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加权双正则函数的一些性质

王龙优[1]， 肖卓峰[2,*]， 王丽萍[1]

(1. 河北师范大学 数学科学学院，河北石家庄 050024;
2. 河北师范大学 附属民族学院，河北石家庄 050091)

摘　要: 加权双正则函数是加权正则函数的进一步发展, 并且加权双正则函数是Clifford分析中一类重要的函数. 在加权正则函数研究的基础上, 结合加权双正则函数的本身特征, 讨论加权双正则函数球外一点的Cauchy积分公式、平均值定理、最大模原理、Morera定理以及Painlevé定理.

关键词: 球外一点的Cauchy积分公式; 平均值定理; 最大模原理; Morera定理; Painlevé定理

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§1　引　言

1878年, Clifford[1]建立了一个新的数学框架, 创造出了一种可结合不可交换的几何代数结构, 并以他的名字将其命名为Clifford代数. 1982年, Brackx等[2]研究了正则函数球外一点的Cauchy积分公式、平均值定理、最大模原理、Morera定理、Painlevé定理、Weierstrass定理以及Taylor展开式及其应用等, 为后续研究Clifford分析打下了坚实的基础. 随后, 实Clifford分析作为新兴起的活跃的分析分支得到了发展. 国内外有很多学者致力于对Clifford分析的探究, 如Gürlebeck等[3]、Begher等[4]、Kähler等[5]以及黄沙等[6-7]、乔玉英等[7-8]、杜金元等[9]、任广斌等[10-11].

2008年, 王海燕等[12]对双正则函数的Morera定理、平均值定理以及Painlevé定理等进行了研究. 2009年, 彭维玲等[13]研究了双正则函数的平均值定理和最大模原理. 2018年, Vanegas等[14]对带有Clifford常数权的Dirac算子的基本解进行研究, 并且详细证明了该类加权正则函数的Cauchy-Pompeiu公式与Cauchy积分公式. 罗利萍等[15-17]在Vanegas等研究的基础上, 对加权Dirac算子进行了进一步探究, 研究了加权正则函数的平均值定理、最大模原理、Weierstrass定理以及Taylor展开式及其应用等.

在上述研究的基础上, 并结合加权双正则函数本身的特征, 本文将研究加权双正则函数球外一点的Cauchy积分公式、平均值定理、最大模原理、Morera定理以及Painlevé定理.

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*通讯作者, E-mail: xzhfwlp@163.com
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§2 预备知识

设$\{e_1,e_2,\cdots,e_n\}$是n维欧氏空间$\mathbf{R}^n$的一组标准正交基, $\mathcal{A}_n(\mathbf{R})$是$2^n$维实Clifford代数空间, 它的基为$\beta=\{e_N|N\in\mathit{\Gamma}_n\}$, 其中$\mathit{\Gamma}_n=\{0,1,\cdots,n,12,13,\cdots,123\cdots n\}$, $\mathcal{A}_n(\mathbf{R})$中的基元素一般可以写为: $e_N=e_{N_1}e_{N_2}\cdots e_{N_r}$, 其中$N=\{N_1,\cdots,N_r\}\subseteq\{1,2,\cdots,n\}$, 且$1\leq N_1<\cdots<N_r\leq n$. 当$N=\varnothing$时, $e_N=e_0=1$. 而且对于任意元素$a\in\mathcal{A}_n(\mathbf{R})$, 都可表示为$a=\sum_N a_Ne_N$, 其中$a_N\in\mathbf{R}$. Clifford代数中乘法运算有法则

$$\begin{cases}e_i^2=-1, & i=1,2,\cdots,n,\\ e_ie_j=-e_je_i, & i,j=1,2,\cdots,n,i<j,\\ \overline{a\cdot b}=\bar{b}\cdot\bar{a}, & a,b\in\mathcal{A}_n(\mathbf{R}).\end{cases}$$

$\mathcal{A}_n(\mathbf{R})$中任一元素$a$的模及共轭分别定义为

$$|a|=\sqrt{[a,a]_0}=\left(\textstyle\sum_N|a_N|^2\right)^{\frac{1}{2}},\quad \overline{a}=\sum_N a_N\overline{e}_N,$$

其中$\overline{e}_N=(-1)^{\#N(\#N+1)/2}e_N$, $\#N$记为N的指标.

设$\mathit{\Omega}\subset\mathbf{R}^n$是一非空连通开集, 则定义在$\mathit{\Omega}$中取值于$\mathcal{A}_n(\mathbf{R})$的函数$f$可表示为$f(x)=\sum_N f_N(x)e_N$, 其中$f_N(x)$为实值函数. $F_{\mathit{\Omega}}^{(r)}$表示$\mathit{\Omega}$中$C^r$函数的全体, 即

$$F_{\mathit{\Omega}}^{(r)}=\{f|f:\mathit{\Omega}\to\mathcal{A}_n(\mathbf{R}),f(x)=\sum_N f_N(x)e_N,\ f_N(x)\in C^r,x\in\mathit{\Omega}\}.$$

加权Dirac算子$\mathcal{D}_\omega$及共轭$\overline{\mathcal{D}}_\omega$分别定义为

$$\mathcal{D}_\omega=\sum_{i=1}^n\psi_i\partial_i,\qquad \overline{\mathcal{D}}_\omega=\sum_{i=1}^n\overline{\psi_i}\partial_i,\tag{2.1}$$

其中$\psi_i(i=1,2,\cdots,n)$是一个Clifford常数.

二阶椭圆微分算子$\widetilde{\mathit{\Delta}}_n$的定义为

$$\widetilde{\mathit{\Delta}}_n=\sum_{i=1}^n B_{ii}\partial_i^2+2\sum_{1\leq i<j\leq n}B_{ij}\partial_i\partial_j,\tag{2.2}$$

其中B_{ij}是矩阵B的元素. 矩阵B是对称的正定矩阵, 从而它的逆矩阵$A=B^{-1}$和它的平方根$B^{\frac{1}{2}}$也是对称的正定矩阵, 且满足$A^{\frac{1}{2}}=B^{-\frac{1}{2}}$. 此外, 矩阵$B$还满足Cholesky分解$B=LL^{\mathrm{T}}$($L$是对角线上元素都大于0的下三角矩阵).

为了让$\widetilde{\mathit{\Delta}}_n=\mathcal{D}_\omega\overline{\mathcal{D}}_\omega=\overline{\mathcal{D}}_\omega\mathcal{D}_\omega$成立, 经过简单计算$\psi_i$需满足

$$\psi_i\overline{\psi_j}+\psi_j\overline{\psi_i}=2B_{ij}\quad\text{或}\quad\overline{\psi_i}\psi_j+\overline{\psi_j}\psi_i=2B_{ij},\quad i,j=1,2,\cdots,n.\tag{2.3}$$

当$\psi_i=e_i$时,

$$e_ie_j+e_je_i=-2\delta_{ij},\quad i,j=1,2,\cdots,n.\tag{2.4}$$

即当(2.3)式中B_{ij}为单位矩阵中的元素时, 加权Dirac算子D_ω便退化为经典Dirac算子

$$D=\textstyle\sum_{i=1}^n e_i\partial_i.$$

下面构造ψ_i, 使其满足(2.3)式. 定义$\mathit{\Gamma}_n^o\subset\mathit{\Gamma}_n$为

$$\mathit{\Gamma}_n^o=\{N\in\mathit{\Gamma}_n:\#N\equiv1(\mathrm{mod}\,4)\text{或}\#N\equiv2(\mathrm{mod}\,4)\}.$$

设$\mathit{\Gamma}_*^-$是$\mathit{\Gamma}_n^o$的非空子集, 满足对$\forall A,B\in\mathit{\Gamma}_*^-$, $A\neq B$ 时, $A\triangle B\in\mathit{\Gamma}_n^o$.

考虑$\mathcal{A}_n(\mathbf{R})$的子空间$\mathcal{A}_n^-(\mathbf{R})$, 定义$\mathcal{A}_n^-(\mathbf{R})=\mathrm{gen}\{e_N:N\in\mathit{\Gamma}_n^-\}$, 其中$\mathit{\Gamma}_n^-=\{0\}\cup\mathit{\Gamma}_*^-$. 令$m=\#\mathit{\Gamma}_*^-$, p_i是$\mathit{\Gamma}_*^-$的第i个元素, $p_0=0$, $i=0,1,\cdots,m$.

引理2.1(见[2, p7]) $\mathcal{A}_n^-(\mathbf{R})$中的元素满足下述运算性质, 即

(1) $\overline{e_N}=-e_N$, $N\in\Gamma_n^o$.

(2) 任意$a\in\mathcal{A}_n^-(\mathbf{R})$, 有$a+\bar{a}=2[a]_0$, $a\bar{a}=\bar{a}a=|a|^2=\sum_{N\in\Gamma_n^-}[a]_N^2$.

(3) $e_N^2=-1$, $e_Ne_M+e_Me_N=0$, $N,M\in\Gamma_*^-$, $N\neq M$.

(4) 若$a,b\in\mathcal{A}_n^-(\mathbf{R})$, 则$[a\bar{b}]_0=\sum_{N\in\Gamma_n^-}[a]_N[b]_N=[\bar{b}a]_0$.

定义ψ_i是矩阵L的第i行嵌入到$\mathcal{A}_n^-(\mathbf{R})$中的元素, 即

$$\psi_i=\sum_{N\in\Gamma_n^-}[\psi_i]_Ne_N=\sum_{k=1}^{m}L_{ik}e_{p_k},\quad i=1,2,\cdots,n. \tag{2.5}$$

将上述性质应用到(2.3)式得

$$\psi_i\overline{\psi_j}+\psi_j\overline{\psi_i}=\psi_i\overline{\psi_j}+\overline{\psi_i\overline{\psi_j}}=2[\psi_i\overline{\psi_j}]_0=2\sum_{k=1}^{m}L_{ik}L_{jk}=2B_{ij},\quad i,j=1,2,\cdots,n.$$

本文中的ψ_i均为(2.5)式的形式.

定义1.1 设$f\in F_\Omega^{(r)}$, 若

$$\begin{cases}\mathcal{D}_{\omega_x}f(x,y)=\displaystyle\sum_{i=1}^{m}\psi_i\frac{\partial f}{\partial x_i}=\sum_{i,N}\psi_ie_N\frac{\partial f_N}{\partial x_i}=0,\\ f(x,y)\mathcal{D}_{\omega_y}=\displaystyle\sum_{j=1}^{k}\frac{\partial f}{\partial y_j}\psi_j=\sum_{j,N}\frac{\partial f_N}{\partial y_j}e_N\psi_j=0.\end{cases}$$

则称$f(x,y)$为加权双正则函数.

