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# 一类带高振荡Bessel核函数的广义积分算法研究

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**摘 要**: 结合Hankel函数的渐近性和Gauss-Laguerre求积法, 文中计算了一类带有对数函数的具有一般振荡子的高振荡无穷积分. 首先通过改变积分路径将积分转化为复平面上的两个线积分, 然后利用Gauss-Laguerre求积公式得到高精度近似值, 最后给出了相应的误差阶, 并用数值例子验证结论的可行性.

**关键词**: Hankel变换; 高振荡; Gauss-Laguerre求积法; 积分路径

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## §1 引 言

很多前沿科学领域都存在高频问题, 此类问题具有深刻的理论价值和重要的应用背景, 如带Bessel核函数的高振荡积分经常应用于天文学、电磁声学、散射问题、物理光学、理论力学、地震学、图像处理和应用数学等领域[1]. 本文计算一类具有一般振荡子的高振荡广义积分

$$I(f)=\int_{\tau}^{\infty}x^{\alpha}\ln(x)f(x)J_m(\omega g(x))\mathrm{d}x, \tag{1}$$

其中$f$和$g$在包含区间$[\tau,+\infty]$的一个无限大单连通复域$D$上是解析的, $J_m(\omega g(x))$是第一类$m$阶Bessel函数, 频率参数$\omega\gg 1,\alpha<0,\tau>0$. 这些含高振荡核函数的无穷积分不能解析计算, 只能使用数值方法. 然而当被积函数变得高振荡时, 常见的数值方法无法得到高精度的数值结果.

关于广义高振荡积分的研究很多. Hascelik[2]通过渐近Filon-type方法计算无穷振荡积分. 基于文[2]的思想, 文[3-4]提出了一种有效的数值方法来逼近振荡的积分$\int_{\tau}^{+\infty}f(x)J_m(\omega x)\,\mathrm{d}x$. 文[5-6]系统介绍了高振荡广义积分$\int_{\tau}^{+\infty}x^{\alpha}f(x)K(\omega x)\,\mathrm{d}x$的相关研究, 其中$K(\omega x)$表示特殊的

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振荡核函数. Kang和Wang用两种不同的方法即改进的Filon型方法[7]和改进的最速下降法[1]研究了高振荡广义积分$\int_\tau^{+\infty} x^\alpha f(x) J_m(\omega g(x))\,\mathrm{d}x$的高精度计算; 改进的Filon型方法可以通过在临界点(包括零点、端点和驻点)处加导数或增加插值节点的数量来达到更高的精度. 但对于复杂的函数$f(x)$, 其导数有时难以求出, 而且仅增加插值点的数量并不能改变误差阶, 于是文[1]在文[7]的基础上将原积分路径转换为复平面的曲线积分方法来计算具有一般振子的无穷积分, 通过增加节点, 可以大大改善得到的误差阶.

然而有些势问题的域具有尖锐边缘和角时, 需求解具有对数函数的高振荡积分[8-9], 而现有的方法对含对数函数的广义积分(1)的计算精度或计算效率不高, 因此本文主要研究这类积分的有效计算. 本文首先通过修正的最速下降法给出了积分(1)的高精度近似计算公式, 然后通过渐近性分析得到算法的误差阶, 最后用数值实例验证了结论的有效性.

## §2 修正的最速下降法

由文献[10, p358]可知Hankel函数和Bessel函数之间有如下关系

$$H_m^{(1)}(x)=J_m(x)+\mathrm{i}Y_m(x),\quad H_m^{(2)}(x)=J_m(x)-\mathrm{i}Y_m(x),\tag{2}$$

其中$H_m^{(1)}(x)$和$H_m^{(2)}(x)$表示第一类和第二类$m$阶Hankel函数, $Y_m(x)$表示第二类$m$阶的Bessel函数. 由(1)和(2)可得

$$I(f)=\frac{1}{2}\int_\tau^{+\infty}x^\alpha\ln(x)f(x)H_m^{(1)}(\omega g(x))\mathrm{d}x+\frac{1}{2}\int_\tau^{+\infty}x^\alpha\ln(x)f(x)H_m^{(2)}(\omega g(x))\mathrm{d}x=\frac{1}{2}I_1+\frac{1}{2}I_2,$$

其中

$$I_1=\int_\tau^{+\infty}x^\alpha\ln(x)f(x)H_m^{(1)}(\omega g(x))\mathrm{d}x;\quad I_2=\int_\tau^{+\infty}x^\alpha\ln(x)f(x)H_m^{(2)}(\omega g(x))\mathrm{d}x.$$

当$m$固定不变且$|x|\to+\infty$时, Hankel函数具有以下渐近性质(见文[1])

$$H_m^{(1)}(x)\sim\sqrt{\frac{2}{\pi x}}\mathrm{e}^{\mathrm{i}(x-\frac{1}{2}m\pi-\frac{\pi}{4})};\quad H_m^{(2)}(x)\sim\sqrt{\frac{2}{\pi x}}\mathrm{e}^{-\mathrm{i}(x-\frac{1}{2}m\pi-\frac{\pi}{4})}.\tag{3}$$

结合Hankel函数的渐近性, 并通过重构积分路径来近似计算广义积分$I_1$和$I_2$.

