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多次正则函数在无界域上的Cauchy积分公式

李冰心， 王丽萍*， 王龙优

(河北师范大学 数学科学学院, 河北石家庄 050024)

摘　要: 多次正则函数是k-正则函数和次正则函数的进一步发展, 是一类重要的函数. 该文利用正则函数在无界域上处理的思路以及多次正则函数本身的特征, 研究了多次正则函数在无界域上的Borel-Pompeiu公式和Cauchy积分公式.

关键词: 实Clifford分析; 多次正则函数; Borel-Pompeiu公式; Cauchy积分公式

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§1　引　言

Clifford代数是复数、四元数、外代数的推广, 是一种可结合但不可交换的代数结构. Clifford分析主要研究定义在n维实或复向量空间上, 取值于Clifford代数上的函数. Clifford分析已经取得了很多研究成果[1-5]. 然而在实际应用过程中, 许多问题是在无界域上给出的. 文献[3, 6]引入了修正的Cauchy核, 使得在任何包含非空开集的无界域上研究Cauchy积分公式成为可能, 并得到了一系列结果. 多次正则函数是指满足迭代“sandwich”方程$D_x^{2k-1}f(x)D_x=0$或$D_xf(x)D_x^{2k-1}=0$的解(k是正整数), 是Clifford分析中一类重要的函数, 是k-正则函数和次正则函数的进一步的发展. 而k-正则函数和次正则函数是正则函数的进一步发展, 并且正则函数、k-正则函数和次正则函数都是多次正则函数, 反之未必成立. 比如, 函数$f(x)=2x^2$是多次正则函数, 但是它不是正则函数, 不是2-正则函数, 也不是次正则函数. 文献[7]给出了无界域上具有高阶核的Clifford分析. 文献[8]给出了次正则函数的Cauchy积分公式. 文献[9]给出了多次正则函数的Cauchy积分公式.

在上述工作的基础之上, 本文利用多次正则函数在有界域上的Borel-Pompeiu公式的证明思路[9]以及应用无界域上$(2k-1)$-正则函数的Cauchy-Pompeiu公式[7], 再结合多次正则函数本身的特征证明了多次正则函数在无界域上的Borel-Pompeiu公式. 最后, 得出了多次正则函数在无界域上的Cauchy积分公式. 本文的结构安排如下: §2给出了Clifford分析的一些基本知识, 与多次正则函数有关的积分算子的定义以及相关的引理. §3研究了多次正则函数在无界域上的Borel-Pompeiu公式和Cauchy积分公式.

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*通讯作者, E-mail: wlpxjj@163.com

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§2 预备知识

设$e_1, e_2, \cdots, e_n$是欧氏空间$\mathbf{R}^n$中的一组标准正交基, $\mathcal{A}_n(\mathbf{R})$为$2^n$维实Clifford代数空间, 它的基为$e_0, e_1, e_2, \cdots, e_n$; $e_1e_2, e_1e_3, \cdots, e_1e_n, \cdots, e_{n-1}e_n$; $\cdots$; $e_1e_2\cdots e_{n-1}e_n$. $\mathcal{A}_n(\mathbf{R})$中的基元素一般表示为$e_A = e_{\alpha_1}\cdots e_{\alpha_h}$, 其中$A = \{\alpha_1, \cdots, \alpha_h\} \subseteq \{1, 2, \cdots, n\}$, 且$1 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_h \leq n$. 当$A = \varnothing$时, $e_A = e_0$为其单位元. 且有

$$\begin{cases} e_i^2 = -1, i = 1, 2, \cdots, n, \\ e_ie_j = -e_je_i, 1 \leq i < j \leq n, \\ e_{\alpha_1}e_{\alpha_2}\cdots e_{\alpha_h} = e_{\alpha_1\alpha_2\cdots\alpha_h}, 1 \leq \alpha_1 < \cdots < \alpha_h \leq n. \end{cases}$$

$$\begin{cases} \overline{e_i} = -e_i, i = 1, 2, \cdots, n, \\ \overline{\lambda\mu} = \overline{\mu}\,\overline{\lambda}, \lambda \in \mathcal{A}_n(\mathbf{R}), \mu \in \mathcal{A}_n(\mathbf{R}). \end{cases}$$

任意元素$a \in \mathcal{A}_n(\mathbf{R})$可表示为

$$a = \sum_A a_Ae_A, a_A \in \mathbf{R}.$$

$\mathcal{A}_n(\mathbf{R})$中的范数为

$$|a| = \sqrt{(a, a)} = \left(\sum_A a_A^2\right)^{\frac{1}{2}}.$$

当$x \in \mathbf{R}^n$时, $|x|^2 = -x^2$.

设$\varOmega \subset \mathbf{R}^n$是一个有界区域, 其边界为$\varGamma$. 下面将考虑定义在$\varOmega$上, 取值于$\mathcal{A}_n(\mathbf{R})$空间的函数$f(x)$, 这样的函数可表示为

$$f(x) = \sum_A f_A(x)e_A,$$

其中$f_A(x)$为实值函数.

