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自对偶Taub-NUT度量的测地线

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摘　要: 研究了部分参数为常数时, 自对偶Taub-NUT度量上的测地线方程, 并给出了几类可以得到测地线方程显式解的情况.

关键词: 自对偶; Taub-NUT度量; 测地线
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§1　引　言

测地线是微分几何中的基本几何对象, 在广义相对论的研究中发挥着至关重要的作用. 其中类光测地线和类时测地线可用来描述光和带质量粒子的运动轨迹. 测地线在各种特定时空结构中已被广泛研究(见[1-2]及其中的参考文献). 最近, 文献[3]分析了Euclidean Schwarzschild几何中的测地线运动问题. 文献[4]探讨了Eguchi-Hanson型度量的测地线. 这两种度量都可以看作是Schwarzschild度量的推广, 并且属于引力瞬子的范畴. 引力瞬子被定义为经典真空爱因斯坦方程或带宇宙常数的爱因斯坦方程的非奇异, 完备, 正定解, 其在量子引力的Euclidean方法中起着重要作用[5]. 因此, 研究其他引力瞬子的测地线具有重要意义. 本文旨在研究自对偶Taub-NUT度量上的测地线, 该度量同样属于引力瞬子的范畴[5].

Taub-NUT度量是广义相对论中爱因斯坦场方程的一个解, 它扩展了Taub解[6], 并融入了Newman-Unti-Tamburino(NUT)度量的元素[7]. 该度量描述了一个由质量和被称为NUT电荷的参数所特征化的四维时空, 这个NUT电荷与“引力磁单极子”或扭曲几何结构相关(见[6, 8-9]). 该度量以其奇异的特性而著称, 包括闭合类时曲线(暗示了时间旅行的理论可能性)和Misner弦. 后者是一种类似于电磁学中Dirac弦的拓扑缺陷(见[8-9]). Taub-NUT时空在引力瞬子、黑洞热力学和弦理论的研究中发挥着关键作用, 并已获得了众多相关成果(见[10]及其参考文献). 该度量的测地完备性已在文献[8-9]中得到证明, 从而可以通过可移除的奇点来求解测地线方程.

本文余下部分安排如下. §2介绍自对偶Taub-NUT度量并给出测地线方程. §3对部分参数为常数时的测地线方程进行求解, 给出其显式解.

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§2 自对偶Taub-NUT度量及测地线方程

自对偶Taub-NUT度量可表示为[5]

$$\mathrm{d}s^2=\frac{r-n}{r+n}(\mathrm{d}\tau+2n\cos\theta\mathrm{d}\varphi)^2+(r^2-n^2)(\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\varphi^2)+\frac{r+n}{r-n}\mathrm{d}r^2.$$

该度量在$r=n$处存在奇点, 该奇点可以看作是超球面极坐标的原点, 是可去奇点. 其相关拓扑性质可见文献[5]. 并且文献[5]指出, 在如下正定向的单位正交基下, 该度量的曲率是自对偶的.

$$\begin{aligned}
\omega^0&=\sqrt{\frac{r+n}{r-n}}\mathrm{d}r,\\
\omega^1&=\sqrt{r^2-n^2}\left(\cos\frac{\tau}{2n}\mathrm{d}\theta+\sin\frac{\tau}{2n}\sin\theta\mathrm{d}\varphi\right),\\
\omega^2&=\sqrt{r^2-n^2}\left(-\sin\frac{\tau}{2n}\mathrm{d}\theta+\cos\frac{\tau}{2n}\sin\theta\mathrm{d}\varphi\right),\\
\omega^3&=\sqrt{\frac{r-n}{r+n}}(\mathrm{d}\tau+2n\cos\theta\mathrm{d}\varphi).
\end{aligned}$$

通过直接计算, 可以得到所有的非平凡度量分量为

$$\begin{aligned}
g_{\tau\tau}&=\frac{r-n}{r+n},\ g_{\tau\varphi}=g_{\varphi\tau}=\frac{2n\cos\theta(r-n)}{r+n},\\
g_{\varphi\varphi}&=\frac{4n^2\cos^2\theta(r-n)}{r+n}+(r^2-n^2)\sin^2\theta,\\
g_{rr}&=\frac{r+n}{r-n},\ g_{\theta\theta}=r^2-n^2,\\
g^{\tau\tau}&=\frac{4n^2\cos^2\theta}{(r^2-n^2)\sin^2\theta}+\frac{r+n}{r-n},\\
g^{\tau\varphi}&=g^{\varphi\tau}=\frac{-2n\cos\theta}{(r^2-n^2)\sin^2\theta},\ g^{\varphi\varphi}=\frac{1}{(r^2-n^2)\sin^2\theta},\\
g^{rr}&=\frac{r-n}{r+n},\ g^{\theta\theta}=\frac{1}{r^2-n^2}.
\end{aligned}$$