**定理1** 假设$g$和$f$在区域$D$都是解析的, 当$x\in[\tau,+\infty)$时, $g(x)\neq0$, $g'(x)\neq0$, 且

$$\lim_{x\to+\infty}g(x)\neq0.$$

设$\triangle_2(p)=g^{-1}(g(\tau)+\mathrm{i}p)$, $\Psi_2(p)=g^{-1}(g(\tau)-\mathrm{i}p)$, 且对任意的$p\in[0,+\infty)$, 满足$|\triangle_2(p)|$, $|\Psi_2(p)|$, $|g'(\triangle_2(p))|$, $|g'(\Psi_2(p))|\geq\eta$, 其中$\eta$是一个正常数. 如果在区域$D$内有$\alpha<0$, $\omega>(\alpha+\mu+1)\omega_1$, 且当$|z|\to+\infty$时满足

(i) 存在$\mu\in\mathbf{R}$使得$|f^{(k)}(z)|=O(|z|^{\mu-k}), k=0,1,2,\cdots$; (4)

(ii) $|\Im(g(z))|\to+\infty$; (5)

(iii) 对$k=0,1,2,\cdots$, 存在$\omega_1\in\mathbf{R}$,使得$|[g^{-1}(z)]^{(k)}|=O(\mathrm{e}^{\omega_1|\Im(z)|})$, (6)

则有

$$I(f)=\frac{1}{2}\mathrm{e}^{\mathrm{i}\omega g(\tau)}\int_0^{+\infty}\triangle_2^\alpha(p)\ln[\triangle_2(p)]f(\triangle_2(p))H_m^{(1)}(\omega(g(\tau)+\mathrm{i}p))\mathrm{e}^{-\mathrm{i}\omega(g(\tau)+\mathrm{i}p)}\triangle_2'(p)\mathrm{e}^{-\omega p}\mathrm{d}p+$$
$$\frac{1}{2}\mathrm{e}^{-\mathrm{i}\omega g(\tau)}\int_0^{+\infty}\Psi_2^\alpha(p)\ln[\Psi_2(p)]f(\Psi_2(p))H_m^{(2)}(\omega(g(\tau)-\mathrm{i}p))\mathrm{e}^{\mathrm{i}\omega(g(\tau)-\mathrm{i}p)}\Psi_2'(p)\mathrm{e}^{-\omega p}\mathrm{d}p.$$

**证** 由(3)可知

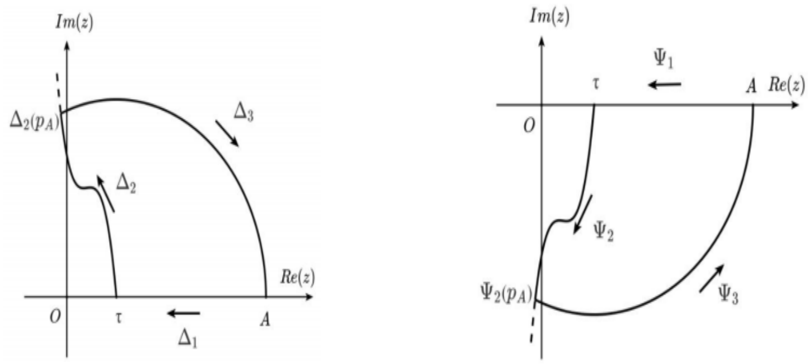


图1　核函数$e^{i\omega g(x)}$中$g(x)$的积分路径(左图); 核函数$e^{-i\omega g(x)}$中$g(x)$的积分路径(右图)

$$H_m^{(1)}(\omega g(x)) \sim \sqrt{\frac{2}{\pi\omega g(x)}}e^{i(\omega g(x)-\frac{1}{2}m\pi-\frac{\pi}{4})}. \tag{7}$$

当$|\omega g(x)| \to \infty$时, 从渐近公式(7)可得$H_m^{(1)}(\omega g(x))e^{i\omega g(x)}$是不振荡的.于是可将$I_1$改成如下只含振荡核函数$e^{i\omega g(x)}$的积分, 即

$$I_1 = \int_\tau^\infty f(x)x^\alpha \ln(x)H_m^{(1)}(\omega g(x))e^{-i\omega g(x)}e^{i\omega g(x)}dx. \tag{8}$$

下面通过在复平面区域中重构如图1所示的积分路径来计算$I[f]$. 在左图中, 设$\triangle_2(p) = g^{-1}(g(\tau)+ip), p \in [0,+\infty), \triangle_2(p)$和圆$\tau+(A-\tau)e^{i\theta}, \theta \in [0,2\pi)$有一个交点$\triangle_2(p_A)$, $\triangle_3$的参数形式可以表示为

$$\triangle_3(\theta) = \tau+(A-\tau)e^{\mathrm{sgn}(\theta_A)i\theta},\ \theta \in [0,|\theta_A|](\theta_A = \arg\triangle_2(p_A)).$$

然后结合Cauchy积分定理[11-12], 可得

$$\begin{aligned}&\textstyle\int_\tau^A f(z)z^\alpha \ln(z)H_m^{(1)}(\omega g(z))e^{-i\omega g(z)}e^{i\omega g(z)}dz = \\ &\textstyle\int_{\triangle_2} f(z)z^\alpha \ln(z)H_m^{(1)}(\omega g(z))e^{-i\omega g(z)}e^{i\omega g(z)}dz + \\ &\textstyle\int_{\triangle_3} f(z)z^\alpha \ln(z)H_m^{(1)}(\omega g(z))e^{-i\omega g(z)}e^{i\omega g(z)}dz.\end{aligned} \tag{9}$$

根据Hankel函数的渐近公式(7), 易知

$$|H_m^{(1)}(x)e^{-ix}| \le M\frac{1}{\sqrt{x}}, \quad |x| \to +\infty, \tag{10}$$