定义1.2 对于$\mathbf{R}^n$内任意两点$\xi=(\xi_1,\xi_2,\cdots,\xi_n)$和$x=(x_1,x_2,\cdots,x_n)$, 定义$x$和$\xi$之间的非欧氏距离$\rho$为

$$\rho^2(x,\xi)=\sum_{i,j=1}^{n}A_{ij}(x_i-\xi_i)(x_j-\xi_j)=\langle(x-\xi),A(x-\xi)\rangle, \tag{2.6}$$

其中A_{ij}是矩阵A的元素. 当$x\neq\xi$时, 设它们之间的欧氏距离为r, 即$r=|x-\xi|$, 则有$x-\xi=ry(|y|=1)$, 把此点y和$(0,\cdots,0)$之间的非欧氏距离记作ρ_0, 则有$\rho_0\geq c>0$. 还可证明$\rho=r\rho_0$, 则$\rho\geq cr$.

引理2.2(见[14, p7]) 设

$$\begin{aligned}E_\omega(x,\xi)&=\frac{1}{\det(B_1)^{\frac{1}{2}}\omega_m}\frac{1}{\rho_1^m}\sum_{i,j=1}^{m}\overline{\psi}_{1_i}A_{1_{ij}}(x_j-\xi_j),\\ E_\omega(y,\tau)&=\frac{1}{\det(B_2)^{\frac{1}{2}}\omega_k}\frac{1}{\rho_2^k}\sum_{i,j=1}^{k}\overline{\psi}_{2_i}A_{2_{ij}}(y_j-\tau_j).\end{aligned} \tag{2.7}$$

则

$$\mathcal{D}_{\omega_x}E_\omega(x,\xi)=E_\omega(x,\xi)\mathcal{D}_{\omega_\xi}=0,\quad \mathcal{D}_{\omega_y}E_\omega(y,\tau)=E_\omega(y,\tau)\mathcal{D}_{\omega_\tau}=0.$$

其中ω_m, ω_k分别表示$\mathbf{R}^m$和$\mathbf{R}^k$中单位球的表面积, $\det(B_1)$, $\det(B_2)$分别表示$m\times m$阶矩阵B_1和$k\times k$阶矩阵B_2的行列式.

引理2.3(见[14, p7])(Stokes公式) 设Ω_1如上所述, $\partial\Omega_1$足够光滑, $u,v:\Omega_1\to\mathcal{A}_m$是$\Omega_1$上的连续可微函数, 则对于加权Dirac算子$\mathcal{D}_\omega$有

$$\int_{\Omega_1}(v\mathcal{D}_\omega\cdot u+v\cdot\mathcal{D}_\omega u)\mathrm{d}x=\int_{\partial\Omega_1}v\mathrm{d}\sigma_xu. \tag{2.8}$$

其中$\mathrm{d}\sigma_x=\sum_{i=1}^{m}\psi_i\mathcal{N}_{1_i}\mathrm{d}\mu_1$是在坐标系$\{\psi_1,\cdots,\psi_m\}$下$\mathcal{A}_m(\mathbf{R})$值的面积微元,

$$\mathcal{N}_1 = (\mathcal{N}_{1_1}, \cdots, \mathcal{N}_{1_m})$$

为单位外法向量, $\mathrm{d}\mu_1$为标量面积微元, 则有$\mathrm{d}\sigma_x = \mathcal{N}_1 \mathrm{d}\mu_1$. $\mathrm{d}x$为体积微元, 且

$$\mathrm{d}x = \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_m.$$

引理2.4(见[16, p23]) 设Ω_1是$\mathbf{R}^m$中一个开集, 若f是Ω_1上的连续函数, 则对于Ω_1上的任意一个有界闭区间I_1,

$$\int_{\partial I_1} \mathrm{d}\sigma_x f(x) = 0 \tag{2.9}$$

成立的充分必要条件是对任意$g \in F^{(1)}_{\Omega_1}$, $g\mathcal{D}_\omega = 0$, 满足

$$C_0 = \int_{\partial I_1} g(x) \mathrm{d}\sigma_x f(x) = 0. \tag{2.10}$$

推论2.1 设Ω_2是$\mathbf{R}^k$中一个开集, 若f是Ω_2上的连续函数, 则对于Ω_2上的任意一个有界闭区间I_2,

$$\int_{\partial I_2} f(y) \mathrm{d}\sigma_y = 0 \tag{2.11}$$

成立的充分必要条件是对任意$h \in F^{(1)}_{\Omega_2}$, $\mathcal{D}_\omega h = 0$, 有下式成立

$$C_0 = \int_{\partial I_2} f(y) \mathrm{d}\sigma_y h(y) = 0, \tag{2.12}$$

其中$\mathrm{d}\sigma_y = \sum_{j=1}^k \psi_j \mathcal{N}_{2_j} \mathrm{d}\mu_2$是在坐标系$\{\psi_1, \cdots, \psi_k\}$下$\mathcal{A}_k(\mathbf{R})$值的面积微元,

$$\mathcal{N}_2 = (\mathcal{N}_{2_1}, \cdots, \mathcal{N}_{2_k})$$

为单位外法向量, $\mathrm{d}\mu_2$为标量面积微元, 则有$\mathrm{d}\sigma_y = \mathcal{N}_2 \mathrm{d}\mu_2$. $\mathrm{d}y$为体积微元, 且$\mathrm{d}y = \mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}y_k$.

设$\Omega_1 \subset \mathbf{R}^m$是一个有界域且$\partial\Omega_1$充分光滑, 对于任意$\xi \in \Omega_1$, 以$\xi$为心, $\varepsilon > 0$为半径, 做m维非欧氏距离超球$U_{1\varepsilon}(\xi) = \{x \in \Omega_1 : \rho(\xi, x) < \varepsilon\}$, $\partial U_{1\varepsilon}(\xi)$的法向量取外法向量, 则曲面$\partial U_{1\varepsilon}(\xi)$的参数方程可表示为

$$x(t) = \varepsilon B_1^{\frac{1}{2}} r_1(t) + \xi, t \in \mathbf{R}^{m-1}, \tag{2.13}$$

其中$r_1(t)$是$\mathbf{R}^m$中欧氏距离下单位球的参数方程.
由[14, p10]可知

$$E_\omega(x, \xi) \cdot \mathrm{d}\sigma_x = \frac{1}{\omega_m} \mathrm{d}\mu_{r_1}. \tag{2.14}$$

设$\Omega_2 \subset \mathbf{R}^k$是一个有界域且$\partial\Omega_2$充分光滑, 对于任意$\tau \in \Omega_2$, 以$\tau$为球心, $\varepsilon > 0$为半径, 做k维非欧氏距离超球$U_{2\varepsilon}(\tau) = \{y \in \Omega_2 : \rho(\tau, y) < \varepsilon\}$, $\partial U_{2\varepsilon}(\tau)$的法向量取外法向量, 则曲面$\partial U_{2\varepsilon}(\tau)$的参数方程可表示为

$$y(t) = \varepsilon B_2^{\frac{1}{2}} r_2(t) + \tau, t \in \mathbf{R}^{k-1}, \tag{2.15}$$

其中$r_2(t)$是$\mathbf{R}^k$中欧氏距离下单位球的参数方程.
同理可知

$$\mathrm{d}\sigma_y \cdot E_\omega(y, \tau) = \frac{1}{\omega_k} \mathrm{d}\mu_{r_2}. \tag{2.16}$$

§3 加权双正则函数的有关性质

引理3.1 设$\Omega = \Omega_1 \times \Omega_2$, $K_1 \subset \Omega_1 \subset \mathbf{R}^m$, Ω_1为非空连通开集, K_1为m维可微紧的定向流形, $2 \le m \le n$, $K_2 \subset \Omega_2 \subset \mathbf{R}^k$, Ω_2为非空连通开集, K_2为k维可微紧的定向流形, $2 \le k \le n$,

$f(x,y)\in F^r_{\Omega}$, $g(x)\in F^r_{\Omega_1}$, $h(y)\in F^r_{\Omega_2}$, $r\geq 2$, ∂K_1, ∂K_2具有给定的诱导定向, 则

$$\int_{\partial K_1} g(x)\mathrm{d}\sigma_x\left[\int_{\partial K_2} f(x,y)\mathrm{d}\sigma_y h(y)\right]=\int_{\partial K_2}\left[\int_{\partial K_1} g(x)\mathrm{d}\sigma_x f(x,y)\right]\mathrm{d}\sigma_y h(y). \tag{3.1}$$

证 令$g(x)=\sum\limits_C g_C(x)e_C$, $f(x,y)=\sum\limits_H f_H(x,y)e_H$, $h(y)=\sum\limits_G h_G(y)e_G$.

$$\int_{\partial K_1} g(x)\mathrm{d}\sigma_x\left[\int_{\partial K_2} f(x,y)\mathrm{d}\sigma_y h(y)\right]=$$
$$\sum_C\sum_{i=1}^m\sum_H\sum_{j=1}^k\sum_G e_C\psi_i e_H\psi_j e_G\int_{\partial K_1} g_C(x)N_{1_i}\mathrm{d}\mu_1\left[\int_{\partial K_2} f_H(x,y)N_{2_j}\mathrm{d}\mu_2 h_G(y)\right].$$

记$S_{C,i,H,j,G}=\sum\limits_C\sum\limits_{i=1}^m\sum\limits_H\sum\limits_{j=1}^k\sum\limits_G e_C\psi_i e_H\psi_j e_G$, 由实Stokes公式可得

$$\int_{\partial K_1} g(x)\mathrm{d}\sigma_x\left[\int_{\partial K_2} f(x,y)\mathrm{d}\sigma_y h(y)\right]=$$
$$S_{C,i,H,j,G}\left\{\int_{\partial K_1} g_C(x)N_{1_i}\mathrm{d}\mu_1\int_{K_2}\left[\frac{\partial f_H}{\partial y_j}(x,y)h_G(y)+f_H(x,y)\frac{\partial h_G}{\partial y_j}(y)\right]\mathrm{d}y\right\}=$$
$$S_{C,i,H,j,G}\left\{\int_{K_1}\frac{\partial g_C}{\partial x_i}(x)\mathrm{d}x\int_{K_2}\left[\frac{\partial f_H}{\partial y_j}(x,y)h_G(y)+f_H(x,y)\frac{\partial h_G}{\partial y_j}(y)\right]\mathrm{d}y+\right.$$
$$\left.\int_{K_1} g_C(x)\mathrm{d}x\int_{K_2}\left[\frac{\partial^2 f_H}{\partial x_i\partial y_j}(x,y)h_G(y)+\frac{\partial f_H}{\partial x_i}(x,y)\frac{\partial h_G}{\partial y_j}(y)\right]\mathrm{d}y\right\}=$$
$$S_{C,i,H,j,G}\int_{K_1\times K_2}\frac{\partial^2(g_C f_H h_G)}{\partial x_i\partial y_j}(x,y)\mathrm{d}x\mathrm{d}y.$$