由此可计算出所有的非平凡Christoffel符号为

$$\begin{aligned}
\varGamma^{\tau}_{\tau r}&=\frac{n}{r^2-n^2},\varGamma^{\tau}_{\tau\theta}=\frac{2n^2\cos\theta}{(r+n)^2\sin\theta},\varGamma^{\tau}_{\varphi r}=-\frac{2n\cos\theta}{r+n},\\
\varGamma^{\tau}_{\varphi\theta}&=\frac{4n^3\cos^2\theta-n\sin^2\theta(r+n)^2-2n(r+n)^2\cos^2\theta}{(r+n)^2\sin\theta},\\
\varGamma^{r}_{\tau\tau}&=-\frac{n(r-n)}{(r+n)^3},\varGamma^{r}_{\tau\varphi}=-\frac{2n^2(r-n)\cos\theta}{(r+n)^3},\varGamma^{r}_{rr}=-\frac{n}{r^2-n^2},\\
\varGamma^{r}_{\theta\theta}&=-\frac{r(r-n)}{r+n},\varGamma^{r}_{\varphi\varphi}=-\left[\frac{4n^3\cos^2\theta}{(r+n)^2}+r\sin^2\theta\right]\frac{r-n}{r+n},\\
\varGamma^{\theta}_{\tau\varphi}&=\frac{n\sin\theta}{(r+n)^2},\varGamma^{\theta}_{r\theta}=\frac{r}{r^2-n^2},\\
\varGamma^{\theta}_{\varphi\varphi}&=\frac{4n^2\cos\theta\sin\theta}{(r+n)^2}-\sin\theta\cos\theta,\\
\varGamma^{\varphi}_{\tau\theta}&=-\frac{n}{(r+n)^2\sin\theta},\varGamma^{\varphi}_{\varphi r}=\frac{r}{r^2-n^2},\\
\varGamma^{\varphi}_{\varphi\theta}&=-\frac{2n^2\cos\theta}{(r+n)^2\sin\theta}+\frac{\cos\theta}{\sin\theta}.
\end{aligned}$$

所以, 以仿射参数t为参数的测地线方程可表示为

$$\frac{\mathrm{d}^2\tau}{\mathrm{d}t^2}+\frac{4n^2\cos\theta}{(r+n)^2\sin\theta}\frac{\mathrm{d}\tau}{\mathrm{d}t}\frac{\mathrm{d}\theta}{\mathrm{d}t}+\frac{2n}{r^2-n^2}\frac{\mathrm{d}\tau}{\mathrm{d}t}\frac{\mathrm{d}r}{\mathrm{d}t}+$$
$$\frac{2[4n^3\cos^2\theta-n\sin^2\theta(r+n)^2-2n(r+n)^2\cos^2\theta]}{(r+n)^2\sin\theta}\frac{\mathrm{d}\varphi}{\mathrm{d}t}\frac{\mathrm{d}\theta}{\mathrm{d}t}-$$
$$\frac{4n\cos\theta}{r+n}\frac{\mathrm{d}\varphi}{\mathrm{d}t}\frac{\mathrm{d}r}{\mathrm{d}t}=0.$$

$$\frac{\mathrm{d}^2\varphi}{\mathrm{d}t^2}+2\left[-\frac{2n^2\cos\theta}{(r+n)^2\sin\theta}+\frac{\cos\theta}{\sin\theta}\right]\frac{\mathrm{d}\varphi}{\mathrm{d}t}\frac{\mathrm{d}\theta}{\mathrm{d}t}+$$
$$\frac{2r}{r^2-n^2}\frac{\mathrm{d}\varphi}{\mathrm{d}t}\frac{\mathrm{d}r}{\mathrm{d}t}-\frac{2n}{(r+n)^2\sin\theta}\frac{\mathrm{d}\tau}{\mathrm{d}t}\frac{\mathrm{d}\theta}{\mathrm{d}t}=0.$$

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}+\frac{2r}{r^2-n^2}\frac{\mathrm{d}r}{\mathrm{d}t}\frac{\mathrm{d}\theta}{\mathrm{d}t}+\left[\frac{4n^2\cos\theta\sin\theta}{(r+n)^2}-\sin\theta\cos\theta\right]\left(\frac{\mathrm{d}\varphi}{\mathrm{d}t}\right)^2+$$
$$\frac{2n\sin\theta}{(r+n)^2}\frac{\mathrm{d}\tau}{\mathrm{d}t}\frac{\mathrm{d}\varphi}{\mathrm{d}t}=0.$$

$$\frac{\mathrm{d}^2r}{\mathrm{d}t^2}-\frac{n}{r^2-n^2}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2-\frac{n(r-n)}{(r+n)^3}\left(\frac{\mathrm{d}\tau}{\mathrm{d}t}\right)^2-\frac{r(r-n)}{r+n}\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2-$$
$$\left[\frac{4n^3\cos^2\theta}{(r+n)^2}+r\sin^2\theta\right]\frac{r-n}{r+n}\left(\frac{\mathrm{d}\varphi}{\mathrm{d}t}\right)^2-\frac{4n^2(r-n)\cos\theta}{(r+n)^3}\frac{\mathrm{d}\tau}{\mathrm{d}t}\frac{\mathrm{d}\varphi}{\mathrm{d}t}=0.$$