其中$M$为正数. 设$\varrho(\theta)=(A-\tau)\mathrm{e}^{\mathrm{sgn}(\theta_A)\mathrm{i}\theta}$, 由$\alpha<0$结合(3)-(4)及(10)易知存在正数$M_1$, 满足

$$\begin{aligned}&\left|\int_{\Delta_3} z^{\alpha}\ln(z)\,f(z)H_m^{(1)}\left(\omega\,g\left(z\right)\right)\mathrm{e}^{-\mathrm{i}\omega\,g(z)}\mathrm{e}^{\mathrm{i}\omega\,g(z)}\mathrm{d}z\right|=\\&|\int_{|\theta_A|}^{0}\ \mathrm{i}\varrho(\theta)\mathrm{sgn}(\theta_A)(\tau+\varrho(\theta))^{\alpha}\ln\left(\tau+\varrho(\theta)\right)\times\\&f\left(\tau+\varrho(\theta)\right)H_m^{(1)}\left(\omega g\left(\tau+\varrho(\theta)\right)\right)\ \mathrm{e}^{-\mathrm{i}\omega g(\tau+\varrho(\theta))}\ \mathrm{e}^{\mathrm{i}\omega g(\tau+\varrho(\theta))}\mathrm{d}\theta\Big|\le\\&||\theta_A|\max_{\theta\in[0,|\theta_A|]}\ (A-\tau)(\tau+\varrho(\theta))^{\alpha}f\left(\tau+\varrho(\theta)\right)\ln\left(\tau+\varrho(\theta)\right)\times\\&H_m^{(1)}\left(\omega g\left(\tau+\varrho(\theta)\right)\right)\mathrm{e}^{-\mathrm{i}\omega g(\tau+\varrho(\theta))}\ \mathrm{e}^{\mathrm{i}\omega g(\tau+\varrho(\theta))}\ |\le\\&|\theta_A M_1\ |A|^{\frac{\alpha}{2}}(A-\tau)^{\mu+1}\tfrac{M}{\sqrt{\omega|g(z)|}}\,\mathrm{e}^{-(\omega|\Im(g(z))|)}\Big|_{z=\tau+\varrho(\theta)}\to 0.\ (\text{其中}A\to+\infty)\end{aligned}$$

这是因为由(5)可知, 当$A\ \to\ +\infty$, 有$|\Im\left(g\left(\tau+\varrho(\theta)\right)\right)|\ \to\ +\infty$和$|\mathrm{g}\left(\tau+\varrho(\theta)\right)|\ \to\ +\infty$. 因此当$A\to+\infty$时, 结合(9)式可得

$$I_1=\mathrm{e}^{\mathrm{i}\omega g(\tau)}\int_0^{+\infty}\Delta_2^{\alpha}(p)\ln\left[\Delta_2(p)\right]f\left[(\Delta_2(p)\right]H_m^{(1)}\left(\omega(g(\tau)+\mathrm{i}p)\right)\mathrm{e}^{-\mathrm{i}\omega(g(\tau)+\mathrm{i}p)}\Delta_2'(p)\mathrm{e}^{-\omega p}\mathrm{d}p. \tag{11}$$

类似地, 在图1的右图中采用$\Psi_2(p)=g^{-1}(g\left(\tau\right)-\mathrm{i}p)$, $p\in(0,+\infty)$, 有

$$I_2=\mathrm{e}^{-\mathrm{i}\omega g(\tau)}\int_0^{+\infty}\Psi_2^{\alpha}(p)\ln\left[\Psi_2(p)\right]f\left[(\Psi_2(p)\right]H_m^{(2)}\left(\omega(g(\tau)-\mathrm{i}p)\right)\mathrm{e}^{\mathrm{i}\omega(g(\tau)-\mathrm{i}p)}\,\Psi'{}_2(p)\mathrm{e}^{-\omega p}\mathrm{d}p.$$

若令$q=\omega p$, 则有

$$\begin{aligned}I_1=\frac{\mathrm{e}^{\mathrm{i}\omega g(\tau)}}{\omega}\int_0^{+\infty}\Delta_2^{\alpha}\left(\frac{q}{\omega}\right)\ln\left[\Delta_2\left(\frac{q}{\omega}\right)\right]f\left[\Delta_2\left(\frac{q}{\omega}\right)\right]\times\\H_m^{(1)}\left[\omega\left(g(\tau)+\frac{\mathrm{i}q}{\omega}\right)\right]\mathrm{e}^{-\mathrm{i}\omega\left(g(\tau)+\frac{\mathrm{i}q}{\omega}\right)}\Delta_2'\left(\frac{q}{\omega}\right)\mathrm{e}^{-q}\mathrm{d}q.\end{aligned}$$

$$I_2=\frac{\mathrm{e}^{-\mathrm{i}\omega g(\tau)}}{\omega}\int_0^{+\infty}\Psi_2^{\alpha}\left(\frac{q}{\omega}\right)\ln\left[\Psi_2\left(\frac{q}{\omega}\right)\right]f\left[\Psi_2\left(\frac{q}{\omega}\right)\right]H_m^{(2)}\left[\omega h(q)\right]\mathrm{e}^{\mathrm{i}\omega h(q)}\,\Psi'{}_2\left(\frac{q}{\omega}\right)\mathrm{e}^{-q}\mathrm{d}q,$$