同理可得

$$\int_{\partial K_2}\left[\int_{\partial K_1} g(x)\mathrm{d}\sigma_x f(x,y)\right]\mathrm{d}\sigma_y h(y)=S_{C,i,H,j,G}\int_{K_1\times K_2}\frac{\partial^2(g_C f_H h_G)}{\partial x_i\partial y_j}(x,y)\mathrm{d}x\mathrm{d}y.$$

综上所述, (3.1)式成立, 即

$$\int_{\partial K_1} g(x)\mathrm{d}\sigma_x\left[\int_{\partial K_2} f(x,y)\mathrm{d}\sigma_y h(y)\right]=\int_{\partial K_2}\left[\int_{\partial K_1} g(x)\mathrm{d}\sigma_x f(x,y)\right]\mathrm{d}\sigma_y h(y).$$

引理3.2(Cauchy积分公式) 设$\Omega,\partial\Omega$如上所述, $\Omega=\Omega_1\times\Omega_2$, $f:\overline{\Omega}\to\mathcal{A}_n(\mathbf{R})$, 若$f$是$\Omega$上的加权双正则函数且$f\in F^{(r)}_{\Omega}$, $r\geq 2$, 则

$$\int_{\partial\Omega_1\times\partial\Omega_2} E_\omega(x,\xi)\mathrm{d}\sigma_x f(x,y)\mathrm{d}\sigma_y E_\omega(y,\tau)=\begin{cases} f(\xi,\tau), & (\xi,\tau)\in\Omega,\\ 0, & (\xi,\tau)\in\overline{\Omega}^c.\end{cases} \tag{3.2}$$

证 当$(\xi,\tau)\in\overline{\Omega}^c$时, 应用引理3.1和引理2.3得

$$\int_{\partial\Omega_1\times\partial\Omega_2} E_\omega(x,\xi)\mathrm{d}\sigma_x f(x,y)\mathrm{d}\sigma_y E_\omega(y,\tau)=$$
$$\int_{\partial\Omega_1} E_\omega(x,\xi)\mathrm{d}\sigma_x\left[\int_{\partial\Omega_2} f(x,y)\mathrm{d}\sigma_y E_\omega(y,\tau)\right]= \tag{3.3}$$
$$\int_{\partial\Omega_1} E_\omega(x,\xi)\mathrm{d}\sigma_x\left[\int_{\Omega_2}(f(x,y)\mathcal{D}_{\omega_y}\cdot E_\omega(y,\tau)+f(x,y)\cdot\mathcal{D}_{\omega_y}E_\omega(y,\tau))\mathrm{d}y\right]=0.$$

当$(\xi,\tau)\in\Omega$时, $\Omega_{1\varepsilon}=\Omega_1\backslash\overline{U_{1\varepsilon}}$, $\Omega_{2\varepsilon}=\Omega_2\backslash\overline{U_{2\varepsilon}}$, 其中$U_{1\varepsilon}=\{x\in\Omega_1:\rho(x,\xi)<\varepsilon\}$, $U_{2\varepsilon}=\{y\in\Omega_2:\rho(y,\tau)<\varepsilon\}$, 则$\partial\Omega_1\bigcup\partial U_{1\varepsilon}$ 是$\Omega_{1\varepsilon}$的边界, $\partial\Omega_2\bigcup\partial U_{2\varepsilon}$是$\Omega_{2\varepsilon}$的边界, 其中$\partial\Omega_1$, $\partial\Omega_2$, $\partial U_{1\varepsilon}$, $\partial U_{2\varepsilon}$的方向都为逆时针. 应用引理3.1和引理2.3得

$$
\begin{aligned}
&\int_{\partial\Omega_1\times\partial\Omega_2} E_\omega(x,\xi)\mathrm{d}\sigma_x f(x,y)\mathrm{d}\sigma_y E_\omega(y,\tau)=\\
&\int_{\partial\Omega_2}\left[\int_{\Omega_{1\varepsilon}} E_\omega(x,\xi)\cdot\mathcal{D}_{\omega_x} f(x,y)\mathrm{d}x+\right.\\
&\left.\int_{\partial U_{1\varepsilon}} E_\omega(x,\xi)\mathrm{d}\sigma_x f(x,y)\right]\mathrm{d}\sigma_y E_\omega(y,\tau)=\\
&\int_{\Omega_{2\varepsilon}}\left[\int_{\Omega_{1\varepsilon}} E_\omega(x,\xi)\cdot\mathcal{D}_{\omega_x} f(x,y)\mathrm{d}x\right]\mathcal{D}_{\omega_y}\cdot E_\omega(y,\tau)\mathrm{d}y+\\
&\int_{\partial U_{2\varepsilon}}\left[\int_{\Omega_{1\varepsilon}} E_\omega(x,\xi)\cdot\mathcal{D}_{\omega_x} f(x,y)\mathrm{d}x\right]\mathrm{d}\sigma_y E_\omega(y,\tau)+\\
&\int_{\Omega_{2\varepsilon}}\left[\int_{\partial U_{1\varepsilon}} E_\omega(x,\xi)\mathrm{d}\sigma_x f(x,y)\right]\mathcal{D}_{\omega_y}\cdot E_\omega(y,\tau)\mathrm{d}y+\\
&\int_{\partial U_{2\varepsilon}}\left[\int_{\partial U_{1\varepsilon}} E_\omega(x,\xi)\mathrm{d}\sigma_x f(x,y)\right]\mathrm{d}\sigma_y E_\omega(y,\tau).
\end{aligned}
\tag{3.4}
$$

由于f为Ω上的加权双正则函数, 因此$\mathcal{D}_{\omega_x} f(x,y)=0$, $f(x,y)\mathcal{D}_{\omega_y}=0$, 则上式可简化为

$$
\begin{aligned}
&\int_{\partial\Omega_1\times\partial\Omega_2} E_\omega(x,\xi)\mathrm{d}\sigma_x f(x,y)\mathrm{d}\sigma_y E_\omega(y,\tau)=\\
&\int_{\partial U_{2\varepsilon}}\left[\int_{\partial U_{1\varepsilon}} E_\omega(x,\xi)\mathrm{d}\sigma_x f(x,y)\right]\mathrm{d}\sigma_y E_\omega(y,\tau).
\end{aligned}
\tag{3.5}
$$

下面计算(3.5)式, 应用引理3.1以及非欧氏距离下曲面的参数方程有

$$
\begin{aligned}
&\int_{\partial U_{1\varepsilon}\times\partial U_{2\varepsilon}} E_\omega(x,\xi)\mathrm{d}\sigma_x f(x,y)\mathrm{d}\sigma_y E_\omega(y,\tau)=\\
&\int_{|r_1|=1}\frac{1}{\omega_m}\mathrm{d}\mu_{r_1}\left[\int_{|r_2|=1}[f(\varepsilon B_1^{\frac12}r_1(t)+\xi,\varepsilon B_2^{\frac12}r_2(t)+\tau)-f(\xi,\tau)]\frac{1}{\omega_k}\mathrm{d}\mu_{r_2}\right]+\\
&\int_{|r_1|=1}\frac{1}{\omega_m}\mathrm{d}\mu_{r_1}\left[\int_{|r_2|=1}\frac{1}{\omega_k}f(\xi,\tau)\mathrm{d}\mu_{r_2}\right]=J_1+J_2.
\end{aligned}
$$

首先讨论J_1, 有下式成立

$$
\begin{aligned}
&|J_1|\le\frac{1}{\omega_m\omega_k}\int_{|r_1|=1}\mathrm{d}\mu_{r_1}\left[\int_{|r_2|=1}\sup_{|r_1|=1,|r_2|=1}|f(\varepsilon B_1^{\frac12}r_1(t)+\xi,\varepsilon B_2^{\frac12}r_2(t)+\tau)-f(\xi,\tau)|\mathrm{d}\mu_{r_2}\right]=\\
&\frac{1}{\omega_m\omega_k}\sup_{|r_1|=1,|r_2|=1}|f(\varepsilon B_1^{\frac12}r_1(t)+\xi,\varepsilon B_2^{\frac12}r_2(t)+\tau)-f(\xi,\tau)|\cdot\int_{|r_1|=1}\mathrm{d}\mu_{r_1}\cdot\int_{|r_2|=1}\mathrm{d}\mu_{r_2}.
\end{aligned}
$$

因为$f:\overline{\Omega}\to\mathcal{A}_n(\mathbf{R})$, 且$f\in F_\Omega^{(r)}$, $r\ge 2$, 所以

$$
\lim_{\varepsilon\to0}\sup_{|r_1|=1,|r_2|=1}|f(\varepsilon B_1^{\frac12}r_1(t)+\xi,\varepsilon B_2^{\frac12}r_2(t)+\tau)-f(\xi,\tau)|=0.
$$

则有

$$
\lim_{\varepsilon\to0}J_1=0,\quad J_2=\frac{1}{\omega_m\omega_k}\int_{|r_1|=1}\mathrm{d}\mu_{r_1}\cdot\int_{|r_2|=1}\mathrm{d}\mu_{r_2}\cdot f(\xi,\tau)=f(\xi,\tau).
$$

故$\lim\limits_{\varepsilon\to0}J_2=f(\xi,\tau)$. 因此

$$
\int_{\partial\Omega_1\times\partial\Omega_2} E_\omega(x,\xi)\mathrm{d}\sigma_x f(x,y)\mathrm{d}\sigma_y E_\omega(y,\tau)=f(\xi,\tau).
$$

结合上述讨论可得到加权双正则函数的Cauchy积分公式.