Dirac算子的定义为

$$D_xf = \sum_{i=1}^n e_i\frac{\partial f}{\partial x_i} = \sum_{i=1}^n\sum_A e_ie_A\frac{\partial f_A}{\partial x_i}, \qquad fD_x = \sum_{i=1}^n \frac{\partial f}{\partial x_i}e_i = \sum_{i=1}^n\sum_A e_Ae_i\frac{\partial f_A}{\partial x_i}.$$

D_x是$\mathbf{R}^n$中Laplace算子$\varDelta_n$的一个分解, 满足$D_x^2 = -\varDelta_n$. 对于$D_x^pfD_x^q = 0$, 其中p和q是正整数. 若p(或q)是偶数，则$D_x^pfD_x^q = (-1)^{\frac{p}{2}}\varDelta_n^{\frac{p}{2}}fD_x^q = fD_x^{p+q} = 0$或

$$D_x^pfD_x^q = (-1)^{\frac{q}{2}}D_x^pf\varDelta_n^{\frac{q}{2}} = D_x^{p+q}f = 0.$$

设$C^k(\varOmega)$表示直到k阶偏导数都在$\varOmega$上连续的函数全体. 若$f(x) \in C^1(\varOmega)$, 对任意的$x \in \varOmega$满足$D_xf(x) = 0(f(x)D_x = 0)$, 则称函数$f(x)$为左(右)正则函数, 通常简称左正则函数为正则函数. 若$f(x) \in C^2(\varOmega)$, 对任意的$x \in \varOmega$满足“sandwich”方程, 即$D_xf(x)D_x = 0$, 则称函数$f(x)$为次正则函数. 若$f(x) \in C^k(\varOmega)$, 对任意的$x \in \varOmega$满足$D_x^kf(x) = 0(f(x)D_x^k = 0)$, 则称函数$f(x)$为左(右)$k$-正则函数, 通常简称左$k$-正则函数为$k$-正则函数. 若$f(x) \in C^{2k}(\varOmega)$, 对任意的$x \in \varOmega$满足迭代“sandwich”方程, 即$D_x^{2k-1}f(x)D_x = 0$或$D_xf(x)D_x^{2k-1} = 0$, 则称函数$f(x)$为多次正则函

数.

由此可见, 正则函数、次正则函数和k-正则函数都是多次正则函数, 但反过来未必成立. 比如函数$f(x)=2x^2$是多次正则函数, 但它不是正则函数, 不是次正则函数, 也不是2-正则函数.

高阶核函数的定义为

$$H_i^*(x)=\frac{A_i}{\omega_n}\frac{\bar{x}^i}{|x|^n}, i<n,$$

$$A_i=\begin{cases}\dfrac{1}{2^{p-1}(p-1)!\prod\limits_{r=1}^{p}(2r-n)}, i=2p, i<n, i=1,2,\cdots,\\[2ex] \dfrac{1}{2^p p!\prod\limits_{r=1}^{p}(2r-n)}, i=2p+1,\ i<n,\ i=0,1,2,\cdots,\end{cases}$$

其中$x\in\mathbf{R}_0^n=\mathbf{R}^n\setminus\{0\}$, $\omega_n=\dfrac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}$ 为$\mathbf{R}^n$中单位球的表面积, 并且可以得到

$$D_xH_{i+1}^*(x)=H_{i+1}^*(x)D_x=H_i^*(x), x\in\mathbf{R}_0^n, 1\le i<n-1.$$

Laplace算子$\varDelta_n$的基本解为

$$H_2^*(x)=\frac{1}{(2-n)\omega_n}\frac{\bar{x}^2}{|x|^n}, x\in\mathbf{R}_0^n,$$

从而D_x的基本解为

$$H_1^*(x)=D_xH_2^*(x)=\frac{1}{\omega_n}\frac{\bar{x}}{|x|^n}, x\in\mathbf{R}_0^n,$$

满足$D_xH_1^*(x)=H_1^*(x)D_x=0, x\in\mathbf{R}_0^n$. 显然高阶核函数$H_k^*(x)$为$k$-正则函数.

带有正则核的左Cauchy型积分算子和左Teodorescu算子的定义分别为

$$\begin{aligned}(C_{\varGamma}^l[f])(x) &= \int_{\varGamma}H_1^*(y-x)n(y)f(y)\mathrm{d}S(y), x\notin\varGamma,\\ (T_{\varOmega}^l[f])(x) &= -\int_{\varOmega}H_1^*(y-x)f(y)\mathrm{d}V(y),\end{aligned}$$

其中$f:\varOmega\to\mathcal{A}_n(\mathbf{R})$.

当$f\in C^1(\varOmega\cup\varGamma)$时, 相应的Borel-Pompeiu公式为

$$f(x) = (C_{\varGamma}^l[f])(x)+(T_{\varOmega}^l[\partial_y f])(x), x\in\varOmega.$$

进而如果f为左正则函数, 则可得到左正则函数的Cauchy积分公式

$$f(x) = \int_{\varGamma}H_1^*(y-x)n(y)f(y)\mathrm{d}S(y), x\in\varOmega.$$

类似地, 带有正则核的右Cauchy型积分算子和右Teodorescu算子的定义分别为

$$\begin{aligned}(C_{\varGamma}^r[f])(x) &= \int_{\varGamma}f(y)n(y)H_1^*(y-x)\mathrm{d}S(y), x\notin\varGamma,\\ (T_{\varOmega}^r[f])(x) &= -\int_{\varOmega}f(y)H_1^*(y-x)\mathrm{d}V(y),\end{aligned}$$

其中$f:\varOmega\to\mathcal{A}_n(\mathbf{R})$.

