j(h)\right]\Big]=0$, 因此$\mathrm{Re}I_4=0$.

对于I_5有

$$\begin{aligned}I_5=\sum_{i=1}^{s}\frac{k_i}{x_i}\sum_{j=s+1}^{n}\frac{k_j}{x_j}\int\limits_{\Omega_1\times\Omega_2}&\Big[T_i(g)e_iT_j(f)e_jh+T_i(g)e_ife_jT_j(h)+\\&ge_iT_i(T_j(f)e_j)h+ge_iT_i(f)e_jT_j(h)\Big]\mathrm{d}x,\end{aligned}$$

$$\begin{aligned}\mathrm{Re}I_5=\sum_{i=1}^{s}\frac{k_i}{x_i}\sum_{j=s+1}^{n}\frac{k_j}{x_j}\int\limits_{\Omega_1\times\Omega_2}&\mathrm{Re}\Big[T_i(g)e_iT_j(f)e_jh+T_i(g)e_ife_jT_j(h)+\\&ge_iT_i(T_j(f)e_j)h+ge_iT_i(f)e_jT_j(h)\Big]\mathrm{d}x,\end{aligned}$$

对于$\forall 1 \le i \le s, \forall s+1 \le j \le n$, 由定理3.1有

$$T_i\left[T_i(g)e_iT_j(f)e_jh + T_i(g)e_ife_jT_j(h) + ge_iT_i(T_j(f))e_jh + ge_iT_i(f)e_jT_j(h)\right] =$$
$$g(-e_i)T_i(T_j(f))e_jh + g(-e_i)T_i(f)e_jT_j(h) + T_i(g)(-e_i)T_j(f)e_jh + T_i(g)(-e_i)fe_jT_j(h) =$$
$$-\left[ge_iT_i(T_j(f))e_jh + ge_iT_i(f)e_jT_j(h) + T_i(g)e_iT_j(f)e_jh + T_i(g)e_ife_jT_j(h)\right],$$
$$T_j\left[T_i(g)e_iT_j(f)e_jh + T_i(g)e_ife_jT_j(h) + ge_iT_i(T_j(f))e_jh + ge_iT_i(f)e_jT_j(h)\right] =$$
$$T_i(g)e_if(-e_j)T_j(h) + T_i(g)e_iT_i(f)(-e_j)h + ge_iT_i(f)(-e_j)T_j(h) + ge_iT_j(T_i(f))(-e_j)h =$$
$$-\left[T_i(g)e_ife_jT_j(h) + T_i(g)e_iT_j(f)e_jh + ge_iT_i(f)e_jT_j(h) + ge_iT_i(T_j(f))e_jh\right],$$

由定理3.3可得

$$\mathrm{Re}\Big[T_i(g)e_iT_j(f)e_jh + T_i(g)e_ife_jT_j(h) + ge_iT_i(T_j(f)e_j)h + ge_iT_i(f)e_jT_j(h)\Big] = 0.$$

因此$\mathrm{Re}I_5 = 0$. 于是

$$\mathrm{Re}\left[\int\limits_{\partial\Omega_1\times\partial\Omega_2} g\mathrm{d}\sigma_1 f\mathrm{d}\sigma_2 h\right] =$$
$$\mathrm{Re}\int\limits_{\Omega_1\times\Omega_2}\left[(gR_1)(fR_2)h + (gR_1)f(L_2h) + g(L_1(fR_2))h + g(L_1f)(L_2h)\right]\mathrm{d}x.$$

推论3.10　当$gR_1 = 0, L_2h = 0, L_1f = 0, fR_2 = 0$时, 有

$$\mathrm{Re}\left[\int\limits_{\partial\Omega_1\times\partial\Omega_2} g\mathrm{d}\sigma_1 f\mathrm{d}\sigma_2 h\right] = 0.$$

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Properties of k-vector monogenic functions and related operators in universal Clifford analysis

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Abstract: In this paper, the T_i operator in universal Clifford analysis is defined and the commutative properties of T_i operator and Dirac operator are studied firstly. Then the L and R operator are constructed by Dirac operator and T_i operator. On the basis, the k-vector monogenic function whose value is on universal Clifford algebra is defined, and the relationship between k-vector monogenic function and other monogenic function classes with harmonic functions is studied. Finally, the generalized Stokes formula related to L and R operators in the universal Clifford analysis space is proved, which is an important basis for finding the integral representation of k-vector monogenic functions.

Keywords: universal Clifford algebra; L operator; R operator; k-vector monogenic function; Stokes formula

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