推论3.1 设$\Omega,\partial\Omega$如引理3.2所述, $f\in F_\Omega^{(r)}$, $r\ge2$, 则

$$
\int_{\partial\Omega_1\times\partial\Omega_2} E_\omega(x,\xi)\mathrm{d}\sigma_x\mathrm{d}\sigma_y E_\omega(y,\tau)=\begin{cases}1, & (\xi,\tau)\in\Omega,\\ 0, & (\xi,\tau)\in\overline{\Omega}^c.\end{cases}
\tag{3.6}
$$

定理3.1(球外一点的Cauchy积分公式) 设f为$(\mathbf{R}^m\backslash\overline{D}(0,R_1))\times(\mathbf{R}^k\backslash\overline{D}(0,R_2))$上的加权双正则函数, 并且满足$\lim\limits_{\xi\to\infty}f(\xi,y)=\lambda$, $\lim\limits_{\tau\to\infty}f(x,\tau)=\lambda$, $\overline{D}(0,R_1)=\{x:\rho(0,x)\le R_1\}$, $\overline{D}(0,R_2)=\{y:\rho(0,y)\le R_2\}$, 那么对任意$\xi\in\mathbf{R}^m\backslash\overline{D}(0,R_1)$, $\tau\in\mathbf{R}^k\backslash\overline{D}(0,R_2)$, 有

$$f(\xi,\tau)=\lambda+\int_{\partial D(0,R_1^{'})\times\partial D(0,R_2^{'})}E_\omega(x,\xi)\mathrm{d}\sigma_x f(x,y)\mathrm{d}\sigma_y E_\omega(y,\tau).\tag{3.7}$$

其中$R_1^{'}$, $R_2^{'}$满足$R_1<R_1^{'}<\rho_\xi$, $R_2<R_2^{'}<\rho_\tau$, ρ_ξ为ξ与$(0,0,\cdots,0)$之间的非欧氏距离, ρ_τ为τ与$(0,0,\cdots,0)$之间的非欧氏距离.

证 存在$R_1^{''}$和$R_2^{''}$使得$R_1<R_1^{'}<\rho_\xi$, $R_2<R_2^{'}<\rho_\tau$和$\overline{D}(0,R_1^{'})\subset\mathring{D}(\xi,R_1^{''})$, $\overline{D}(0,R_2^{'})\subset\mathring{D}(\tau,R_2^{''})$, 其中$\overline{D}(0,R_1^{'})=\{x:\rho(0,x)\le R_1^{'}\}$, $\overline{D}(0,R_2^{'})=\{y:\rho(0,y)\le R_2^{'}\}$, $\mathring{D}(\xi,R_1^{''})=\{x:\rho(\xi,x)<R_1^{''}\}$, $\mathring{D}(\tau,R_2^{''})=\{y:\rho(\tau,y)<R_2^{''}\}$. 设$\partial D(\xi,R_1^{''})$, $\partial D(\tau,R_2^{''})$, $\partial D(0,R_1^{'})$和$\partial D(0,R_2^{'})$的方向均为逆时针方向, 对$(\overline{D}(\xi,R_1^{''})\backslash\mathring{D}(0,R_1^{'}))\times(\overline{D}(\tau,R_2^{''})\backslash\mathring{D}(0,R_2^{'}))$应用引理3.2有

$$\begin{aligned}f(\xi,\tau)\;=\;&\int_{\partial(D(\xi,R_1^{''})\backslash D(0,R_1^{'}))\times\partial(D(\tau,R_2^{''})\backslash D(0,R_2^{'}))}E_\omega(x,\xi)\mathrm{d}\sigma_x f(x,y)\mathrm{d}\sigma_y E_\omega(y,\tau)=\\&\int_{\partial D(\xi,R_1^{''})\times\partial D(\tau,R_2^{''})}E_\omega(x,\xi)\mathrm{d}\sigma_x f(x,y)\mathrm{d}\sigma_y E_\omega(y,\tau)-\\&\int_{\partial D(0,R_1^{'})\times\partial D(\tau,R_2^{''})}E_\omega(x,\xi)\mathrm{d}\sigma_x f(x,y)\mathrm{d}\sigma_y E_\omega(y,\tau)-\\&\int_{\partial D(\xi,R_1^{''})\times\partial D(0,R_2^{'})}E_\omega(x,\xi)\mathrm{d}\sigma_x f(x,y)\mathrm{d}\sigma_y E_\omega(y,\tau)+\\&\int_{\partial D(0,R_1^{'})\times\partial D(0,R_2^{'})}E_\omega(x,\xi)\mathrm{d}\sigma_x f(x,y)\mathrm{d}\sigma_y E_\omega(y,\tau)=I_1-I_2-I_3+I_4.\end{aligned}\tag{3.8}$$

由于$D(\xi,R_1^{''})$, $D(\tau,R_2^{''})$是在非欧氏距离下的球, 所以参数方程可分别表示为

$$x(t)=R_1^{''}B_1^{\frac{1}{2}}r_1(t)+\xi,\quad y(t)=R_2^{''}B_2^{\frac{1}{2}}r_2(t)+\tau.$$

并且$E_\omega(x,\xi)\cdot\mathrm{d}\sigma_x=\dfrac{1}{\omega_m}\mathrm{d}\mu_{r_1}$, $\mathrm{d}\sigma_y\cdot E_\omega(y,\tau)=\dfrac{1}{\omega_k}\mathrm{d}\mu_{r_2}$. 首先计算$I_1$得

$$\begin{aligned}\lim_{R_1^{''}\to\infty}I_1\;=\;&\lim_{R_1^{''}\to\infty}\int_{\partial D(\tau,R_2^{''})}\left[\int_{\partial D(\xi,R_1^{''})}E_\omega(x,\xi)\mathrm{d}\sigma_x f(x,y)\right]\mathrm{d}\sigma_y E_\omega(y,\tau)=\\&\lim_{R_1^{''}\to\infty}\int_{\partial D(\tau,R_2^{''})}\left[\int_{|r_1|=1}\frac{1}{\omega_m}f(R_1^{''}B_1^{\frac{1}{2}}r_1(t)+\xi,y)\mathrm{d}\mu_{r_1}\right]\mathrm{d}\sigma_y E_\omega(y,\tau)=\\&\lambda\int_{\partial D(\tau,R_2^{''})}\mathrm{d}\sigma_y E_\omega(y,\tau)=\lambda\int_{|r_2|=1}\frac{1}{\omega_k}\mathrm{d}\mu_{r_2}=\lambda.\end{aligned}$$

其次计算I_2得

$$\begin{aligned}\lim_{R_2^{''}\to\infty}I_2\;=\;&\lim_{R_2^{''}\to\infty}\int_{\partial D(0,R_1^{'})}E_\omega(x,\xi)\mathrm{d}\sigma_x\left[\int_{\partial D(\tau,R_2^{''})}f(x,y)\mathrm{d}\sigma_y E_\omega(y,\tau)\right]=\\&\int_{\partial D(0,R_1^{'})}E_\omega(x,\xi)\mathrm{d}\sigma_x\left[\int_{|r_2|=1}\lim_{R_2^{''}\to\infty}f(x,R_2^{''}B_2^{\frac{1}{2}}r(t)+\tau)\frac{1}{\omega_k}\mathrm{d}\mu_{r_2}\right]=\\&\lambda\int_{\partial D(0,R_1^{'})}E_\omega(x,\xi)\mathrm{d}\sigma_x.\end{aligned}$$

由Stokes公式可知

$$\lambda\int_{\partial D(0,R_1^{'})}E_\omega(x,\xi)\mathrm{d}\sigma_x=\lambda\int_{D(0,R_1^{'})}E_\omega(x,\xi)\mathcal{D}_{\omega_x}\mathrm{d}x=0.$$

同理可计算得$I_3=0$. 对(3.8)式等号两边取极限$R_1^{''}\to\infty$, $R_2^{''}\to\infty$得到

$$f(\xi,\tau)=\lambda+\int_{\partial D(0,R_1^{'})\times\partial D(0,R_2^{'})}E_\omega(x,\xi)\mathrm{d}\sigma_x f(x,y)\mathrm{d}\sigma_y E_\omega(y,\tau).$$

定理3.2(平均值定理) 设$\Omega,\partial\Omega$如上所述, $f\in F_{\Omega}^{(r)}$, $r\geq 2$, 若f是Ω上的加权双正则函数, 则对于任意$(\xi,\tau)\in\Omega$, 满足$\overline{D}_1(\xi,R_1)\subset\Omega_1$, $\overline{D}_2(\tau,R_2)\subset\Omega_2$, 那么有

$$f(\xi,\tau)=\frac{1}{\det(B_1)^{\frac{1}{2}}\det(B_2)^{\frac{1}{2}}R_1^mR_2^kV_mV_k}\int_{D_1(\xi,R_1)\times D_2(\tau,R_2)}f(x,y)\mathrm{d}x\mathrm{d}y, \tag{3.9}$$

其中$\overline{D}_1(\xi,R_1)=\{x\in\Omega_1:\rho(\xi,x)\leq R_1\}$, $\overline{D}_2(\tau,R_2)=\{y\in\Omega_2:\rho(\tau,y)\leq R_2\}$, $V_m=\dfrac{\omega_m}{m}$是$\mathbf{R}^m$中m维单位球的体积, $V_k=\dfrac{\omega_k}{k}$是$\mathbf{R}^k$中k维单位球的体积.