当$f\in C^1(\Omega\cup\Gamma)$时, 相应的Borel-Pompeiu公式为

$$f(x) = (C_\Gamma^r[f])(x)+(T_\Omega^r[f\partial_y])(x), x\in\Omega.$$

进而如果f为右正则函数, 则可得到如下右正则函数的Cauchy积分公式

$$f(x) = \int_\Gamma f(y)n(y)H_1^*(y-x)\mathrm{d}S(y), x\in\Omega.$$

带有k-正则核的左Cauchy型积分算子和左Teodorescu算子的定义分别为

$$\begin{aligned}(C_\Gamma^{l,k}[f])(x) &= \sum_{i=0}^{k-1}(-1)^i\int_\Gamma H_{i+1}^*(y-x)n(y)\partial_y^i f(y)\mathrm{d}S(y), x\notin\Gamma,\\ (T_\Omega^{l,k}[f])(x) &= (-1)^k\int_\Omega H_k^*(y-x)f(y)\mathrm{d}V(y),\end{aligned}$$

其中$f:\Omega\to\mathcal{A}_n(\mathbf{R})$.

当$f\in C^k(\Omega\cup\Gamma)$时, 相应的Borel-Pompeiu公式为

$$f(x) = (C_\Gamma^{l,k}[f])(x)+(T_\Omega^{l,k}[\partial_y^k f])(x), x\in\Omega.$$

进而如果f为左k-正则函数, 则可得到如下左k-正则函数的Cauchy积分公式

$$f(x) = \sum_{i=0}^{k-1}(-1)^i\int_\Gamma H_{i+1}^*(y-x)n(y)\partial_y^i f(y)\mathrm{d}S(y), x\in\Omega.$$

类似地, 带有k-正则核的右Cauchy型积分算子和右Teodorescu算子的定义分别为

$$\begin{aligned}(C_\Gamma^{r,k}[f])(x) &= \sum_{i=0}^{k-1}(-1)^i\int_\Gamma \left(f(y)\partial_y^i\right)n(y)H_{i+1}^*(y-x)\mathrm{d}S(y), x\notin\Gamma,\\ (T_\Omega^{r,k}[f])(x) &= (-1)^k\int_\Omega f(y)H_k^*(y-x)\mathrm{d}V(y),\end{aligned}$$

其中$f:\Omega\to\mathcal{A}_n(\mathbf{R})$.

当$f\in C^k(\Omega\cup\Gamma)$时, 相应的高阶Borel-Pompeiu公式为

$$f(x) = (C_\Gamma^{r,k}[f])(x)+(T_\Omega^{r,k}[f\partial_y^k])(x), x\in\Omega.$$

进而如果f为右k-正则函数, 则可得到如下右k-正则函数的Cauchy积分公式

$$f(x) = \sum_{i=0}^{k-1}(-1)^i\int_\Gamma \left(f(y)\partial_y^i\right)n(y)H_{i+1}^*(y-x)\mathrm{d}S(y), x\in\Omega.$$

下面引入算子$\Psi(f)=\sum\limits_{i=1}^n e_i f e_i$.

与次正则函数有关的Cauchy型积分算子和Teodorescu算子的定义分别为

$$\begin{aligned}(C_\Gamma^{infra}[f])(x) &= \frac{1}{2}\left\{\int_\Gamma H_1^*(y-x)n(y)f(y)(y-x)\mathrm{d}S(y)+\right.\\ &\quad\left.\Psi\left(\int_\Gamma H_2^*(y-x)n(y)f(y)\mathrm{d}S(y)\right)\right\}, x\notin\Gamma,\\ (T_\Omega^{infra}[f])(x) &= -\frac{1}{2}\left\{\int_\Omega H_1^*(y-x)f(y)(y-x)\mathrm{d}V(y)+\right.\\ &\quad\left.\Psi\left(\int_\Omega H_2^*(y-x)f(y)\mathrm{d}V(y)\right)\right\},\end{aligned}$$

其中$f:\Omega\to\mathcal{A}_n(\mathbf{R})$.