这里$h(q)=g(\tau)-\frac{\mathrm{i}q}{\omega}$. 接下来采用$n$点的Gauss-Laguerre求积法则计算$I[f]$的数值解. 设$\xi(x_k)=g(\tau)+\frac{\mathrm{i}x_k}{\omega}$, $\phi(x_k)=g(\tau)-\frac{\mathrm{i}x_k}{\omega}$, $\{a_k,x_k\}_{k=1}^{n}$分别是权函数$\mathrm{e}^{-q}$的点和权, 则可得到$I_1$和$I_2$的点近似求积公式

$$Q_{n,1}\left[f\right]=\frac{\mathrm{e}^{\mathrm{i}\omega g(\tau)}}{\omega}\sum_{k=1}^{n}a_k\Delta_2^{\alpha}\left(\frac{x_k}{\omega}\right)\ln\left(\Delta_2\left(\frac{x_k}{\omega}\right)\right)f\left(\Delta_2\left(\frac{x_k}{\omega}\right)\right)H_m^{(1)}\left[\omega\xi(x_k)\right]\mathrm{e}^{-\mathrm{i}\omega\xi(x_k)}\Delta_2'\left(\frac{x_k}{\omega}\right),$$

$$Q_{n,2}\left[f\right]=\frac{\mathrm{e}^{-\mathrm{i}\omega g(\tau)}}{\omega}\sum_{k=1}^{n}a_k\,\Psi_2^{\alpha}\left(\frac{x_k}{\omega}\right)\ln\left(\Psi_2\left(\frac{x_k}{\omega}\right)\right)f\left(\Psi_2\left(\frac{x_k}{\omega}\right)\right)H_m^{(2)}\left[\omega\phi(x_k)\right]\mathrm{e}^{\mathrm{i}\omega\phi(x_k)}\,\Psi_2'\left(\frac{x_k}{\omega}\right).$$

从而得到了公式(1)的近似计算公式

$$Q_n\left[f\right]=\frac{1}{2}\left(Q_{n,1}\left[f\right]+Q_{n,2}\left[f\right]\right). \tag{12}$$

# §3 数值算法误差分析

本节通过函数的渐近性来分析改进的最速下降法计算$I[f]$所得到的误差阶.

**定理2** 在定理1的假设条件下, 公式(12)近似计算公式(1)的误差阶为

$$I\left[f\right]-Q_n\left[f\right]=O\left(\omega^{-2n-\frac{3}{2}}\right),\quad \omega\to+\infty. \tag{13}$$

**证**　由文献[13]可知, 当$|z|$很大时, 有

$$H_m^{(1)}(z)\,\mathrm{e}^{-\mathrm{i}z}=\sqrt{\frac{2}{\pi z}}\mathrm{e}^{-\mathrm{i}\left(\frac{m\pi}{2}+\frac{\pi}{4}\right)}\sum_{j=0}^{2n+1}\frac{\left(\frac{1}{2}+m\right)_j\left(\frac{1}{2}-m\right)_j}{j!(2\mathrm{i}z)^j}+O\left(|z|^{-2n-\frac{5}{2}}\right), \tag{14}$$

其中

$$(x)_j=\frac{\Gamma(x+j)}{\Gamma(x)}=\begin{cases}1, & j=0,\\ x(x+1)(x+2)\cdots(x+j-1), & j\geq 1.\end{cases}$$

设$E(u)=\Delta_2^{\alpha}(u)\ln(\Delta_2(u))f(\Delta_2(u))\Delta_2'(u)$, $u=\frac{q}{\omega}$, $G(q)=\sum\limits_{j=0}^{2n+1}\frac{\left(\frac{1}{2}+m\right)_j\left(\frac{1}{2}-m\right)_j}{j!(2i)^j(\omega g(\tau)+\mathrm{i}q)^{j+\frac{1}{2}}}$. 利用Hermite插值求函数$E\left(\frac{q}{\omega}\right)G(q)$, 构造插值多项式$P_{2n-1}\left(\frac{q}{\omega}\right)$, 满足

$$P_{2n-1}\left(\frac{x_k}{\omega}\right)=E\left(\frac{x_k}{\omega}\right)G(x_k),\quad P_{2n-1}'\left(\frac{x_k}{\omega}\right)=\left[E\left(\frac{q}{\omega}\right)G(q)\right]'|_{q=x_k},$$

其中$\{x_k\},\{a_k\},k=1,2,\cdots,n$分别是Gauss-Laguerre点和权函数, 则有$n$点Gauss-Laguerre求积公式

$$\int_0^{+\infty}P_{2n-1}\left(\frac{q}{\omega}\right)\mathrm{e}^{-q}\mathrm{d}q=\sum_{k=1}^{n}a_kP_{2n-1}\left(\frac{x_k}{\omega}\right)=\sum_{k=1}^{n}a_kE\left(\frac{x_k}{\omega}\right)G(x_k).$$

绝对误差为

$$AE=\int_0^{+\infty}E\left(\frac{q}{\omega}\right)G(q)\,\mathrm{e}^{-q}\mathrm{d}q-\sum_{k=1}^{n}a_kE\left(\frac{x_k}{\omega}\right)G(x_k),$$

则由Hermite的插值余项公式可得误差

$$\int_0^{+\infty}\left[E\left(\frac{q}{\omega}\right)G(q)-P_{2n-1}\left(\frac{q}{\omega}\right)\right]\mathrm{e}^{-q}\mathrm{d}q=\int_0^{+\infty}\frac{\left[E\left(\frac{q}{\omega}\right)G(q)\right]^{(2n)}|_{q=\gamma}}{(2n)\,!}Ln^2(q)\,\mathrm{e}^{-q}\mathrm{d}q, \tag{15}$$