证 根据加权双正则函数的Cauchy积分公式和Stokes公式, 有

$$\begin{aligned}
f(\xi,\tau)\ =\ &\int_{\partial D_1(\xi,R_1)\times\partial D_2(\tau,R_2)}E_\omega(x,\xi)\mathrm{d}\sigma_xf(x,y)\mathrm{d}\sigma_yE_\omega(y,\tau)=\\
&\int_{\partial D_1(\xi,R_1)}\frac{1}{\det(B_1)^{\frac{1}{2}}\omega_m}\frac{1}{\rho_1^m}\sum_{i,j=1}^m\overline{\psi}_{1_i}A_{1_{ij}}(x_j-\xi_j)\mathrm{d}\sigma_x\times\\
&\left[\int_{\partial D_2(\tau,R_2)}f(x,y)\mathrm{d}\sigma_y\frac{1}{\det(B_2)^{\frac{1}{2}}\omega_k}\frac{1}{\rho_2^k}\sum_{i,j=1}^k\overline{\psi}_{2_i}A_{2_{ij}}(y_j-\tau_j)\right]=\\
&\frac{1}{\det(B_1)^{\frac{1}{2}}\det(B_2)^{\frac{1}{2}}\omega_m\omega_kR_1^mR_2^k}\int_{\partial D_1(\xi,R_1)}\sum_{i,j=1}^m\overline{\psi}_{1_i}A_{1_{ij}}(x_j-\xi_j)\mathrm{d}\sigma_x\times\\
&\left[\int_{D_2(\tau,R_2)}\left(f(x,y)\mathcal{D}_{\omega_y}\times\sum_{i,j=1}^k\overline{\psi_{2_i}}A_{2_{ij}}(y_j-\tau_j)+f(x,y)\times\sum_{i,j=1}^k\overline{\psi}_{2_i}A_{2_{ij}}(y_j-\tau_j)\mathcal{D}_{\omega_y}\right)\mathrm{d}y\right]=\\
&\frac{1}{\det(B_1)^{\frac{1}{2}}\det(B_2)^{\frac{1}{2}}\omega_m\omega_kR_1^mR_2^k}\int_{D_1(\xi,R_1)}\mathcal{D}_{\omega_x}\sum_{i,j=1}^m\overline{\psi}_{1_i}A_{1_{ij}}(x_j-\xi_j)\times\\
&\left[\int_{D_2(\tau,R_2)}\left(f(x,y)\mathcal{D}_{\omega_y}\times\sum_{i,j=1}^k\overline{\psi}_{2_i}A_{2_{ij}}(y_j-\tau_j)+f(x,y)\times\sum_{i,j=1}^k\overline{\psi}_{2_i}A_{2_{ij}}(y_j-\tau_j)\mathcal{D}_{\omega_y}\right)\mathrm{d}y\right]\mathrm{d}x+\\
&\frac{1}{\det(B_1)^{\frac{1}{2}}\det(B_2)^{\frac{1}{2}}\omega_m\omega_kR_1^mR_2^k}\int_{D_1(\xi,R_1)}\sum_{i,j=1}^m\overline{\psi}_{1_i}A_{1_{ij}}(x_j-\xi_j)\times\\
&\left[\int_{D_2(\tau,R_2)}\left(\mathcal{D}_{\omega_x}f(x,y)\mathcal{D}_{\omega_y}\times\sum_{i,j=1}^k\overline{\psi}_{2_i}A_{2_{ij}}(y_j-\tau_j)+\mathcal{D}_{\omega_x}f(x,y)\times\sum_{i,j=1}^k\overline{\psi}_{2_i}A_{2_{ij}}(y_j-\tau_j)\mathcal{D}_{\omega_y}\right)\mathrm{d}y\right]\mathrm{d}x.
\end{aligned}$$

因为f是Ω上的加权双正则函数, 所以对任意$(x,y)\in D_1\times D_2\subset\Omega$, 有$\mathcal{D}_{\omega_x}f(x,y)=0$, $f(x,y)\mathcal{D}_{\omega_y}=0$, $\mathcal{D}_{\omega_x}f(x,y)\mathcal{D}_{\omega_y}=0$, 则上式化简为

$$\begin{aligned}
f(\xi,\tau)=&\frac{1}{\det(B_1)^{\frac{1}{2}}\det(B_2)^{\frac{1}{2}}\omega_m\omega_kR_1^mR_2^k}\int_{D_1(\xi,R_1)}\mathcal{D}_{\omega_x}\sum_{i,j=1}^m\overline{\psi}_{1_i}A_{1_{ij}}(x_j-\xi_j)\times\\
&\left[\int_{D_2(\tau,R_2)}f(x,y)\cdot\sum_{i,j=1}^k\overline{\psi}_{2_i}A_{2_{ij}}(y_j-\tau_j)\mathcal{D}_{\omega_y}\mathrm{d}y\right]\mathrm{d}x.
\end{aligned}\tag{3.10}$$

由于[15, p67]可知

$$\mathcal{D}_{\omega_x}\sum_{i,j=1}^m\overline{\psi_{1_i}}A_{1_{ij}}(x_j-\xi_j)=m,$$

同理可证

$$\sum_{i,j=1}^k\overline{\psi_{2_i}}A_{2_{ij}}(y_j-\tau_j)\mathcal{D}_{\omega_y}=k.$$

因此上式为

$$f(\xi,\tau) = \frac{mk}{\det(B_1)^{\frac{1}{2}}\det(B_2)^{\frac{1}{2}}\omega_m\omega_k R_1^m R_2^k}\int_{D_1(\xi,R_1)\times D_2(\tau,R_2)} f(x,y)\mathrm{d}x\mathrm{d}y = \frac{1}{\det(B_1)^{\frac{1}{2}}\det(B_2)^{\frac{1}{2}}R_1^m R_2^k V_m V_k}\int_{D_1(\xi,R_1)\times D_2(\tau,R_2)} f(x,y)\mathrm{d}x\mathrm{d}y.$$

定理3.3(最大模原理) 设$\Omega,\partial\Omega$如上所述, f是Ω上的加权双正则函数, 若存在$(a,b)\in\Omega$, 使得

$$|f(\xi,\tau)|\le|f(a,b)| \tag{3.11}$$

对所有$(\xi,\tau)\in\Omega$都成立, 则f在Ω上为常函数.

证 设$|f(a,b)|=\lambda$, 令$\Omega_\lambda=\{(\xi,\tau)\in\Omega_1\times\Omega_2||f(\xi,\tau)|=\lambda\}$. 要证结论成立, 首先证$\Omega_\lambda=\Omega$.

由于$(a,b)\in\Omega_\lambda$, 则Ω_λ非空. 对于任意$(\xi,\tau)\in\Omega\backslash\Omega_\lambda$, 有$|f(\xi,\tau)|<\lambda$, 又由于$|f(\xi,\tau)|$在$\Omega$上连续, 则存在以$(\xi,\tau)$为中心的某邻域$U$, 对于任意的$(u,v)\in U$有$|f(u,v)|<\lambda$, 所以$U\subset\Omega\backslash\Omega_\lambda$, 即$\Omega\backslash\Omega_\lambda$是开集, 这说明$\Omega_\lambda$是相对闭的.

任取$(\xi,\tau)\in\Omega_\lambda$, 作$m$维超球$D(\xi,R_1)=\{x\in\Omega_1:\rho(\xi,x)<R_1\}$, 作$k$维超球$D(\tau,R_2)=\{y\in\Omega_2:\rho(\tau,y)<R_2\}$, 且$\overline{D}(\xi,R_1)\subset\Omega_1$, $\overline{D}(\tau,R_2)\subset\Omega_2$, 根据定理3.2, 有

$$f(\xi,\tau)=\frac{1}{\det(B_1)^{\frac{1}{2}}\det(B_2)^{\frac{1}{2}}R_1^m R_2^k V_m V_k}\int_{D(\xi,R_1)\times D(\tau,R_2)} f(x,y)\mathrm{d}x\mathrm{d}y.$$

则

$$\lambda^2=|f(\xi,\tau)|^2=\frac{1}{\det(B_1)\det(B_2)R_1^{2m}R_2^{2k}V_m^2V_k^2}\sum_N\left(\int_{D(\xi,R_1)\times D(\tau,R_2)} f_N(x,y)\mathrm{d}x\mathrm{d}y\right)^2.$$

由Hölder不等式, 有

$$\lambda^2\le\frac{1}{\det(B_1)\det(B_2)R_1^{2m}R_2^{2k}V_m^2V_k^2}\sum_N\left(\int_{D(\xi,R_1)\times D(\tau,R_2)}\mathrm{d}x\mathrm{d}y\right)\left(\int_{D(\xi,R_1)\times D(\tau,R_2)} f_N^2(x,y)\mathrm{d}x\mathrm{d}y\right).$$

由定理3.2, 当$f\equiv 1$时, 有

$$1=\frac{1}{\det(B_1)^{\frac{1}{2}}\det(B_2)^{\frac{1}{2}}R_1^m R_2^k V_m V_k}\int_{D(\xi,R_1)\times D(\tau,R_2)}\mathrm{d}x\mathrm{d}y.$$

因而

$$\lambda^2=\frac{1}{\det(B_1)^{\frac{1}{2}}\det(B_2)^{\frac{1}{2}}R_1^m R_2^k V_m V_k}\int_{D(\xi,R_1)\times D(\tau,R_2)}\lambda^2\mathrm{d}x\mathrm{d}y.$$

且

$$\int_{D(\xi,R_1)\times D(\tau,R_2)}\mathrm{d}x\mathrm{d}y=\det(B_1)^{\frac{1}{2}}\det(B_2)^{\frac{1}{2}}R_1^m R_2^k V_m V_k,$$

则

$$\lambda^2\le\frac{1}{\det(B_1)^{\frac{1}{2}}\det(B_2)^{\frac{1}{2}}R_1^m R_2^k V_m V_k}\int_{D(\xi,R_1)\times D(\tau,R_2)}|f(x,y)|^2\mathrm{d}x\mathrm{d}y,$$

因此

$$0\le\frac{1}{\det(B_1)^{\frac{1}{2}}\det(B_2)^{\frac{1}{2}}R_1^m R_2^k V_m V_k}\int_{D(\xi,R_1)\times D(\tau,R_2)}(|f(x,y)|^2-\lambda^2)\mathrm{d}x\mathrm{d}y\le 0.$$

所以对于任意$(x,y)\in D(\xi,R_1)\times D(\tau,R_2)$, 有$|f(x,y)|=\lambda$, 即$D(\xi,R_1)\times D(\tau,R_2)\subset\Omega_\lambda$, 因此$\Omega_\lambda$是$\Omega$中的开集.

综上所述$\Omega_\lambda=\Omega$, 则对任意$(\xi,\tau)\in\Omega$, 有$|f(\xi,\tau)|=\lambda$. 下证f为常函数.

当$\lambda=0$时, 显然有$f(\xi,\tau)=0$, 即f在Ω上为常数.

当$\lambda>0$时, 由[15, p68]可知$\partial_{\xi_i}f_N=0, i=1,2,\cdots,m$, 同理可证得$\partial_{\tau_j}f_N=0, j=$

$1,2,\cdots,k$. 所以f 在Ω上为常函数.

推论3.2 设$\Omega,\partial\Omega$如上所述, 若f是Ω上的加权双正则函数, 且在$\overline{\Omega}$内连续, 对于任意的$(\xi,\tau)\in\overline{\Omega}$, 有$|f(\xi,\tau)|\leq M_1$, 则除$f$为常函数外, $|f(\xi,\tau)|<M_1$ $((\xi,\tau)\in\Omega)$.

推论3.3 设$\Omega,\partial\Omega$如上所述, 若f是Ω上的加权双正则函数, 且在$\overline{\Omega}$上连续, 则

$$\sup_{(\xi,\tau)\in\overline{\Omega}}|f(\xi,\tau)|=\sup_{(\xi,\tau)\in\partial\Omega}|f(\xi,\tau)|. \tag{3.12}$$

证 如果f是常函数, 则结论成立.

假设f是非常函数的加权双正则函数, 由于Ω是有界域, 且f在$\overline{\Omega}$上连续, 则存在点$(a,b)\in\overline{\Omega}$, 使得

$$\sup_{(\xi,\tau)\in\overline{\Omega}}|f(\xi,\tau)|=|f(a,b)|.$$

如果$(a,b)\in\partial\Omega_1\times\partial\Omega_2$, 则

$$\sup_{(\xi,\tau)\in\partial\Omega}|f(\xi,\tau)|=|f(a,b)|.$$

则结论成立.