当$f\in C^2(\Omega\cup\Gamma)$时, 相应的Borel-Pompeiu公式为

$$f(x) = (C_\Gamma^r[f])(x)+(C_\Gamma^{infra}[f\partial_y])(x)+(T_\Omega^{infra}[\partial_y f\partial_y])(x), x\in\Omega.$$

进而如果f为次正则函数, 则可得到如下次正则函数的Cauchy积分公式

$$f(x) = (C_\Gamma^r[f])(x)+(C_\Gamma^{infra}[f\partial_y])(x), x\in\Omega.$$

与多次正则函数有关的Cauchy型积分算子和Teodorescu算子的定义分别为

$$\begin{aligned}(C_\Gamma^{ifp,k}[f])(x) &= \sum_{i=0}^{2k-2}(-1)^i\int_\Gamma H_{i+1}^*(y-x)n(y)\partial_y^i((f(y)\partial_y)(y-x))\mathrm{d}S(y)+\\ &\quad \Psi\left(\int_\Gamma H_{2k}^*(y-x)n(y)\partial_y^{2k-2}f(y)\partial_y\mathrm{d}S(y)\right), x\notin\Gamma,\\ (T_\Omega^{ifp,k}[f])(x) &= -\frac{1}{2k}\left\{\int_\Omega H_{2k-1}^*(y-x)f(y)(y-x)\mathrm{d}V(y)+\right.\\ &\quad \left.\Psi\left(\int_\Omega H_{2k}^*(y-x)f(y)\mathrm{d}V(y)\right)\right\},\end{aligned}$$

其中$f:\Omega\to\mathcal{A}_n(\mathbf{R})$.

当$f\in C^{2k}(\Omega\cup\Gamma)$, 相应的Borel-Pompeiu公式为

$$\begin{aligned}f(x) &= \frac{1}{2k}\Big[2(C_\Gamma^{r,2k-1}[f])(x)+(2k-2)(C_\Gamma^{l,2k-1}[f])(x)+\\ &\quad (C_\Gamma^{ifp,k}[f])(x)+2k(T_\Omega^{ifp,k}[\partial_y^{2k-1}f\partial_y])(x)\Big].\end{aligned}$$

进而如果f为多次正则函数, 则可得到如下多次正则函数的Cauchy积分公式

$$f(x) = \frac{1}{2k}\left[2(C_\Gamma^{r,2k-1}[f])(x)+(2k-2)(C_\Gamma^{l,2k-1}[f])(x)+(C_\Gamma^{ifp,k}[f])(x)\right], x\in\Omega.$$

§3所研究的$\mathbf{R}^n$中的无界域U, 它的边界满足Lipschitz连续的条件, 且U的余集中包含非空开集, $0\notin\partial U$, 以及对任意的$t\in\partial U$, 0是不在t处的切平面上的.

左$(2k-1)$-正则函数和右$(2k-1)$-正则函数在无界域U上的Cauchy型积分算子的定义分别为

$$(C_{\partial U}^{r,2k-1}[f])(x)=\sum_{i=0}^{2k-2}(-1)^i\int_{\partial U}(f(y)D_y^i)n(y)G_{i+1}^*(y,x)\mathrm{d}S(y),$$
$$(C_{\partial U}^{l,2k-1}[f])(x)=\sum_{i=0}^{2k-2}(-1)^i\int_{\partial U}G_{i+1}^*(y,x)n(y)D_y^if(y)\mathrm{d}S(y),$$

其中$f:U\to\mathcal{A}_n(\mathbf{R})$, $G_{i+1}^*(y,x)=H_{i+1}^*(y-x)-H_{i+1}^*(y-x_0)$, $x_0\in\mathbf{R}^n\backslash\overline{U}$为固定的点.

多次正则函数在无界域U上的积分算子$(C_{\partial U}^{ifp,k}[f])(x)$和Teodorescu算子的定义分别为

$$\begin{aligned}(C_{\partial U}^{ifp,k}[f])(x) &= \sum_{i=0}^{2k-2}(-1)^i\int_{\partial U}G_{i+1}^*(y,x)n(y)D_y^i((f(y)D_y)(y-x))\mathrm{d}S(y)+\\ &\quad \Psi\left(\int_{\partial U}G_{2k}^*(y,x)n(y)D_y^{2k-2}f(y)D_y\mathrm{d}S(y)\right),\\ (T_U^{ifp,k}[D_y^{2k-1}fD_y])(x) &= -\frac{1}{2k}\left\{\int_U G_{2k-1}^*(y,x)(D_y^{2k-1}f(y)D_y)(y-x)\mathrm{d}V(y)+\right.\\ &\quad \left.\Psi\left(\int_U G_{2k}^*(y,x)(D_y^{2k-1}f(y)D_y)\mathrm{d}V(y)\right)\right\},\end{aligned}$$

其中f, $G^*_{i+1}(y,x)$如上所述.

引理2.1(Stokes公式)[4] 设Ω, Γ如上所述, u, $v\in C^1(\Omega\cup\Gamma)$, 则

$$\int_\Gamma u(y)n(y)v(y)\mathrm{d}S(y) = \int_\Omega [(u(y)D_y)v(y)+u(y)(D_yv(y))]\,\mathrm{d}V(y),$$

其中$n(y)$是Γ上点y处的单位外法向量, $\mathrm{d}S$和$\mathrm{d}V$分别表示面积微元和体积微元.

引理2.2[9] 以下等式成立.

(1) $D_x\Psi(f(x))=-\Psi(D_xf(x))-2f(x)D_x$, $(\Psi(f(x)))D_x=-\Psi(f(x)D_x)-2D_xf(x)$;

(2) $D_x(f(x)x)=(D_xf(x))x+\Psi(f(x))$;

(3) 如果g是一个向量值函数, 则$g(x)\Psi(f(x))=-\Psi(g(x)f(x))-2f(x)g(x)$;

(4) $D_x^2(f(x)x)=(D_x^2f(x))x-2f(x)D_x$.