其中$Ln(x)=\prod_{j=1}^{n}(x-x_j)$, $\gamma\in(0,\max(q,x_n))$. 根据(4)和(5), 考虑误差估计

$$\begin{aligned}
&I_1[f]-Q_{n,1}[f]=\\
&\frac{\mathrm{e}^{\mathrm{i}\omega g(\tau)}}{\omega}\int_0^{+\infty}\Delta_2^{\alpha}\left(\tfrac{q}{\omega}\right)\ln\left(\tfrac{q}{\omega}\right)f\left(\Delta_2\left(\tfrac{q}{\omega}\right)\right)H_m^{(1)}\left[\omega\left(g(\tau)+\mathrm{i}\tfrac{q}{\omega}\right)\right]\mathrm{e}^{-\mathrm{i}\omega\left(g(\tau)+\mathrm{i}\frac{q}{\omega}\right)}{\Delta'}_2\left(\tfrac{q}{\omega}\right)\mathrm{e}^{-q}\mathrm{d}q-\\
&\frac{\mathrm{e}^{\mathrm{i}\omega g(\tau)}}{\omega}\sum_{k=1}^{n}a_k\Delta_2^{\alpha}\left(\tfrac{x_k}{\omega}\right)\ln\left(\tfrac{x_k}{\omega}\right)f\left(\Delta_2\left(\tfrac{x_k}{\omega}\right)\right)H_m^{(1)}\left[\omega\left(g(\tau)+\mathrm{i}\tfrac{x_k}{\omega}\right)\right]\mathrm{e}^{-\mathrm{i}\omega\left(g(\tau)+\mathrm{i}\frac{x_k}{\omega}\right)}\Delta_2'\left(\tfrac{x_k}{\omega}\right)=\\
&\frac{\mathrm{e}^{\mathrm{i}\omega g(\tau)}}{\omega}\int_0^{+\infty}\Delta_2^{\alpha}\left(\tfrac{q}{\omega}\right)\ln\left(\tfrac{q}{\omega}\right)f\left(\Delta_2\left(\tfrac{q}{\omega}\right)\right)\sqrt{\tfrac{2}{\pi}}\mathrm{e}^{-\mathrm{i}\left(\frac{m\pi}{2}+\frac{\pi}{4}\right)}\sum_{j=0}^{2n+1}\frac{\left(\frac{1}{2}+m\right)_j\left(\frac{1}{2}-m\right)_j\mathrm{e}^{-q}}{j!(2\mathrm{i})^j(\omega g(\tau)+\mathrm{i}q)^{j+\frac{1}{2}}}\Delta_2'\left(\tfrac{q}{\omega}\right)\mathrm{d}q+\\
&\quad\frac{\mathrm{e}^{\mathrm{i}\omega g(\tau)}}{\omega}\int_0^{+\infty}\Delta_2^{\alpha}\left(\frac{q}{\omega}\right)\ln\left(\frac{q}{\omega}\right)f\left(\Delta_2\left(\frac{q}{\omega}\right)\right)\Delta_2'\left(\frac{q}{\omega}\right)\mathrm{e}^{-q}O\left(|\omega g(\tau)+\mathrm{i}q|^{-2n-\frac{5}{2}}\right)\mathrm{d}q-\\
&\qquad\frac{\mathrm{e}^{\mathrm{i}\omega g(\tau)}}{\omega}\sum_{k=1}^{n}a_k\Delta_2^{\alpha}\left(\tfrac{x_k}{\omega}\right)\ln\left(\tfrac{x_k}{\omega}\right)f\left(\Delta_2\left(\tfrac{x_k}{\omega}\right)\right)\Delta_2'\left(\tfrac{x_k}{\omega}\right)\left[\sqrt{\tfrac{2}{\pi}}\mathrm{e}^{-\mathrm{i}\left(\frac{m\pi}{2}+\frac{\pi}{4}\right)}\times\right.\\
&\qquad\left.\sum_{j=0}^{2n+1}\frac{\left(\frac{1}{2}+m\right)_j\left(\frac{1}{2}-m\right)_j}{j!(2\mathrm{i})^j(\omega g(\tau)+\mathrm{i}x_k)^{j+\frac{1}{2}}}+O\left(|\omega g(\tau)+\mathrm{i}x_k|^{-2n-\frac{5}{2}}\right)\right]=\\
&\qquad\frac{\mathrm{e}^{\mathrm{i}\omega g(\tau)}}{\omega}\sqrt{\tfrac{2}{\pi}}\mathrm{e}^{-\mathrm{i}\left(\frac{m\pi}{2}+\frac{\pi}{4}\right)}\int_0^{+\infty}\frac{\left[E\left(\frac{q}{\omega}\right)G(q)\right]^{(2n)}|_{q=\gamma}}{(2n)\,!}Ln^2(q)\,\mathrm{e}^{-q}\mathrm{d}q+\\
&\qquad\frac{\mathrm{e}^{\mathrm{i}\omega g(\tau)}}{\omega}\int_0^{+\infty}\Delta_2^{\alpha}\left(\tfrac{q}{\omega}\right)\ln\left(\tfrac{q}{\omega}\right)f\left(\Delta_2\left(\tfrac{q}{\omega}\right)\right)\Delta_2'\left(\tfrac{q}{\omega}\right)\mathrm{e}^{-q}O\left(|\omega g(\tau)+\mathrm{i}q|^{-2n-\frac{5}{2}}\right)\mathrm{d}q-\\
&\qquad\quad\frac{\mathrm{e}^{\mathrm{i}\omega g(\tau)}}{\omega}O\left(|\omega g(\tau)+\mathrm{i}x_k|^{-2n-\frac{5}{2}}\right),
\end{aligned}\tag{16}$$