假设$(a,b)\in\Omega_1\times\Omega_2$, 将$\Omega$分解为$\Omega=\Omega_1'\cup\Omega_2'\cup\cdots$, 其中$\Omega_j'$是有界的连通开集, $j=1,2,\cdots$, 则存在j使得$(a,b)\in\Omega_j'$. 由于任意$(\xi,\tau)\in\Omega_j'$, 有$|f(\xi,\tau)|\leq|f(a,b)|$, f是加权双正则函数, 根据定理3.3, 则f在Ω_j'上为常函数, 与已知矛盾, 因此

$$\sup_{(\xi,\tau)\in\overline{\Omega}}|f(\xi,\tau)|=|f(a,b)|=\sup_{(\xi,\tau)\in\partial\Omega_j'}|f(\xi,\tau)|\leq\sup_{(\xi,\tau)\in\partial\Omega}|f(\xi,\tau)|\leq\sup_{(\xi,\tau)\in\overline{\Omega}}|f(\xi,\tau)|,$$

即

$$\sup_{(\xi,\tau)\in\overline{\Omega}}|f(\xi,\tau)|=\sup_{(\xi,\tau)\in\partial\Omega}|f(\xi,\tau)|.$$

推论3.4 设$\Omega,\partial\Omega$如上所述, 若f是Ω上非常量的加权双正则函数, 则对Ω上任意一点(ξ,τ), 有

$$|f(\xi,\tau)|<\sup_{(\zeta,\eta)\in\partial\Omega}\ \overline{\lim_{\substack{(u,v)\to(\zeta,\eta)\\(u,v)\in\Omega}}}|f(u,v)|. \tag{3.13}$$

证 不妨设$\sup\limits_{(\zeta,\eta)\in\partial\Omega}\ \overline{\lim\limits_{(u,v)\to(\zeta,\eta),(u,v)\in\Omega}}|f(u,v)|<+\infty$, 否则显然成立. 令

$$\varphi(\xi,\tau)=\begin{cases}|f(\xi,\tau)|, & (\xi,\tau)\in\Omega,\\ \overline{\lim\limits_{\substack{(u,v)\to(\xi,\tau)\\(u,v)\in\Omega}}}|f(u,v)|, & (\xi,\tau)\in\partial\Omega.\end{cases}$$

由$\varphi(\xi,\tau)$的上半连续性和$\overline{\Omega}$为紧集, 则存在一点$(a,b)\in\overline{\Omega}$, 使得$\varphi(a,b)=\{\sup\varphi(u,v),(u,v)\in\overline{\Omega}\}$. 由于$f$是$\Omega$上的非常量的加权双正则函数, 则$\varphi$也是$\Omega$上的非常量加权双正则函数. 由定理3.3, 则$(a,b)\in\partial\Omega$. 即

$$|f(\xi,\tau)|=\varphi(\xi,\tau)<\varphi(a,b)=\sup_{(\zeta,\eta)\in\partial\Omega}\ \overline{\lim_{\substack{(u,v)\to(\zeta,\eta)\\(u,v)\in\Omega}}}|f(u,v)|,(\xi,\tau)\in\Omega.$$

从而$|f(\xi,\tau)|<\sup\limits_{(\zeta,\eta)\in\partial\Omega}\ \overline{\lim\limits_{\substack{(u,v)\to(\zeta,\eta)\\(u,v)\in\Omega}}}|f(u,v)|$.

定理3.4 设$\Omega,\partial\Omega$如上所述, 若f是Ω中的连续函数, 则对于任意有界闭区间$I_1\subset\Omega_1$, $I_2\subset\Omega_2$,

$$\int_{\partial I_1\times\partial I_2}\mathrm{d}\sigma_x f(x,y)\mathrm{d}\sigma_y=0 \tag{3.14}$$

成立的充分必要条件是对任意$g \in F_{\Omega_1}^{(1)}$, $g\mathcal{D}_\omega = 0$, 且任意$h \in F_{\Omega_2}^{(1)}$, $\mathcal{D}_\omega h = 0$, 满足

$$C_0 = \int_{\partial I_1 \times \partial I_2} g(x)\mathrm{d}\sigma_x f(x,y)\mathrm{d}\sigma_y h(y) = 0. \tag{3.15}$$

证 充分性. 当$g \equiv 1$, $h \equiv 1$时, $g\mathcal{D}_w = 0$, $\mathcal{D}_w h = 0$, 根据已知条件, 可得

$$\int_{\partial I_1 \times \partial I_2} \mathrm{d}\sigma_x f(x,y)\mathrm{d}\sigma_y = 0.$$

必要性. 对任意给定的有界闭区间I_1, I_2, 将I_1, I_2分别划分为2^m, 2^k个小闭区间, 记划分得到的小闭区间为$I_{i1}^{(1)} \times I_{j2}^{(1)}$, $i = 1, 2, \cdots, 2^m$, $j = 1, 2, \cdots, 2^k$.

$$C_0 = \sum_{i=1}^{2^m}\sum_{j=1}^{2^k}\int_{\partial I_{i1}^{(1)} \times \partial I_{j2}^{(1)}} g(x)\mathrm{d}\sigma_x f(x,y)\mathrm{d}\sigma_y h(y) = \sum_{i=1}^{2^m}\sum_{j=1}^{2^k} C_{i,j}^1.$$

则存在$i_1, j_1 (1 \le i_1 \le 2^m, 1 \le j_1 \le 2^k)$, 使得

$$|C_{i_1,j_1}^1| \ge \frac{1}{2^m 2^k}|C_0|.$$

记$\partial I_{i_1 1}^{(1)} \times \partial I_{j_1 2}^{(1)} = \partial I_{11} \times \partial I_{12}$, $C_{i_1,j_1}^1 = C_1$. 对$I_{11} \times I_{12}$继续做同样的分法, q次划分之后, 得到

$$|C_q| \ge \frac{1}{2^{qm}2^{qk}}|C_0|$$

和闭区间套$\{I_{q_1} \times I_{q_2}\}$的一交点$(a,b)$.

因为f在点(a,b)处连续, 则对任意$\varepsilon > 0$, 存在$\delta_1(\varepsilon) > 0$, $\delta_2(\varepsilon) > 0$, 当$\rho(x,a) < \delta_1(\varepsilon)$, $\rho(y,b) < \delta_2(\varepsilon)$时, 有$|f(x,y) - f(a,b)| < \varepsilon$, 选充分大的$q'$, 使$\delta(I_{q'1}) < \delta_1(\varepsilon)$, $\delta(I_{q'2}) < \delta_2(\varepsilon)$, 则

$$\begin{aligned} C_{q'} \;=\;& \int_{\partial I_{q'1} \times \partial I_{q'2}} [g(x) - g(a)]\mathrm{d}\sigma_x f(x,y)\mathrm{d}\sigma_y [h(y) - h(b)] + \\ & \int_{\partial I_{q'1} \times \partial I_{q'2}} [g(x) - g(a)]\mathrm{d}\sigma_x f(x,y)\mathrm{d}\sigma_y h(b) + \\ & \int_{\partial I_{q'1} \times \partial I_{q'2}} g(a)\mathrm{d}\sigma_x f(x,y)\mathrm{d}\sigma_y [h(y) - h(b)] + \\ & \int_{\partial I_{q'1} \times \partial I_{q'2}} g(a)\mathrm{d}\sigma_x f(x,y)\mathrm{d}\sigma_y h(b) = \\ & C_{q'}^{(1)} + C_{q'}^{(2)} + C_{q'}^{(3)} + C_{q'}^{(4)}. \end{aligned}$$

对于$C_{q'}^{(1)}$有

$$\begin{aligned} C_{q'}^{(1)} \;=\;& \int_{\partial I_{q'1} \times \partial I_{q'2}} [g(x) - g(a)]\mathrm{d}\sigma_x f(a,b)\mathrm{d}\sigma_y [h(y) - h(b)] + \\ & \int_{\partial I_{q'1} \times \partial I_{q'2}} \sum_{i=1}^{m}(x_i - a_i)\partial_{x_i} g[a - t_1(a-x)]\mathrm{d}\sigma_x \times \\ & [f(x,y) - f(a,b)]\mathrm{d}\sigma_y \sum_{j=1}^{k}(y_j - b_j)\partial_{y_j} h[b - t_2(b-y)]. \end{aligned}$$

其中$0 < t_1 < 1$, $0 < t_2 < 1$, 对第一项由引理3.1和Stokes公式得其为0. 对第二项, 因为$|x_i - a_i| \le r_1 = \dfrac{1}{\rho_0}\rho(x,a)$, $|y_j - b_j| \le r_2 = \dfrac{1}{\rho_0'}\rho(y,b)$, $|\partial_{x_i} g[a - t_1(a-x)]| \le M_2$, $|\partial_{y_j} h[b - t_2(b-y)]| \le M_3$, 其中$M_2 > 0$, $M_3 > 0$是一个常数, $i = 1, 2, \cdots, m$, $j = 1, 2, \cdots, k$, 且$|\mathrm{d}\sigma_x| = |N_{\psi_1}|\mathrm{d}\mu_1 = \mathrm{d}\mu_1$, $|\mathrm{d}\sigma_y| = |N_{\psi_2}|\mathrm{d}\mu_2 = \mathrm{d}\mu_2$.

$$|C_{q'}^{(1)}| \le m \cdot k \cdot \frac{1}{\rho_0\rho_0'} \cdot \frac{\delta(I_1)}{2^{q'}} \cdot M_2 \cdot \frac{V(\partial I_1)}{2^{q'(m-1)}} \cdot \varepsilon \cdot \frac{V(\partial I_2)}{2^{q'(k-1)}} \cdot \frac{\delta(I_2)}{2^{q'}} \cdot M_3.$$

对于$C_{q'}^{(2)}$有

$$\begin{aligned}C_{q'}^{(2)} &= \int_{\partial I_{q'1}\times\partial I_{q'2}}[g(x)-g(a)]\mathrm{d}\sigma_x f(a,b)\mathrm{d}\sigma_y h(b)+\\ &\int_{\partial I_{q'1}\times\partial I_{q'2}}[g(x)-g(a)]\mathrm{d}\sigma_x[f(x,y)-f(a,b)]\mathrm{d}\sigma_y h(b).\end{aligned}$$

对第一项, 由引理3.1和Stokes公式得其为0, 故

$$\begin{aligned}|C_{q'}^{(2)}| &= \left|\int_{\partial I_{q'1}\times\partial I_{q'2}}\sum_{i=1}^{m}(x_i-a_i)\partial_{x_i}g[a-t_1(a-x)]\mathrm{d}\sigma_x[f(x,y)-f(a,b)]\mathrm{d}\sigma_y h(b)\right| \le\\ &|h(b)|\cdot m\cdot M_2\cdot\frac{1}{\rho_0}\cdot\frac{\delta(I_1)}{2^{q'}}\cdot\varepsilon\cdot\frac{V(\partial I_1)}{2^{q'(m-1)}}\cdot\frac{V(\partial I_2)}{2^{q'(k-1)}}.\end{aligned}$$

同理可证对于$C_{q'}^{(3)}$有

$$|C_{q'}^{(3)}| \le |g(a)|\cdot k\cdot M_3\cdot\frac{1}{\rho_0'}\cdot\frac{\delta(I_2)}{2^{q'}}\cdot\varepsilon\cdot\frac{V(\partial I_1)}{2^{q'(m-1)}}\cdot\frac{V(\partial I_2)}{2^{q'(k-1)}}.$$

对$C_{q'}^{(4)}$有

$$C_{q'}^{(4)} = g(a)\left(\int_{\partial I_{q'1}\times\partial I_{q'2}}\mathrm{d}\sigma_x f(x,y)\mathrm{d}\sigma_y\right)h(b)=0.$$

综上所述, 存在$M_4>0$, 使得

$$|C_0|\le 2^{q'm}2^{q'k}|C_{q'}|\le M_4\cdot\varepsilon.$$

所以$C_0=0$.