注 在本文余下部分中, 始终定义$\epsilon=\pm1$. 并用$\mathrm{sgn}(x)$表示符号函数, 即

$$\mathrm{sgn}(x)=\begin{cases}1, & x>0,\\ 0, & x=0,\\ -1, & x<0.\end{cases}$$

§3 测地线方程求解

定理3.1 设参数τ,θ,φ 为常数. 假设$r_1\neq0$且当$r\geq n$ 时满足条件

$$\lim_{r\to n}t=t_1,\quad \lim_{r\to n}\sqrt{\frac{r+n}{r-n}}\frac{\mathrm{d}r}{\mathrm{d}t}=r_1.$$

则过$r=n$的测地线可表示为

$$t(r)=t_1+\frac{1}{r_1}\left[\sqrt{r^2-n^2}+2n\ln\left(\sqrt{r+n}+\sqrt{r-n}\right)-n\ln 2n\right].$$

证 当τ,θ,φ为常数且$r>n$时, 测地线方程化为

$$\frac{\mathrm{d}^2r}{\mathrm{d}t^2}-\frac{n}{r^2-n^2}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2=0. \tag{1}$$

则$\frac{\mathrm{d}}{\mathrm{d}t}\left(\sqrt{\frac{r+n}{r-n}}\frac{\mathrm{d}r}{\mathrm{d}t}\right)=0$, 由定理条件可以解得$\sqrt{\frac{r+n}{r-n}}\frac{\mathrm{d}r}{dt}=r_1$, 移项可得

$$\frac{\mathrm{d}t}{\mathrm{d}r}=\frac{1}{r_1}\sqrt{\frac{r+n}{r-n}}.$$

将上式从n到r积分即可得到结论.

定理3.2 设$\theta,\varphi,r_1,\tau_0$为常数, 且$r_1>0$. 令

$$R_1=n+\frac{2\tau_0^2n}{r_1^2}\geq n. \tag{2}$$

如果当$r \geq R_1 > n$时满足条件

$$\lim_{r\to R_1} t = t_1,\ \lim_{r\to R_1} \tau = \tau_1,\ \lim_{r\to R_1} \frac{r-n}{r+n}\frac{\mathrm{d}\tau}{\mathrm{d}t} = \tau_0,\ \lim_{r\to R_1} \left[\frac{r+n}{r-n}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 + g_1\right] = r_1^2,$$

则测地线方程可表示为

$$\begin{aligned}\tau(r) =&\tau_1 + \frac{\epsilon\tau_0}{r_1}\Bigg\{(5n+R_1)\left[\ln(\sqrt{r-R_1}+\sqrt{r+n}) - \ln\sqrt{R_1+n}\right] + \\ &\frac{2(2n)^{\frac{3}{2}}}{\sqrt{R_1-n}}\arctan\left(\frac{\sqrt{2n}\sqrt{r-R_1}}{\sqrt{R_1-n}\sqrt{r+n}}\right) + \sqrt{r-R_1}\sqrt{r+n}\Bigg\},\\ t(r) =&t_1 + \frac{\epsilon}{r_1}\left\{\sqrt{r-R_1}\sqrt{r+n} + (n+R_1)\left[\ln\left(\sqrt{r-R_1}+\sqrt{r+n}\right) - \ln\sqrt{R_1+n}\right]\right\}.\end{aligned}$$

证 当θ,φ为常数时, 测地线方程化为

$$\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} - \frac{n}{r^2-n^2}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 - \frac{n(r-n)}{(r+n)^3}\left(\frac{\mathrm{d}\tau}{\mathrm{d}t}\right)^2 = 0. \tag{3}$$

$$\frac{\mathrm{d}^2\tau}{\mathrm{d}t^2} + \frac{2n}{r^2-n^2}\frac{\mathrm{d}\tau}{\mathrm{d}t}\frac{\mathrm{d}r}{\mathrm{d}t} = 0. \tag{4}$$

其中方程(3)可变形为$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{r-n}{r+n}\frac{\mathrm{d}\tau}{\mathrm{d}t}\right) = 0$. 应用定理条件可得

$$\frac{\mathrm{d}\tau}{\mathrm{d}t} = \tau_0\frac{r+n}{r-n}. \tag{5}$$

将(5)式代入到方程(4)中可得

$$\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} - \frac{n}{r^2-n^2}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 - \frac{\tau_0^2 n}{r^2-n^2} = 0.$$

则上式等价于$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{r+n}{r-n}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 + \frac{2\tau_0^2 n}{r-n}\right] = 0$. 所以$\frac{r+n}{r-n}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 + \frac{2\tau_0^2 n}{r-n} = r_1^2$. 结合(2)式和(5)式可得, 当$r > R_1$时,

$$\frac{\mathrm{d}t}{\mathrm{d}r} = \frac{\epsilon}{r_1}\sqrt{\frac{r+n}{r-R_1}},\quad \frac{\mathrm{d}\tau}{\mathrm{d}r} = \frac{\mathrm{d}\tau}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}r} = \frac{\epsilon\tau_0}{r_1}\frac{(r+n)^{\frac{3}{2}}}{(r-n)\sqrt{r-R_1}}.$$

将上面两个式子从R_1到r积分, 即可得到定理结论.