其中$\gamma\in(0,\max(q,x_n))$. 对于(16)倒数第二行的积分有

$$\left|\int_0^{+\infty}\Delta_2^{\alpha}\left(\frac{q}{\omega}\right)\ln\left(\frac{q}{\omega}\right)f\left(\Delta_2\left(\frac{q}{\omega}\right)\right)\Delta'_2\left(\frac{q}{\omega}\right)\mathrm{e}^{-q}O\left(|\omega g(\tau)+\mathrm{i}q|^{-2n-\frac{5}{2}}\right)\mathrm{d}q\right|\le$$

$$\int_0^{+\infty}\left|\Delta_2^{\alpha}\left(\frac{q}{\omega}\right)\ln\left(\frac{q}{\omega}\right)f\left(\Delta_2\left(\frac{q}{\omega}\right)\right)\Delta'_2\left(\frac{q}{\omega}\right)\mathrm{e}^{-q}\right|O\left(\omega^{-2n-\frac{5}{2}}\right)\left[g^2(\tau)+\left(\frac{q}{\omega}\right)^2\right]^{-n-\frac{5}{4}}\mathrm{d}q\le$$

$$\omega^{-2n-\frac{5}{2}}M_2\int_0^{+\infty}\left|\Delta_2^{\alpha}\left(\frac{q}{\omega}\right)\ln\left(\frac{q}{\omega}\right)f\left(\Delta_2\left(\frac{q}{\omega}\right)\right)\Delta'_2\left(\frac{q}{\omega}\right)\right|\mathrm{e}^{-q}\left[g^2(\tau)+\left(\frac{q}{\omega}\right)^2\right]^{-n-\frac{5}{4}}\mathrm{d}q=$$

$$\omega^{-2n-\frac{5}{2}}M_2\left\{\int_0^{1}\left|\Delta_2^{\alpha}\left(\frac{q}{\omega}\right)\ln\left(\frac{q}{\omega}\right)f\left(\Delta_2\left(\frac{q}{\omega}\right)\right)\Delta'_2\left(\frac{q}{\omega}\right)\right|\mathrm{e}^{-q}\left[g^2(\tau)+\left(\frac{q}{\omega}\right)^2\right]^{-n-\frac{5}{4}}\mathrm{d}q\right\}+$$

$$\int_1^{+\infty}\left|\Delta_2^{\alpha}\left(\frac{q}{\omega}\right)\ln\left(\frac{q}{\omega}\right)f\left(\Delta_2\left(\frac{q}{\omega}\right)\right)\Delta'_2\left(\frac{q}{\omega}\right)\right|\mathrm{e}^{-q}\left[g^2(\tau)+\left(\frac{q}{\omega}\right)^2\right]^{-n-\frac{5}{4}}\mathrm{d}q,\tag{17}$$

其中$M_2$是一个正数, (17)倒数第二行中的积分为定积分, 一定收敛. 由下式可知(17)最后一行的无穷积分也是收敛的. 由于$j=0,1,\cdots$,有

$$(\ln(z))^{(j)}=O(z^{-j}),\ (\ln(z)f(z))^{(j)}=\sum_{k=0}^{j}C_j^k\ln^{(k)}(z)f^{(j-k)}(z)=\sum_{k=0}^{j}C_j^k\frac{(-1)^{k+1}}{z^k}f^{(j-k)}(z),$$

则结合(4)易得$(\ln(z)f(z))^{(j)}=O\left(|z|^{\mu-j}\right)$. 由(5)和(7)可知

$$\left|\Delta_2^{\alpha}\left(\frac{q}{\omega}\right)\ln\left(\frac{q}{\omega}\right)f\left(\Delta_2\left(\frac{q}{\omega}\right)\right)\Delta'_2\left(\frac{q}{\omega}\right)\right|=O\left(\mathrm{e}^{\frac{(\alpha+\mu+1)\omega_1 q}{\omega}}\right),\ q\to+\infty.$$

若$\frac{(\alpha+\mu+1)\omega_1}{\omega}-1<0$, 即$\omega>(\alpha+\mu+1)\omega_1,\left|g'\left(\Delta_2\left(\frac{q}{\omega}\right)\right)\right|\ge\eta>0$, 且$\left|\Delta_2\left(\frac{q}{\omega}\right)\right|\ge\eta>0$, 则存在$\beta>1$, 从而结合(5)和(7)可得

$$\lim_{q\to+\infty}q^{\beta}\cdot\left|\Delta_2^{\alpha}\left(\frac{q}{\omega}\right)\ln\left(\frac{q}{\omega}\right)f\left[\Delta_2\left(\frac{q}{\omega}\right)\right]\Delta'_2\left(\frac{q}{\omega}\right)\right|\mathrm{e}^{-q}\left[g^2(\tau)+\left(\frac{q}{\omega}\right)^2\right]^{-n-\frac{5}{4}}=$$

$$\lim_{q\to+\infty}q^{\beta}\cdot O\left(\mathrm{e}^{\left[\frac{(\alpha+\mu+1)\omega_1}{\omega}-1\right]q}\right)\left[g^2(\tau)+\left(\frac{q}{\omega}\right)^2\right]^{-n-\frac{5}{4}}=0.$$