定理3.5 设$\Omega,\partial\Omega$如上所述, $f\in F_{\Omega}^{(r)}$, $r\ge 2$, 且满足对任意有界闭区间$I_1\subset\Omega_1$, $I_2\subset\Omega_2$有

$$\int_{\partial I_1}\mathrm{d}\sigma_x f(x,y)=0,\quad \int_{\partial I_2}f(x,y)\mathrm{d}\sigma_y=0. \tag{3.16}$$

则对任意$(\xi,\tau)\in\mathring{I}_1\times\mathring{I}_2$有

$$f(\xi,\tau)=\int_{\partial I_1\times\partial I_2}E_\omega(x,\xi)\mathrm{d}\sigma_x f(x,y)\mathrm{d}\sigma_y E_\omega(y,\tau). \tag{3.17}$$

证 任取给定的有界闭区间$I_1\subset\Omega_1$, $I_2\subset\Omega_2$, 任取$(\xi,\tau)\in\mathring{I}_1\times\mathring{I}_2$. 由于$f$在点$(\xi,\tau)$处连续, 则对任给$\varepsilon>0$, 存在$\delta_1>0$, $\delta_2>0$, 若$(\xi+\eta_1,\tau+\eta_2)\in\mathring{I}_1\times\mathring{I}_2$, 当$|\eta_1|<\delta_1$, $|\eta_2|<\delta_2$时, 有

$$|f(\xi+\eta_1,\tau+\eta_2)-f(\xi,\tau)|<\varepsilon.$$

以ξ为中心, $l_1<\delta_1(\varepsilon)$为边长, 作闭区间$J_1$, 使$\overline{J}_1\subset\mathring{I}_1$, 以$\tau$为中心, $l_2<\delta_2(\varepsilon)$为边长, 作闭区间$J_2$, 使$\overline{J}_2\subset\mathring{I}_2$. 由于$f$连续, 当$x\in\partial J_1$, $y\in\partial J_2$时, 有$|f(x,y)-f(\xi,\tau)|<\varepsilon$.

由引理3.2有

$$\begin{aligned}&\int_{\partial I_1\times\partial I_2}E_\omega(x,\xi)\mathrm{d}\sigma_x f(x,y)\mathrm{d}\sigma_y E_\omega(y,\tau)=\\ &f(\xi,\tau)+\int_{\partial I_1\times\partial I_2}E_\omega(x,\xi)\mathrm{d}\sigma_x[f(x,y)-f(\xi,\tau)]\mathrm{d}\sigma_y E_\omega(y,\tau)=\\ &f(\xi,\tau)+I_5(\xi,\tau).\end{aligned}$$

将$I_1\backslash\mathring{J}_1$分成$k_1$个小闭区间, 将$I_2\backslash\mathring{J}_2$分成$k_2$个小闭区间, 即

$$I_1\backslash\mathring{J}_1=\sum_{i=1}^{k_1}J_{1i},\ I_2\backslash\mathring{J}_2=\sum_{j=1}^{k_2}J_{2j},$$

考虑

$$\begin{aligned}
I_5(\xi,\tau) \quad = \quad & \int_{\partial(I_1\setminus\mathring{J}_1)\times\partial(I_2\setminus\mathring{J}_2)} E_\omega(x,\xi)\mathrm{d}\sigma_x[f(x,y)-f(\xi,\tau)]\mathrm{d}\sigma_y E_\omega(y,\tau)+\\
& \int_{\partial(I_1\setminus\mathring{J}_1)\times\partial J_2} E_\omega(x,\xi)\mathrm{d}\sigma_x[f(x,y)-f(\xi,\tau)]\mathrm{d}\sigma_y E_\omega(y,\tau)+\\
& \int_{\partial J_1\times\partial(I_2\setminus\mathring{J}_2)} E_\omega(x,\xi)\mathrm{d}\sigma_x[f(x,y)-f(\xi,\tau)]\mathrm{d}\sigma_y E_\omega(y,\tau)+\\
& \int_{\partial J_1\times\partial J_2} E_\omega(x,\xi)\mathrm{d}\sigma_x[f(x,y)-f(\xi,\tau)]\mathrm{d}\sigma_y E_\omega(y,\tau)=\\
& \sum_{i=1}^{k_1}\sum_{j=1}^{k_2}\int_{\partial J_{1i}\times\partial J_{2j}} E_\omega(x,\xi)\mathrm{d}\sigma_x[f(x,y)-f(\xi,\tau)]\mathrm{d}\sigma_y E_\omega(y,\tau)+\\
& \sum_{i=1}^{k_1}\int_{\partial J_{1i}\times\partial J_2} E_\omega(x,\xi)\mathrm{d}\sigma_x[f(x,y)-f(\xi,\tau)]\mathrm{d}\sigma_y E_\omega(y,\tau)+\\
& \sum_{j=1}^{k_2}\int_{\partial J_1\times\partial J_{2j}} E_\omega(x,\xi)\mathrm{d}\sigma_x[f(x,y)-f(\xi,\tau)]\mathrm{d}\sigma_y E_\omega(y,\tau)+\\
& \int_{\partial J_1\times\partial J_2} E_\omega(x,\xi)\mathrm{d}\sigma_x[f(x,y)-f(\xi,\tau)]\mathrm{d}\sigma_y E_\omega(y,\tau)=\\
& I_6(\xi,\tau)+I_7(\xi,\tau)+I_8(\xi,\tau)+I_9(\xi,\tau).
\end{aligned}$$

首先讨论$I_6(\xi,\tau)$, 当$(x,y)\in\partial J_{1i}\times\partial J_{2j}$时, $(x,y)\neq(\xi,\tau)$, 由(3.16)式可知

$$\int_{\partial J_{1i}\times\partial J_{2j}}\mathrm{d}\sigma_x f(x,y)\mathrm{d}\sigma_y=\int_{\partial J_{2j}}\left[\int_{\partial J_{1i}}\mathrm{d}\sigma_x f(x,y)\right]\mathrm{d}\sigma_y=0.$$

因此由定理3.4得

$$\sum_{i=1}^{k_1}\sum_{j=1}^{k_2}\int_{\partial J_{1i}\times\partial J_{2j}} E_\omega(x,\xi)\mathrm{d}\sigma_x f(x,y)\mathrm{d}\sigma_y E_\omega(y,\tau)=0.$$

由推论3.1可知$\sum_{i=1}^{k_1}\sum_{j=1}^{k_2}\int_{\partial J_{1i}\times\partial J_{2j}} E_\omega(x,\xi)\mathrm{d}\sigma_x f(\xi,\tau)\mathrm{d}\sigma_y E_\omega(y,\tau)=0$. 则$I_6(\xi,\tau)=0$.

其次讨论$I_7(\xi,\tau)$, 由引理3.1得

$$I_7(\xi,\tau) \quad = \quad \sum_{i=1}^{k_1}\int_{\partial J_2}\left[\int_{\partial J_{1i}} E_\omega(x,\xi)\mathrm{d}\sigma_x[f(x,y)-f(\xi,\tau)]\right]\mathrm{d}\sigma_y E_\omega(y,\tau).$$

由引理2.4可知$\int_{\partial J_{1i}} E_\omega(x,\xi)\mathrm{d}\sigma_x f(x,y)=0$. 由Stokes公式

$$\int_{\partial J_{1i}} E_\omega(x,\xi)\mathrm{d}\sigma_x f(\xi,\tau)=\int_{J_{1i}} E_\omega(x,\xi)\mathcal{D}_{\omega_x}\cdot f(\xi,\tau)\mathrm{d}x+\int_{J_{1i}} E_\omega(x,\xi)\cdot\mathcal{D}_{\omega_x} f(\xi,\tau)\mathrm{d}x=0.$$

因此

$$I_7(\xi,\tau)=0.$$

同理可证得$I_8(\xi,\tau)=0$.

综上所述, 并利用推论3.1有

$$\begin{aligned}
&\left|f(\xi,\tau)-\int_{\partial I_1\times\partial I_2} E_\omega(x,\xi)\mathrm{d}\sigma_x f(x,y)\mathrm{d}\sigma_y E_\omega(y,\tau)\right|=\\
&\left|\int_{\partial J_1\times\partial J_2} E_\omega(x,\xi)\mathrm{d}\sigma_x[f(x,y)-f(\xi,\tau)]\mathrm{d}\sigma_y E_\omega(y,\tau)\right|\leq\\
&\left|\int_{\partial J_1\times\partial J_2} E_\omega(x,\xi)\mathrm{d}\sigma_x\mathrm{d}\sigma_y E_\omega(y,\tau)\right|\cdot\varepsilon=\varepsilon.
\end{aligned}$$

所以$f(\xi,\tau)=\int_{\partial I_1\times\partial I_2}E_\omega(x,\xi)\mathrm{d}\sigma_x f(x,y)\mathrm{d}\sigma_y E_\omega(y,\tau)$.