注 由(2)式易知$R_1 = n$当且仅当$\tau_0 = 0$. 所以, 当$\tau_0 = 0$时, 测地线可经过$r = n$.

假设θ,τ为常数, 其中$\theta \in (0,\pi)$, 则此时测地线方程化为

$$\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} - \frac{n}{r^2-n^2}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 - \left[\frac{4n^3\cos^2\theta}{(r+n)^2} + r\sin^2\theta\right]\frac{r-n}{r+n}\left(\frac{\mathrm{d}\varphi}{\mathrm{d}t}\right)^2 = 0, \tag{6}$$

$$\begin{gathered}\frac{\mathrm{d}^2\varphi}{\mathrm{d}t^2} + \frac{2r}{r^2-n^2}\frac{\mathrm{d}\varphi}{\mathrm{d}t}\frac{\mathrm{d}r}{\mathrm{d}t} = 0.\\ \frac{4n\cos\theta}{r+n}\frac{\mathrm{d}\varphi}{\mathrm{d}t}\frac{\mathrm{d}r}{\mathrm{d}t} = 0. \qquad (7)\\ \left[\frac{4n^2\cos\theta\sin\theta}{(r+n)^2} - \sin\theta\cos\theta\right]\left(\frac{\mathrm{d}\varphi}{\mathrm{d}t}\right)^2 = 0.\end{gathered}$$

由方程(7)可知, $\frac{\mathrm{d}\varphi}{\mathrm{d}t} = 0$, $\frac{\mathrm{d}r}{\mathrm{d}t} = 0$ 或$\theta = \frac{\pi}{2}$. 若$\frac{\mathrm{d}\varphi}{\mathrm{d}t} = 0$, 则化为定理3.1的情况. 若$\frac{\mathrm{d}r}{\mathrm{d}t} = 0$, 由方程(6)可得$\frac{\mathrm{d}\varphi}{\mathrm{d}t} = 0$. 此时$\tau, r, \theta, \varphi$均为常数. 故接下来处理$\theta = \frac{\pi}{2}$的情况. 对此, 有如下定理.

定理3.3 设τ, r_1, φ_0为常数, 且$\theta = \dfrac{\pi}{2}, r_1 > 0$. 令

$$R_2 = \sqrt{n^2 + \frac{\varphi_0^2}{r_1^2}}. \tag{8}$$

则$R_2 \geq n$. 若当$r \geq R_2 > n$时满足条件

$$\lim_{r\to R_2} t = t_1,\ \lim_{r\to R_2} \varphi = \varphi_1,\ \lim_{r\to R_2} (r^2-n^2)\frac{\mathrm{d}\varphi}{\mathrm{d}t} = \varphi_0,\quad \lim_{r\to R_2}\left[\frac{r+n}{r-n}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 + \frac{\varphi_0^2}{r^2-n^2}\right] = r_1^2.$$

则测地线可表示为

$$t(r) = t_1 + \frac{\epsilon}{r_1}\left[\sqrt{r^2-R_2^2} + n\ln\left(r + \sqrt{r^2-R_2^2}\right) - n\ln R_2\right],$$

$$\varphi(r) = \varphi_1 + \epsilon\arctan\frac{r_1(rn-R_2^2)}{\varphi_0\sqrt{r^2-R_2^2}} + \frac{\pi\epsilon}{2}\mathrm{sgn}(\varphi_0).$$

证 此时测地线方程转化为

$$\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} - \frac{n}{r^2-n^2}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 - \frac{r(r-n)}{r+n}\left(\frac{\mathrm{d}\varphi}{\mathrm{d}t}\right)^2 = 0. \tag{9}$$

$$\frac{\mathrm{d}^2\varphi}{\mathrm{d}t^2} + \frac{2r}{r^2-n^2}\frac{\mathrm{d}\varphi}{\mathrm{d}t}\frac{\mathrm{d}r}{\mathrm{d}t} = 0. \tag{10}$$

由方程(10)可得$\dfrac{\mathrm{d}}{\mathrm{d}t}\left[(r^2-n^2)\dfrac{\mathrm{d}\varphi}{\mathrm{d}t}\right] = 0$. 结合定理条件得

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t} = \frac{\varphi_0}{r^2-n^2}. \tag{11}$$