引理2.3[9] 设$k\geq 1$, 则

$$D_x^{2k-1}(f(x)x) = (D_x^{2k-1}f(x))x+\Psi(D_x^{2k-2}f(x))-A_k(x),$$

其中

$$A_k(x)=\begin{cases}0, & k=1,\\ (2k-2)D_xf(x)D_x^{2k-3}, & k>1.\end{cases}$$

引理2.4[10] 设$k\geq 1$, 则

$$D_x^{2k}(f(x)x)=(D_x^{2k}f(x))x-2kD_x^{2k-2}f(x)D_x.$$

引理2.5(高阶Cauchy积分公式)[7] 设U, ∂U如上所述, 函数f为U上的左$(2k-1)$-正则函数, 且满足$|D_y^if(y)|\leq C|y|^{-i+s}$, $s\in(0,1)$, $i=0,1,\cdots,2k-2$, C为正实常数, 则有

$$\sum_{i=0}^{2k-2}(-1)^i\int_{\partial U}G^*_{i+1}(y,x)\mathrm{d}\sigma_yD_y^if(y)=\begin{cases}f(x), & x\in U,\\ \\ 0, & x\in\mathbf{R}^n\backslash\overline{U},\end{cases}$$

或

$$\sum_{i=0}^{2k-2}(-1)^i\int_{\partial U}(f(y)D_y^i)\mathrm{d}\sigma_yG^*_{i+1}(y,x)=\begin{cases}f(x), & x\in U,\\ \\ 0, & x\in\mathbf{R}^n\backslash\overline{U},\end{cases}$$

其中$G^*_{i+1}(y,x)$如上所述.

引理2.6(Cauchy-Pompeiu公式)[7] 设U, ∂U如上所述, $f\in C^{2k-1}(U\bigcup\partial U)$, $1\leq k<\dfrac{n}{2}$, 且$|D_y^if(y)|\leq C|y|^{-i+s}$, $s\in(0,1)$, $i=0,1,\cdots,2k-2$, C为正实常数, 对任意的$x\in U$有

$$f(x)=\sum_{i=0}^{2k-2}(-1)^i\int_{\partial U}G^*_{i+1}(y,x)\mathrm{d}\sigma_yD_y^if(y)+(-1)^{2k-1}\int_UG^*_{2k-1}(y,x)(D_y^{2k-1}f(y))\mathrm{d}y^n,$$

或

$$f(x)=\sum_{i=0}^{2k-2}(-1)^i\int_{\partial U}f(y)D_y^i\mathrm{d}\sigma_yG^*_{i+1}(y,x)+(-1)^{2k-1}\int_U(f(y)D_y^{2k-1})G^*_{2k-1}(y,x)\mathrm{d}y^n,$$

其中$G^*_{i+1}(y,x)$如上所述.

§3 多次正则函数在无界域上的Borel-Pompeiu公式和Cauchy积分公式

定理3.1(Borel-Pompeiu公式) 设$U\subset\mathbf{R}^n$是一个Jordan域, 具有足够光滑的∂U, 且$f\in C^{2k}(U\bigcup\partial U)$, $|D_y^if(y)|\le C|y|^{-i+s}$, $s\in(0,1)$, $i=0,1,2,\cdots 2k-2$, C为正实常数, 则对于$x\in U$有

$$\begin{aligned}f(x) &= \frac{1}{2k}\Big[2(C_{\partial U}^{r,2k-1}[f])(x)+(2k-2)(C_{\partial U}^{l,2k-1}[f])(x)+\\ &\quad (C_{\partial U}^{ifp,k}[f])(x)+2k(T_U^{ifp,k}[D_y^{2k-1}fD_y])(x)\Big].\end{aligned}\tag{3.1}$$

其中$G_{i+1}^*(y,x)=H_{i+1}^*(y-x)-H_{i+1}^*(y-x_0)$, $x_0\in\mathbf{R}^n\backslash\overline{U}$为固定的点.

证 由引理2.3可以推导出

$$\begin{aligned}D_y^{2k-1}((f(y)D_y)(y-x)) &= (D_y^{2k-1}f(y)D_y)(y-x)+\Psi\left(D_y^{2k-2}f(y)D_y\right)-\\ &\quad (2k-2)D_y(f(y)D_y)D_y^{2k-3},\end{aligned}$$

因此两边同乘向量核$G_{2k-1}^*(y,x)$得

$$\begin{aligned}G_{2k-1}^*(y,x)D_y^{2k-1}((f(y)D_y)(y-x)) &= G_{2k-1}^*(y,x)(D_y^{2k-1}f(y)D_y)(y-x)+\\ &\quad G_{2k-1}^*(y,x)\Psi\left(D_y^{2k-2}f(y)D_y\right)-\\ &\quad (2k-2)G_{2k-1}^*(y,x)D_yf(y)D_y^{2k-2}.\end{aligned}\tag{3.2}$$