根据Cauchy判别法, (17)的最后一行积分也是收敛的. 故对于(16)的最后一行积分, 可推导出

$$\int_0^{+\infty}\Delta_2^{\alpha}\left(\frac{q}{\omega}\right)\ln\left(\frac{q}{\omega}\right)f\left(\Delta_2\left(\frac{q}{\omega}\right)\right)\Delta'_2\left(\frac{q}{\omega}\right)\mathrm{e}^{-q}O\left(|\omega g(\tau)+\mathrm{i}q|^{-2n-\frac{5}{2}}\right)\mathrm{d}q=$$

$$O\left(\omega^{-2n-\frac{5}{2}}\right),\ \omega\to+\infty.\tag{18}$$

因此当$\omega\to+\infty$时, 可以得到

$$\frac{\mathrm{d}}{\mathrm{d}q}G(q)=-\mathrm{i}\sum_{j=0}^{2n+1}\frac{\left(j+\frac{1}{2}\right)\left(\frac{1}{2}+m\right)_j\left(\frac{1}{2}-m\right)_j}{j!(2\mathrm{i})^j(\omega g(\tau)+\mathrm{i}q)^{j+\frac{3}{2}}}=O\left(\omega^{-\frac{3}{2}}\right).$$

同样的可得

$$\frac{\mathrm{d}^k}{\mathrm{d}q^k}G(q)=O\left(\omega^{-k-\frac{1}{2}}\right),k\in\mathbf{N}^+,\omega\to+\infty.\tag{19}$$

在$E(u)$中令$u=\frac{q}{\omega}$, 结合公式(5)和(7)可得

$$E(u)\ =\ \Delta_2^{\alpha}(u)\ln(\Delta_2(u))f(\Delta_2(u))\ \Delta'_2(u)=O\left(\mathrm{e}^{\frac{(\alpha+\mu+1)\omega_1 q}{\omega}}\right),\ q\to+\infty.$$

$$E'(u)=\alpha\Delta_2^{\alpha-1}(u)\left[\Delta'_2(u)\right]^2f(\Delta_2(u))\ln(\Delta_2(u))+\Delta_2^{\alpha}(u)\left[\Delta'_2(u)\right]^2f'(\Delta_2(u))\ln(\Delta_2(u))+$$

$$\Delta_2^{\alpha}(u)\Delta''_2(u)f(\Delta_2(u))\ln(\Delta_2(u))+\Delta_2^{\alpha-1}(u)\left(\Delta'_2(u)\right)^2f(\Delta_2(u))=O\left(\mathrm{e}^{\frac{(\alpha+\mu+1)\omega_1 q}{\omega}}\right),$$

$q \to +\infty$.

$$E''(u) = \alpha(\alpha-1)\varDelta_2^{\alpha-2}(u)[\varDelta'_2(u)]^3 f(\varDelta_2(u))\ln(\varDelta_2(u)) +$$
$$2\alpha\varDelta_2^{\alpha-1}(u)\varDelta'_2(u)[\varDelta''_2(u)] f(\varDelta_2(u))\ln(\varDelta_2(u)) +$$
$$\alpha\varDelta_2^{\alpha-1}(u)[\varDelta'_2(u)]^3 f'(\varDelta_2(u))\ln(\varDelta_2(u)) + \alpha\varDelta_2^{\alpha-2}(u) f(\varDelta_2(u))[\varDelta'_2(u)]^3 +$$
$$\alpha\varDelta_2^{\alpha-1}(u)[\varDelta'_2(u)]^3 f'(\varDelta_2(u))\ln(\varDelta_2(u)) + 2\varDelta_2^{\alpha}(u)\varDelta'_2(u)\varDelta''_2(u) f'(\varDelta_2(u))\ln(\varDelta_2(u)) \times$$
$$(\varDelta_2(u))\ln(\varDelta_2(u)) + \varDelta_2^{\alpha}(u) f'(\varDelta_2(u))[\varDelta'_2(u)]^3 +$$
$$\alpha\varDelta_2^{\alpha-1}(u)\varDelta'_2(u)\varDelta''_2(u) f(\varDelta_2(u))\ln(\varDelta_2(u)) + \varDelta_2^{\alpha}(u)\varDelta'''_2(u) f(\varDelta_2(u))\ln(\varDelta_2(u)) +$$
$$\varDelta_2^{\alpha}(u)\varDelta''_2(u) f'(\varDelta_2(u))\varDelta'_2(u)\ln(\varDelta_2(u)) + \varDelta_2^{\alpha-1}(u)\varDelta''_2(u) f(\varDelta_2(u))\varDelta'_2(u) +$$
$$(\alpha-1)\varDelta_2^{\alpha-2}(u)[\varDelta'_2(u)]^3 f(\varDelta_2(u)) + 2\varDelta_2^{\alpha-1}(u)\varDelta'_2(u)\varDelta''_2(u) f(\varDelta_2(u)) +$$
$$\varDelta_2^{\alpha-1}(u)[\varDelta'_2(u)]^3 f'(\varDelta_2(u)) =$$
$$O\left(\mathrm{e}^{\frac{(\alpha+\mu+1)\omega_1 q}{\omega}}\right),\ q \to \infty. \tag{20}$$

类似的, 可得到各阶误差阶估计为

$$E^{(k)}(u) = O\left(\mathrm{e}^{\frac{(\alpha+\mu+1)\omega_1 q}{\omega}}\right),\ q \to \infty,\ k = 0, 1, 2, \cdots,$$