将上式代入到方程(9)中有

$$\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} - \frac{n}{r^2-n^2}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 - \frac{r\varphi_0^2}{(r-n)(r+n)^3} = 0.$$

则$\dfrac{\mathrm{d}}{\mathrm{d}t}\left[\dfrac{r+n}{r-n}\left(\dfrac{\mathrm{d}r}{\mathrm{d}t}\right)^2 + \dfrac{\varphi_0^2}{r^2-n^2}\right] = 0$. 根据定理条件可得$\dfrac{\mathrm{d}r}{\mathrm{d}t} = \epsilon\dfrac{\sqrt{r_1^2(r^2-n^2)-\varphi_0^2}}{r+n}$. 再结合(8)式和(11)式可得, 当$r > R_2$时,

$$\frac{\mathrm{d}t}{\mathrm{d}r} = \frac{\epsilon}{r_1}\frac{r+n}{\sqrt{r^2-R_2^2}},\quad \frac{\mathrm{d}\varphi}{\mathrm{d}r} = \frac{\mathrm{d}\varphi}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}r} = \frac{\epsilon\varphi_0}{r_1(r-n)\sqrt{r^2-R_2^2}}.$$

将上两式从R_2到r积分即可得到定理结论.

注 易知$R_2 = n$当且仅当$\varphi_0 = 0$, 所以当$\varphi_0 = 0$时, 测地线可经过$r = n$.

定理3.4 设$\tau, \varphi, r_1, \theta_0$为常数, 且$r_1 > 0$. 令

$$R_3 = \sqrt{n^2 + \frac{\theta_0^2}{r_1^2}}. \tag{12}$$

则$R_3 \geq n$. 若当$r \geq R_3 > n$时满足条件

$$\lim_{r\to R_3} t = t_1,\ \lim_{r\to R_3} \theta = \theta_1,\ \lim_{r\to R_3} (r^2-n^2)\frac{\mathrm{d}\theta}{\mathrm{d}t} = \theta_0,$$

$$\lim_{r\to R_3}\left[\frac{r+n}{r-n}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 + \frac{\theta_0^2}{r^2-n^2}\right] = r_1^2.$$

则测地线可表示为

$$t(r) = t_1 + \frac{\epsilon}{r_1}\left[\sqrt{r^2-R_3^2} + n\ln\left(r + \sqrt{r^2-R_3^2}\right) - n\ln R_3\right],$$

$$\theta(r) = \theta_1 + \epsilon\arctan\frac{r_1(rn-R_3^2)}{\theta_0\sqrt{r^2-R_3^2}} + \frac{\pi\epsilon}{2}\mathrm{sgn}(\theta_0).$$

证 当τ和φ为常数时，测地线方程化为

$$\frac{\mathrm{d}^2 r}{\mathrm{d}t^2}-\frac{n}{r^2-n^2}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2-\frac{r(r-n)}{r+n}\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2=0. \tag{13}$$

$$\frac{\mathrm{d}^2 \theta}{\mathrm{d}t^2}+\frac{2r}{r^2-n^2}\frac{\mathrm{d}r}{\mathrm{d}t}\frac{\mathrm{d}\theta}{\mathrm{d}t}=0. \tag{14}$$

由方程(14)可得

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[(r^2-n^2)\frac{\mathrm{d}\theta}{\mathrm{d}t}\right]=0.$$

结合定理条件可解得

$$\frac{\mathrm{d}\theta}{\mathrm{d}t}=\frac{\theta_0}{r^2-n^2}. \tag{15}$$

将上式代回(13)式，可得

$$\frac{\mathrm{d}^2 r}{\mathrm{d}t^2}-\frac{n}{r^2-n^2}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2-\frac{r\theta_0^2}{(r-n)(r+n)^3}=0.$$

化简得

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{r+n}{r-n}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2+\frac{\theta_0^2}{r^2-n^2}\right]=0.$$

根据定理条件及(12)式可得

$$\frac{\mathrm{d}r}{\mathrm{d}t}=\frac{\epsilon r_1}{r+n}\sqrt{r^2-R_3^2}.$$

再结合(15)式，当$r>R_3$时，有

$$\frac{\mathrm{d}t}{\mathrm{d}r}=\frac{\epsilon(r+n)}{r_1\sqrt{r^2-R_3^2}},$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}r}=\frac{\mathrm{d}\theta}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}r}=\frac{\epsilon\theta_0}{r_1(r-n)\sqrt{r^2-R_3^2}}.$$

将上述两式从R_3到r积分即可得定理结论.

定理3.5 设r为常数，则测地线方程的解τ,θ,φ均为常数.