利用引理2.2的(3), 其中$g=G_{2k-1}^*(y,x)$, 可以得到

$$\begin{aligned}G_{2k-1}^*(y,x)\Psi\left(D_y^{2k-2}f(y)D_y\right) &= -\Psi\left(G_{2k-1}^*(y,x)D_y^{2k-2}f(y)D_y\right)-\\ &\quad 2(D_y^{2k-2}f(y)D_y)G_{2k-1}^*(y,x).\end{aligned}\tag{3.3}$$

由(3.2)和(3.3)得

$$\begin{aligned}G_{2k-1}^*(y,x)D_y^{2k-1}((f(y)D_y)(y-x)) &= G_{2k-1}^*(y,x)(D_y^{2k-1}f(y)D_y)(y-x)-\\ &\quad \Psi\left(G_{2k-1}^*(y,x)D_y^{2k-2}f(y)D_y\right)-\\ &\quad 2(D_y^{2k-2}f(y)D_y)G_{2k-1}^*(y,x)-\\ &\quad (2k-2)G_{2k-1}^*(y,x)D_yf(y)D_y^{2k-2}.\end{aligned}\tag{3.4}$$

接下来, 对于$x\in U$, 构造以0为心, r为半径的球$E(0,r)$. 当r充分大时, 有$x\in U\bigcap E(0,r)$. 记$U_r=U\bigcap E(0,r)$. 取$\varepsilon>0$, 使得$\overline{E}(x,\varepsilon)=\{y\in\mathbf{R}^n,\ |y-x|\le\varepsilon\}\subset U_r$. 令$U_{r,\varepsilon}=U_r\backslash\overline{E}(x,\varepsilon)$, 记

$$\begin{aligned}I_1(x) &= \int_{U_{r,\varepsilon}}G_{2k-1}^*(y,x)D_y^{2k-1}((f(y)D_y)(y-x))\mathrm{d}V(y),\\ T_1(x) &= \int_{U_{r,\varepsilon}}G_{2k-1}^*(y,x)(D_y^{2k-1}f(y)D_y)(y-x)\mathrm{d}V(y),\\ I_2(x) &= \int_{U_{r,\varepsilon}}\Psi\left(G_{2k-1}^*(y,x)D_y^{2k-2}f(y)D_y\right)\mathrm{d}V(y),\\ I_3(x) &= 2\int_{U_{r,\varepsilon}}(D_y^{2k-2}f(y)D_y)G_{2k-1}^*(y,x)\mathrm{d}V(y),\\ I_4(x) &= (2k-2)\int_{U_{r,\varepsilon}}G_{2k-1}^*(y,x)D_y^{2k-1}f(y)\mathrm{d}V(y).\end{aligned}$$

接下来对(3.4)两边进行积分得

$$I_1(x)=T_1(x)-I_2(x)-I_3(x)-I_4(x).\tag{3.5}$$

根据引理2.1得

$$\int_{U_{r,\varepsilon}} G^*_{2k-1}(y,x)D_y^{2k-2}f(y)D_y\mathrm{d}V(y) = \int_{\partial U_{r,\varepsilon}} G^*_{2k}(y,x)n(y)D_y^{2k-2}f(y)D_y\mathrm{d}S(y) - \int_{U_{r,\varepsilon}} G^*_{2k}(y,x)D_y^{2k-1}f(y)D_y\mathrm{d}V(y),$$

两边同时作用Ψ, 得

$$I_2(x)=\Psi\left(\int_{\partial U_{r,\varepsilon}} G^*_{2k}(y,x)n(y)D_y^{2k-2}f(y)D_y\mathrm{d}S(y)\right)-\Psi\left(\int_{U_{r,\varepsilon}} G^*_{2k}(y,x)D_y^{2k-1}f(y)D_y\mathrm{d}V(y)\right). \tag{3.6}$$

另一方面, 对于下面的式子分别应用引理2.1可得

$$-\int_{\partial U_{r,\varepsilon}} G^*_2(y,x)n(y)D_y((f(y)D_y)(y-x))\mathrm{d}S(y) = -\int_{U_{r,\varepsilon}} G^*_1(y,x)D_y((f(y)D_y)(y-x))\mathrm{d}V(y)-\int_{U_{r,\varepsilon}} G^*_2(y,x)D_y^2((f(y)D_y)(y-x))\mathrm{d}V(y),$$

$$\int_{\partial U_{r,\varepsilon}} G^*_3(y,x)n(y)D_y^2((f(y)D_y)(y-x))\mathrm{d}S(y) = \int_{U_{r,\varepsilon}} G^*_2(y,x)D_y^2((f(y)D_y)(y-x))\mathrm{d}V(y)+\int_{U_{r,\varepsilon}} G^*_3(y,x)D_y^3((f(y)D_y)(y-x))\mathrm{d}V(y),$$

$$\cdots$$

$$\int_{\partial U_{r,\varepsilon}} G^*_{2k-1}(y,x)n(y)D_y^{2k-2}((f(y)D_y)(y-x))\mathrm{d}S(y) = \int_{U_{r,\varepsilon}} G^*_{2k-2}(y,x)D_y^{2k-2}((f(y)D_y)(y-x))\mathrm{d}V(y)+\int_{U_{r,\varepsilon}} G^*_{2k-1}(y,x)D_y^{2k-1}((f(y)D_y)(y-x))\mathrm{d}V(y).$$