定理3.6(Morera定理) 设$\Omega,\partial\Omega$如上所述, $f\in F_\Omega^{(r)}$, $r\geq 2$, 则f是Ω上的加权双正则函数的充分必要条件是对任意闭区间$I_1\subset\Omega_1$, $I_2\subset\Omega_2$, 有
$$\int_{\partial I_1}\mathrm{d}\sigma_x f(x,y)=0,\quad \int_{\partial I_2}f(x,y)\mathrm{d}\sigma_y=0. \tag{3.18}$$
证 必要性. 由于f是Ω上的加权双正则函数, 由Stokes公式可知
$$\int_{\partial I_1}\mathrm{d}\sigma_x f(x,y)=0,\quad \int_{\partial I_2}f(x,y)\mathrm{d}\sigma_y=0.$$
充分性. 任取$(\xi,\tau)\in\Omega_1\times\Omega_2$, 则存在闭区间$I_1\subset\Omega_1$, $I_2\subset\Omega_2$, 使得$(\xi,\tau)\in\mathring{I}_1\times\mathring{I}_2$, 则由定理3.5知$f(\xi,\tau)=\int_{\partial I_1\times\partial I_2}E_\omega(x,\xi)\mathrm{d}\sigma_x f(x,y)\mathrm{d}\sigma_y E_\omega(y,\tau)$. 则由加权双正则函数的定义以及$(\xi,\tau)$的任意性可知, f是Ω上的加权双正则函数.

定理3.7(Painlevé定理) 设$\Omega,\partial\Omega$如上所述, 令$\widetilde{\Omega}=\widetilde{\Omega_1}\times\widetilde{\Omega_2}$, $\widetilde{\Omega_1}=\Omega_1\cap\{(x_1,x_2,\cdots,x_{m-1},a_m)\}$, 其中$(x_1,x_2,\cdots,x_{m-1})\in\mathbf{R}^{m-1}$, a_m是定值, $\widetilde{\Omega_2}=\Omega_2\cap\{(y_1,y_2,\cdots,y_{k-1},b_k)\}$, 其中$(y_1,y_2,\cdots,y_{k-1})\in\mathbf{R}^{k-1}$, b_k是定值, 若f在$\Omega\backslash\widetilde{\Omega}$上是加权双正则函数, 且$f\in F_\Omega^r$, $r\geq 2$, 则f在Ω上是加权双正则函数.

证 分别在Ω_1, Ω_2内任取闭区间I_1, I_2, 记$I=I_1\times I_2$.

(1) 若$I\cap\widetilde{\Omega}=\varnothing$, 则$I\subset\Omega\backslash\widetilde{\Omega}$. 由于$f$在$I$上是加权双正则函数, 所以
$$\int_{\partial I_1}\mathrm{d}\sigma_x f(x,y)=0,\quad \int_{\partial I_2}f(x,y)\mathrm{d}\sigma_y=0.$$
(2) 若$I\cap\widetilde{\Omega}\neq\varnothing$, 即$(I_1\times I_2)\cap(\widetilde{\Omega_1}\times\widetilde{\Omega_2})\neq\varnothing$, 令
$$I_{1\varepsilon}=\{(x_1,x_2,\cdots,x_{m-1},x_m)|(x_1,x_2,\cdots,x_{m-1},x_m)\in I_1,a_m-\varepsilon<x_m<a_m+\varepsilon\},$$
$$I_{2\varepsilon}=\{(y_1,y_2,\cdots,y_{k-1},y_k)|(y_1,y_2,\cdots,y_{k-1},y_k)\in I_2,b_k-\varepsilon<y_k<b_k+\varepsilon\}.$$
下面分三种情况进行讨论.

$1'$ $I_1\cap\widetilde{\Omega_1}=\varnothing$, $I_2\cap\widetilde{\Omega_2}\neq\varnothing$. 则$I_1\subset\Omega_1\backslash\widetilde{\Omega_1}$, $(I_2\backslash I_{2\varepsilon})\subset\Omega_2\backslash\widetilde{\Omega_2}$, 由于$f$在$I_1\times(I_2\backslash I_{2\varepsilon})$上是加权双正则函数, 因此
$$\int_{\partial I_1}\mathrm{d}\sigma_x f(x,y)=\int_{I_1}\mathcal{D}_{\omega_x}f(x,y)=0,$$
$$\int_{\partial I_2}f(x,y)\mathrm{d}\sigma_y=\lim_{\varepsilon\to 0}\int_{\partial(I_2\backslash I_{2\varepsilon})}f(x,y)\mathrm{d}\sigma_y=\lim_{\varepsilon\to 0}\int_{I_2\backslash I_{2\varepsilon}}f(x,y)\mathcal{D}_{\omega_y}=0.$$
$2'$ $I_1\cap\widetilde{\Omega_1}\neq\varnothing$, $I_2\cap\widetilde{\Omega_2}=\varnothing$. 则$(I_1\backslash I_{1\varepsilon})\subset\Omega_1\backslash\widetilde{\Omega_1}$, $I_2\subset\Omega_2\backslash\widetilde{\Omega_2}$, 由于$f$在$(I_1\backslash I_{1\varepsilon})\times I_2$上是加权双正则函数, 因此
$$\int_{\partial I_1}\mathrm{d}\sigma_x f(x,y)=\lim_{\varepsilon\to 0}\int_{\partial(I_1\backslash I_{1\varepsilon})}\mathrm{d}\sigma_x f(x,y)=\lim_{\varepsilon\to 0}\int_{I_1\backslash I_{1\varepsilon}}\mathcal{D}_{\omega_x}f(x,y)=0,$$
$$\int_{\partial I_2}f(x,y)\mathrm{d}\sigma_y=\int_{I_2}f(x,y)\mathcal{D}_{\omega_y}=0.$$
$3'$ $I_1\cap\widetilde{\Omega_1}\neq\varnothing$, $I_2\cap\widetilde{\Omega_2}\neq\varnothing$. 则$(I_1\backslash I_{1\varepsilon})\subset\Omega_1\backslash\widetilde{\Omega_1}$, $(I_2\backslash I_{2\varepsilon})\subset\Omega_2\backslash\widetilde{\Omega_2}$, 由于$f$在$(I_1\backslash I_{1\varepsilon})\times(I_2\backslash I_{2\varepsilon})$上是加权双正则函数, 因此
$$\int_{\partial I_1}\mathrm{d}\sigma_x f(x,y)=\lim_{\varepsilon\to 0}\int_{\partial(I_1\backslash I_{1\varepsilon})}\mathrm{d}\sigma_x f(x,y)=\lim_{\varepsilon\to 0}\int_{I_1\backslash I_{1\varepsilon}}\mathcal{D}_{\omega_x}f(x,y)=0,$$
$$\int_{\partial I_2}f(x,y)\mathrm{d}\sigma_y=\lim_{\varepsilon\to 0}\int_{\partial(I_2\backslash I_{2\varepsilon})}f(x,y)\mathrm{d}\sigma_y=\lim_{\varepsilon\to 0}\int_{I_2\backslash I_{2\varepsilon}}f(x,y)\mathcal{D}_{\omega_y}=0.$$
由上述的(1)和(2)和Morera定理可得f是Ω上的加权双正则函数.

参考文献:

[1] Clifford W K. Applications of Grassmann's extensive algebra[J]. American Journal of Mathematics, 1878, 1(4): 350-358.

[2] Brackx F, Delanghe R, Sommen F. Clifford Analysis[M]. London: Pitman, 1982.

[3] Gürlebeck K, Kähler U, Ryan J, et al. Clifford analysis over unbounded domains[J]. Advances in Applied Mathematics, 1997, 19(2): 216–239.

[4] Begehr H, Dai Daoqing. Spatial Riemann problem for analytic functions of two complex variables[J]. Zeitschrift für Analysis und ihre Anwendungen, 1999, 18(4): 827-837.

[5] Cerejeiras P, Kähler U, Ren Guangbin. Clifford analysis for finite reflection groups[J]. Complex Variables Elliptic Equations, 2006, 51(5-6): 487-495.

[6] 黄沙. Clifford分析中广义双正则函数的(线性)非线性边值问题[J]. 数学学报, 1997, 40(6): 114-121.

[7] Huang Sha, Qiao Yuying, Wen Guochun. Real and Complex Clifford Analysis[M]. New York: Springer, 2006.

[8] 乔玉英, 贾美枝. 实Clifford分析中超正则函数列和函数空间的性质[J]. 高校应用数学学报, 2006, 21(4): 477-483.

[9] 杜金元, 许娜, 张忠祥. Boundary behavior of Cauchy-type integrals in Clifford analysis[J]. Acta Mathematica Scientia, 2009, 29(1): 210-224.

[10] 任广斌. 非交换非结合的多复变[J]. 大学数学, 2017, 33(5): 1-7.

[11] 任广斌, 王谢平, 徐正华. Slice正则函数理论[J]. 中国科学: 数学, 2015, 45(11): 1779-1790.

[12] 王海燕, 乔玉英, 李小伶, 等. 实Clifford分析中双正则函数的一些性质[J]. 河北师范大学学报(自然科学版), 2008, 32(3): 281-286.

[13] 彭维玲, 王丽萍. 实Clifford分析中双正则函数的最大模原理[J]. 河北师范大学学报(自然科学版), 2009, 33(2): 170-175.

[14] Vanegas J, Vargas F. On weighted Dirac operators and their fundamental solutions for anisotropic media[J]. Advances in Applied Clifford Algebra, 2018, 28(2): 1-13.

[15] 罗利萍, 王丽萍, 邱芬. 加权正则函数的一些性质[J]. 高校应用数学学报, 2022, 37(1): 61-78.

[16] 罗利萍. 加权正则函数的有关性质[D]. 石家庄: 河北师范大学, 2022.

[17] Wang Liping, Luo Liping, Qiu Fen. The Taylor expansion of weighted monogenic functions[J]. Complex Variables Elliptic Equations, 2024, 69(3): 383-402.

Some properties of weighted biregular functions

WANG Long-you[1], XIAO Zhuo-feng[2], WANG Li-ping[1]

(1. School of Mathematical Science, Hebei Normal University, Shijiazhuang 050024, China;
2. Institute for Ethnic Groups Attached to Hebei Normal University, Shijiazhuang 050091, China)

Abstract: Weighted biregular function is a further development of weighted regular function, and weighted biregular function is an important class of functions in Clifford analysis. On the basis of the study of weighted regular function, combined with the characteristics of weighted biregular function, Cauchy integral formula for a point outside the sphere, mean value theorem, maximum modulus principle, Morera theorem, Painlevé theorem of weighted biregular function are discussed.

Keywords: Cauchy integral formula for a point outside the sphere; mean value theorem; maximum modulus principle; Morera theorem; Painlevé theorem

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