证 此时测地线方程转化为

$$\frac{\mathrm{d}^2\tau}{\mathrm{d}t^2}+\frac{4n^2\cos\theta}{(r+n)^2\sin\theta}\frac{\mathrm{d}\tau}{\mathrm{d}t}\frac{\mathrm{d}\theta}{\mathrm{d}t}+$$
$$\frac{2[4n^3\cos^2\theta-n\sin^2\theta(r+n)^2-2n(r+n)^2\cos^2\theta]}{(r+n)^2\sin\theta}\frac{\mathrm{d}\varphi}{\mathrm{d}t}\frac{\mathrm{d}\theta}{\mathrm{d}t}=0.$$

$$\frac{\mathrm{d}^2\varphi}{\mathrm{d}t^2}+2\left[-\frac{2n^2\cos\theta}{(r+n)^2\sin\theta}+\frac{\cos\theta}{\sin\theta}\right]\frac{\mathrm{d}\varphi}{\mathrm{d}t}\frac{\mathrm{d}\theta}{\mathrm{d}t}-\frac{2n}{(r+n)^2\sin\theta}\frac{\mathrm{d}\tau}{\mathrm{d}t}\frac{\mathrm{d}\theta}{\mathrm{d}t}=0.$$

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}+\left[\frac{4n^2\cos\theta\sin\theta}{(r+n)^2}-\sin\theta\cos\theta\right]\left(\frac{\mathrm{d}\varphi}{\mathrm{d}t}\right)^2+\frac{2n\sin\theta}{(r+n)^2}\frac{\mathrm{d}\tau}{\mathrm{d}t}\frac{\mathrm{d}\varphi}{\mathrm{d}t}=0.$$

$$\frac{n(r-n)}{(r+n)^3}\left(\frac{\mathrm{d}\tau}{\mathrm{d}t}\right)^2+\left[\frac{4n^3\cos^2\theta}{(r+n)^2}+r\sin^2\theta\right]\frac{r-n}{r+n}\left(\frac{\mathrm{d}\varphi}{\mathrm{d}t}\right)^2+$$
$$\frac{r(r-n)}{r+n}\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2+\frac{4n^2(r-n)\cos\theta}{(r+n)^3}\frac{\mathrm{d}\tau}{\mathrm{d}t}\frac{\mathrm{d}\varphi}{\mathrm{d}t}=0. \tag{16}$$

对方程(16)配方可得

$$\frac{n(r-n)}{(r+n)^3}\left(\frac{\mathrm{d}\tau}{\mathrm{d}t}+2n\cos\theta\frac{\mathrm{d}\varphi}{\mathrm{d}t}\right)^2+\frac{r(r-n)\sin^2\theta}{r+n}\left(\frac{\mathrm{d}\varphi}{\mathrm{d}t}\right)^2+\frac{r(r-n)}{r+n}\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2=0.$$

上式中的三项均非负，所以有$\frac{\mathrm{d}\tau}{\mathrm{d}t}=\frac{\mathrm{d}\varphi}{\mathrm{d}t}=\frac{\mathrm{d}\theta}{\mathrm{d}t}=0$. 即$\tau,\theta,\varphi$均为常数，此时其他方程自动成立，

故定理得证.

定理3.6 设φ为常数, 则有τ或θ为常数.

证 此时测地线方程化为

$$\frac{\mathrm{d}^2\tau}{\mathrm{d}t^2}+\frac{4n^2\cos\theta}{(r+n)^2\sin\theta}\frac{\mathrm{d}\tau}{\mathrm{d}t}\frac{\mathrm{d}\theta}{\mathrm{d}t}+\frac{2n}{r^2-n^2}\frac{\mathrm{d}\tau}{\mathrm{d}t}\frac{\mathrm{d}r}{\mathrm{d}t}=0.$$

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}+\frac{2r}{r^2-n^2}\frac{\mathrm{d}r}{\mathrm{d}t}\frac{\mathrm{d}\theta}{\mathrm{d}t}=0.$$

$$\frac{\mathrm{d}^2r}{\mathrm{d}t^2}-\frac{n}{r^2-n^2}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2-\frac{n(r-n)}{(r+n)^3}\left(\frac{\mathrm{d}\tau}{\mathrm{d}t}\right)^2-\frac{r(r-n)}{r+n}\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2=0.$$

$$-\frac{2n}{(r+n)^2\sin\theta}\frac{\mathrm{d}\tau}{\mathrm{d}t}\frac{\mathrm{d}\theta}{\mathrm{d}t}=0. \tag{17}$$

由方程(17)可得$\dfrac{\mathrm{d}\tau}{\mathrm{d}t}=0$或$\dfrac{\mathrm{d}\theta}{\mathrm{d}t}=0$. 即$\tau$或$\theta$为常数.

注 当$\dfrac{\mathrm{d}\tau}{\mathrm{d}t}=0$时, 测地线方程转化为定理3.2的情形. 当$\dfrac{\mathrm{d}\theta}{\mathrm{d}t}=0$时, 转化为定理3.3的情形.