将上述各式相加得

$$\sum_{i=1}^{2k-2}(-1)^i\int_{\partial U_{r,\varepsilon}} G^*_{i+1}(y,x)n(y)D_y^i((f(y)D_y)(y-x))\mathrm{d}S(y) = -\int_{U_{r,\varepsilon}} G^*_1(y,x)D_y((f(y)D_y)(y-x))\mathrm{d}V(y)+\int_{U_{r,\varepsilon}} G^*_{2k-1}(y,x)D_y^{2k-1}((f(y)D_y)(y-x))\mathrm{d}V(y).$$

进而可得

$$\begin{aligned} I_1(x) &= \int_{U_{r,\varepsilon}} G^*_{2k-1}(y,x)D_y^{2k-1}((f(y)D_y)(y-x))\mathrm{d}V(y) = \\ &\int_{U_{r,\varepsilon}} G^*_1(y,x)\left[D_y((f(y)D_y)(y-x))\right]\mathrm{d}V(y)+ \\ &\sum_{i=1}^{2k-2}(-1)^i\int_{\partial U_{r,\varepsilon}} G^*_{i+1}(y,x)n(y)D_y^i((f(y)D_y)(y-x))\mathrm{d}S(y). \end{aligned} \tag{3.7}$$

又由引理2.1得

$$\begin{aligned} &\int_{U_{r,\varepsilon}} G^*_1(y,x)\left[D_y((f(y)D_y)(y-x))\right]\mathrm{d}V(y) = \\ &\int_{U_{r,\varepsilon}} G^*_1(y,x)\left[D_y((f(y)D_y)(y-x))\right]+\left[G^*_1(y,x)D_y\right](f(y)D_y)(y-x)\mathrm{d}V(y) = \\ &\int_{\partial U_{r,\varepsilon}} G^*_1(y,x)n(y)(f(y)D_y)(y-x)\mathrm{d}S(y). \end{aligned}$$

因此

$$I_1(x) = \sum_{i=0}^{2k-2}(-1)^i\int_{\partial U_{r,\varepsilon}} G^*_{i+1}(y,x)n(y)D^i_y((f(y)D_y)(y-x))\mathrm{d}S(y). \tag{3.8}$$

将(3.6)和(3.8)代入(3.5)得

$$\begin{aligned}&\sum_{i=0}^{2k-2}(-1)^i\int_{\partial U_{r,\varepsilon}} G^*_{i+1}(y,x)n(y)D^i_y((f(y)D_y)(y-x))\mathrm{d}S(y)=\\&T_1(x)-\varPsi\left(\int_{\partial U_{r,\varepsilon}} G^*_{2k}(y,x)D_y^{2k-2}f(y)D_y\mathrm{d}S(y)\right)+\\&\varPsi\left(\int_{U_{r,\varepsilon}} G^*_{2k}(y,x)D_y^{2k-1}f(y)D_y\mathrm{d}V(y)\right)-I_3(x)-I_4(x).\end{aligned} \tag{3.9}$$

让(3.9)中的$\varepsilon\to 0$得

$$\begin{aligned}&\sum_{i=0}^{2k-2}(-1)^i\int_{\partial U_r} G^*_{i+1}(y,x)n(y)D^i_y((f(y)D_y)(y-x))\mathrm{d}S(y)=\\&\int_{U_r} G^*_{2k-1}(y,x)(D_y^{2k-1}f(y)D_y)(y-x)\mathrm{d}V(y)-\\&\varPsi\left(\int_{\partial U_r} G^*_{2k}(y,x)D_y^{2k-2}f(y)D_y\mathrm{d}S(y)\right)+\\&\varPsi\left(\int_{U_r} G^*_{2k}(y,x)D_y^{2k-1}f(y)D_y\mathrm{d}V(y)\right)-\\&2\int_{U_r}(D_y^{2k-2}f(y)D_y)G^*_{2k-1}(y,x)\mathrm{d}V(y)-\\&(2k-2)\int_{U_r} G^*_{2k-1}(y,x)D_y^{2k-1}f(y)\mathrm{d}V(y).\end{aligned} \tag{3.10}$$

让(3.10)中的$r\to\infty$得

$$\begin{aligned}&\sum_{i=0}^{2k-2}(-1)^i\int_{\partial U} G^*_{i+1}(y,x)n(y)D^i_y((f(y)D_y)(y-x))\mathrm{d}S(y)=\\&\int_{U} G^*_{2k-1}(y,x)(D_y^{2k-1}f(y)D_y)(y-x)\mathrm{d}V(y)-\\&\varPsi\left(\int_{\partial U} G^*_{2k}(y,x)D_y^{2k-2}f(y)D_y\mathrm{d}S(y)\right)+\\&\varPsi\left(\int_{U} G^*_{2k}(y,x)D_y^{2k-1}f(y)D_y\mathrm{d}V(y)\right)-\\&\qquad 2\int_{U}(D_y^{2k-2}f(y)D_y)G^*_{2k-1}(y,x)\mathrm{d}V(y)-\\&\qquad (2k-2)\int_{U} G^*_{2k-1}(y,x)D_y^{2k-1}f(y)\mathrm{d}V(y).\end{aligned} \tag{3.11}$$