设$\theta\in(0,\pi)$为常数, 此时测地线方程化为

$$\frac{\mathrm{d}^2\tau}{\mathrm{d}t^2}+\frac{2n}{r^2-n^2}\frac{\mathrm{d}\tau}{\mathrm{d}t}\frac{\mathrm{d}r}{\mathrm{d}t}-\frac{4n\cos\theta}{r+n}\frac{\mathrm{d}\varphi}{\mathrm{d}t}\frac{\mathrm{d}r}{\mathrm{d}t}=0.$$

$$\frac{\mathrm{d}^2r}{\mathrm{d}t^2}-\frac{n}{r^2-n^2}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2-\frac{n(r-n)}{(r+n)^3}\left(\frac{\mathrm{d}\tau}{\mathrm{d}t}\right)^2-$$

$$\left[\frac{4n^3\cos^2\theta}{(r+n)^2}+r\sin^2\theta\right]\frac{r-n}{r+n}\left(\frac{\mathrm{d}\varphi}{\mathrm{d}t}\right)^2-\frac{4n^2(r-n)\cos\theta}{(r+n)^3}\frac{\mathrm{d}\tau}{\mathrm{d}t}\frac{\mathrm{d}\varphi}{\mathrm{d}t}=0. \tag{18}$$

$$\frac{\mathrm{d}^2\varphi}{\mathrm{d}t^2}+\frac{2r}{r^2-n^2}\frac{\mathrm{d}\varphi}{\mathrm{d}t}\frac{\mathrm{d}r}{\mathrm{d}t}=0. \tag{19}$$

$$-\left[\sin\theta\cos\theta-\frac{4n^2\cos\theta\sin\theta}{(r+n)^2}\right]\left(\frac{\mathrm{d}\varphi}{\mathrm{d}t}\right)^2+\frac{2n\sin\theta}{(r+n)^2}\frac{\mathrm{d}\tau}{\mathrm{d}t}\frac{\mathrm{d}\varphi}{\mathrm{d}t}=0. \tag{20}$$

由方程(20)可得

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t}\left\{\left[\frac{4n^2\cos\theta\sin\theta}{(r+n)^2}-\sin\theta\cos\theta\right]\frac{\mathrm{d}\varphi}{\mathrm{d}t}+\frac{2n\sin\theta}{(r+n)^2}\frac{\mathrm{d}\tau}{\mathrm{d}t}\right\}=0.$$

所以$\dfrac{\mathrm{d}\varphi}{\mathrm{d}t}=0$或

$$\left[\frac{4n^2\cos\theta\sin\theta}{(r+n)^2}-\sin\theta\cos\theta\right]\frac{\mathrm{d}\varphi}{\mathrm{d}t}+\frac{2n\sin\theta}{(r+n)^2}\frac{\mathrm{d}\tau}{\mathrm{d}t}=0. \tag{21}$$

当$\frac{\mathrm{d}\varphi}{\mathrm{d}t}=0$时, 求解测地线方程转化为定理3.2的情形. 所以下面来求解(21)式的情形.

引理3.1 设θ, r_1, φ_0为常数, 且$\theta\in(0,\pi), r_1>0$. 令

$$F(r)=2nr_1^2r^2-\varphi_0^2\cos^2\theta r-(2n^3r_1^2+n\varphi_0^2\cos^2\theta+2n\varphi_0^2\sin^2\theta). \tag{22}$$

则$F(r)=0$存在两个实根$R_+\geq R_-$, 具体形式如下

$$R_\pm=\frac{\varphi_0^2\cos^2\theta}{4nr_1^2}\pm\sqrt{n^2+\frac{\varphi_0^4\cos^4\theta+8n^2\varphi_0^2r_1^2\cos^2\theta+16n^2\varphi_0^2r_1^2\sin^2\theta}{16n^2r_1^4}}.$$

此外$R_+\geq n\geq R_-$, 且$R_+=n$当且仅当$\varphi_0=0$.

证 通过直接计算即可证明.

定理3.7 设$\theta\in(0,\pi)$为常数, $r_1>0$. 如果当$r\geq R_+>n>R_-$时满足条件

$$\lim_{r\to R_+} t=t_1,\ \lim_{r\to R_+}\tau=\tau_1,\ \lim_{r\to R_+}(r^2-n^2)\frac{\mathrm{d}\varphi}{\mathrm{d}t}=\varphi_0,$$

$$\lim_{r\to R_+}\varphi=\varphi_1,\ \lim_{r\to R_+}\left[\frac{r+n}{r-n}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2+\frac{\varphi_0^2\cos^2\theta}{2n(r-n)}+\frac{\varphi_0^2\sin^2\theta}{r^2-n^2}\right]=r_1^2.$$