因为$2k-2$是偶数, 所以

$$D_y^{2k-2}f(y)D_y = f(y)D_y^{2k-1}.$$

进而在(3.11)的最后两个积分里应用引理2.6可得

$$
\begin{aligned}
&-2\int_U (D_y^{2k-2}f(y)D_y)G_{2k-1}^*(y,x)\mathrm{d}V(y)-(2k-2)\int_U G_{2k-1}^*(y,x)D_y^{2k-1}f(y)\mathrm{d}V(y)=\\
&-2\int_U (f(y)D_y^{2k-1})G_{2k-1}^*(y,x)\mathrm{d}V(y)-(2k-2)\int_U G_{2k-1}^*(y,x)D_y^{2k-1}f(y)\mathrm{d}V(y)=\\
&-2\left[(C_{\partial U}^{r,2k-1}[f])(x)-f(x)\right]-(2k-2)\left[(C_{\partial U}^{l,2k-1}[f])(x)-f(x)\right]=\\
&-2(C_{\partial U}^{r,2k-1}[f])(x)-(2k-2)(C_{\partial U}^{l,2k-1}[f])(x)+2kf(x).
\end{aligned}
$$

所以(3.11)可变为

$$
\begin{aligned}
&\sum_{i=0}^{2k-2}(-1)^i\int_{\partial U}G_{i+1}^*(y,x)n(y)D_y^i((f(y)D_y)(y-x))\mathrm{d}S(y)=\\
&\int_U G_{2k-1}^*(y,x)(D_y^{2k-1}f(y)D_y)(y-x)\mathrm{d}V(y)-\\
&\Psi\left(\int_{\partial U}G_{2k}^*(y,x)D_y^{2k-2}f(y)D_y\mathrm{d}S(y)\right)+\\
&\Psi\left(\int_U G_{2k}^*(y,x)D_y^{2k-1}f(y)D_y\mathrm{d}V(y)\right)-\\
&2(C_{\partial U}^{r,2k-1}[f])(x)-(2k-2)(C_{\partial U}^{l,2k-1}[f])(x)+2kf(x).
\end{aligned}
$$

移项可得

$$
\begin{aligned}
f(x) \;=\; &\frac{1}{2k}\Big[2(C_{\partial U}^{r,2k-1}[f])(x)+(2k-2)(C_{\partial U}^{l,2k-1}[f])(x)+\\
&(C_{\partial U}^{ifp,k}[f])(x)+2k(T_U^{ifp,k}[D_y^{2k-1}fD_y])(x)\Big].
\end{aligned}
$$

定理3.2(Cauchy积分公式) 设$U\subset\mathbf{R}^n$是一个Jordan域, 具有足够光滑的∂U, 且$f\in C^{2k}(U\bigcup\partial U)$. 若$f$为$U$上的多次正则函数, 并且$|D_y^i f(y)|\le C|y|^{-i+s}$, $s\in(0,1)$, $i=0,1,2,\cdots 2k-2$, C为正实常数, 则对于$x\in U$有

$$
f(x)=\frac{1}{2k}\left[2(C_{\partial U}^{r,2k-1}[f])(x)+(2k-2)(C_{\partial U}^{l,2k-1}[f])(x)+(C_{\partial U}^{ifp,k}[f])(x)\right],
$$

其中$G_{i+1}^*(y,x)$如定理3.1所述.

证 由定理3.1得

$$
\begin{aligned}
f(x) \;=\; &\frac{1}{2k}\Big[2(C_{\partial U}^{r,2k-1}[f])(x)+(2k-2)(C_{\partial U}^{l,2k-1}[f])(x)+\\
&(C_{\partial U}^{ifp,k}[f])(x)+2k(T_U^{ifp,k}[D_y^{2k-1}fD_y])(x)\Big].
\end{aligned}
$$

因为f为U上的多次正则函数, 所以

$$
(T_U^{ifp,k}[D_y^{2k-1}fD_y])(x)=0.
$$

因此

$$
f(x)=\frac{1}{2k}\left[2(C_{\partial U}^{r,2k-1}[f])(x)+(2k-2)(C_{\partial U}^{l,2k-1}[f])(x)+(C_{\partial U}^{ifp,k}[f])(x)\right].
$$

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Cauchy integral formula for infrapolymonogenic functions over unbounded domains

LI Bing-xin, WANG Li-ping, WANG Long-you
(School of Mathematical Science, Hebei Normal University, Shijiazhuang 050024, China)

Abstract: Infrapolymonogenic function is a further development of k-monogenic function and inframonogenic function, which is an important kind of function. Based on the idea of treating monogenic function on unbounded domain and the features of infrapolymonogenic function itself, this paper studies the Borel-Pompeiu formula and Cauchy integral formula of inframonogenic function over unbounded domains.

Keywords: real Clifford analysis; infrapolymonogenic functions; Borel-Pompeiu formula; Cauchy integral formula

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