则测地线可表示为

$$t(r)=t_1+\frac{\epsilon}{r_1}\left[\sqrt{(r-R_+)(r-R_-)}+\left(\frac{\varphi_0^2\cos^2\theta}{2nr_1^2}+2n\right)\sinh^{-1}\left(\sqrt{\frac{r-R_+}{R_+-R_-}}\right)\right],$$

$$\varphi(r)=\varphi_1+\frac{2\epsilon\varphi_0}{r_1\sqrt{(R_+-n)(n-R_-)}}\tanh^{-1}\left(\sqrt{\frac{(r-R_+)(n-R_-)}{(r-R_-)(R_+-n)}}\right),$$

$$\tau(r)=\tau_1+\frac{\epsilon\varphi_0\cos\theta}{2nr_1}\left[\sqrt{(r-R_+)(r-R_-)}+\left(\frac{\varphi_0^2\cos^2\theta}{2nr_1^2}+6n\right)\sinh^{-1}\left(\sqrt{\frac{r-R_+}{R_+-R_-}}\right)\right].$$

证 与定理3.3的证明类似, 由方程(19)可得

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t}=\frac{\varphi_0}{r^2-n^2}. \tag{23}$$

将上式代入(21)式, 有

$$\frac{\mathrm{d}\tau}{\mathrm{d}t}=\frac{\varphi_0(r+3n)\cos\theta}{2n(r+n)}. \tag{24}$$

代入方程(18)可得

$$\frac{\mathrm{d}^2r}{\mathrm{d}t^2}-\frac{n}{r^2-n^2}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2-\frac{\varphi_0^2r\sin^2\theta}{(r+n)^3(r-n)}-\frac{\varphi_0^2\cos^2\theta}{4n(r^2-n^2)}=0.$$

对上式变形可得

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{r+n}{r-n}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2+\frac{\varphi_0^2\cos^2\theta}{2n(r-n)}+\frac{\varphi_0^2\sin^2\theta}{r^2-n^2}\right]=0.$$

应用(22)式解得

$$\frac{\mathrm{d}r}{\mathrm{d}t}=\frac{\epsilon\sqrt{F(r)}}{\sqrt{2n}(r+n)}=\frac{\epsilon r_1\sqrt{(r-R_+)(r-R_-)}}{r+n}.$$

结合(23)式和(24)式, 当$r>R_+$时, 有

$$\frac{\mathrm{d}t}{\mathrm{d}r}=\frac{\epsilon(r+n)}{r_1\sqrt{(r-R_+)(r-R_-)}},$$

$$\frac{\mathrm{d}\varphi}{\mathrm{d}r}=\frac{\mathrm{d}\varphi}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}r}=\frac{\epsilon\varphi_0}{r_1(r-n)}\frac{1}{\sqrt{(r-R_+)(r-R_-)}},$$

$$\frac{\mathrm{d}\tau}{\mathrm{d}r}=\frac{\mathrm{d}\tau}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}r}=\frac{\epsilon\varphi_0(r+3n)\cos\theta}{2nr_1}\frac{1}{\sqrt{(r-R_+)(r-R_-)}}.$$

将上述三式从R_+到r积分, 即可得到定理结论.

注 由引理3.1可知, $R_+=n$当且仅当$\varphi_0=0$. 所以当$\varphi_0=0$时, 测地线可经过$r=n$.

参考文献:

[1] Chandrasekhar S. The Mathematical Theory of Black Holes[M]. New York: Oxford University Press, 1983.

[2] Ciéslik A, Mach P. Revisiting timelike and null geodesics in the Schwarzschild spacetime: general expressions in terms of Weierstrass elliptic functions[J]. Class Quantum Grav, 2022, 39: 225003.
[3] Battista E, Esposito G. Geodesic motion in Euclidean Schwarzschild geometry. Eur Phys J C, 2022, 82: 1088.
[4] Yang Yekun, Zhang Xiao. Geodesics on metrics of Eguchi-Hanson type[J]. Eur Phys J C,2023, 83: 574.
[5] Gibbons G W, Hawking S W. Classification of Gravitational Instanton Symmetries[J]. Commun Math Phys, 1979, 66: 291-310.
[6] Taub A H. Empty space-times admitting a three parameter group of motions[J]. Ann Math,1951, 53: 472-490.
[7] Newman E, Tamburino L, Unti T. Empty space generalization of the Schwarzschild metric[J]. J Math Phys, 1963, 4: 915.
[8] Clément G, Gal'tsov D, Guenouche M. Rehabilitating spacetimes with NUTs[J]. Phys Lett B, 2015, 750: 591-594.
[9] Clément G, Gal'tsov D, Guenouche M. NUT wormholes[J]. Phys Rev D, 2016, 93(2): 024048.
[10] Liu Junfei, Liu Haishan. Thermodynamics of Taub-NUT-AdS spacetimes[J]. Eur Phys J C, 2024, 84: 515.

Geodesics on metrics of self-dual Taub-NUT type

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Abstract: Geodesic equations on metrics of self-dual Taub-NUT type were studied when some of the parameters were constants , and several cases were presented in which the explicit solutions to geodesic equations could be obtained.

Keywords: self-dual; Taub-NUT metrics; geodesics

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