由(19)和(20), 利用莱布尼兹定理, 可得(16)第七行的积分

$$\int_0^{+\infty} \frac{\left[E\left(\frac{q}{\omega}\right)G(q)\right]^{(2n)}|_{q=\gamma}}{(2n)\,!} Ln^2(q)\,\mathrm{e}^{-q}\mathrm{d}q = \frac{1}{\omega^{2n+\frac{1}{2}}}\int_0^{+\infty}\frac{R(\gamma)}{(2n)!}Ln^2(q)\,\mathrm{e}^{-q}\mathrm{d}q, \tag{21}$$

其中$\left[E\left(\frac{q}{\omega}\right)G(q)\right]^{(2n)}|_{q=\gamma} = \frac{1}{\omega^{2n+\frac{1}{2}}}R(\gamma)$, $R(\gamma)$是一个很长的函数表达式. 利用(18)类似的证明方式以及Cauchy判断法, 再结合下面条件, 即当$\omega > (\alpha+\mu+1)\,\omega_1$时, $\left|\varDelta_2\left(\frac{q}{\omega}\right)\right| \geq \eta > 0$, 且$\left|g'\left(\varDelta_2\left(\frac{q}{\omega}\right)\right)\right| \geq \eta > 0$,可以验证(20)的积分$\int_0^{+\infty}\frac{R(\gamma)}{(2n)!}Ln^2(q)\ \mathrm{e}^{-q}\mathrm{d}q$也是收敛的, 从而有

$$\int_0^{+\infty} \frac{\left[E\left(\frac{q}{\omega}\right)G(q)\right]^{(2n)}|_{q=\gamma}}{(2n)\,!} Ln^2(q)\,\mathrm{e}^{-q}\mathrm{d}q = O\left(\omega^{-2n-\frac{1}{2}}\right),\ \omega \to +\infty.$$

因此由(16), (18)和(21)可得

$$I_1[f] - Q_{n,1}[f] = O\left(\omega^{-2n-\frac{3}{2}}\right),\ \omega \to +\infty.$$

同理可得

$$I_2[f] - Q_{n,2}[f] = O\left(\omega^{-2n-\frac{3}{2}}\right),\ \omega \to +\infty.$$

因此可得修正的最速下降法的绝对误差阶为

$$AE = |I[f] - Q_n[f]| = \left|\tfrac{1}{2}\left(I_1[f] - Q_{n.1}[f] + I_2[f] - Q_{n,2}[f]\right)\right| = O\left(\omega^{-2n-\frac{3}{2}}\right),\ \omega \to +\infty.$$

## §4　数值实验

接下来, 通过几个数值例子来说明数值算法的有效性以及误差估计的准确性.

**例1**　用公式(12)近似计算积分$\int_1^{\infty} x^{-4}\ln(x)\sin\left(\frac{1}{x}\right)J_2(\omega x)\mathrm{d}x$.

**例2**　用公式(12)来近似计算积分$\int_1^{\infty} x^{-2}\frac{\ln(x)}{1+x^2}J_1(\omega x^3)\mathrm{d}x$.

**例3**　用公式(12)来近似计算积分$\int_1^{\infty} x^{-3}\ln(x)\mathrm{e}^{-x}J_2(\omega x^2)\mathrm{d}x$.

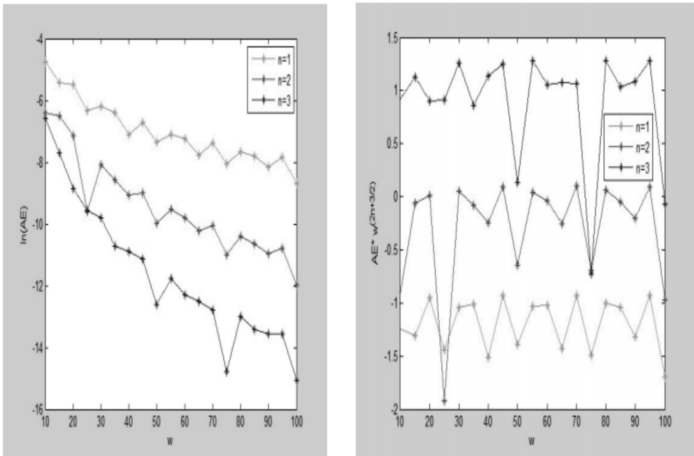


图2 例2中当$n$取1, 2, 3, $\omega$取不同值时, 左图为最速下降法计算积分产生的绝对误差用$\ln(AE)$表示; 右图表示$AE \times \omega^{2n+3/2}$, 其中$AE$为绝对误差

表1 例1中当$n$取1, 2, 3, $\omega$取不同值时, 最速下降法计算积分的绝对误差估计

| $\omega$ | $n=1$ | $n=2$ | $n=3$ |
|---|---|---|---|
| 20 | 2.8657e-05 | 4.8214e-06 | 4.9412e-07 |
| 50 | 4.0913e-06 | 5.2529e-08 | 6.5101e-10 |
| 80 | 3.9406e-07 | 1.5166e-09 | 9.7384e-13 |
| 100 | 3.7471e-07 | 1.1892e-09 | 3.0865e-12 |

表2 例2中当$n$取1, 2, 3, $\omega$取不同值时, 最速下降法计算积分的绝对误差估计

| $\omega$ | $n=1$ | $n=2$ | $n=3$ |
|---|---|---|---|
| 10 | 1.7789e-05 | 4.0062e-07 | 2.6651e-07 |
| 30 | 6.1688e-07 | 8.4659e-09 | 1.5143e-10 |
| 50 | 4.5668e-08 | 1.0291e-10 | 2.4748e-13 |
| 70 | 4.1010e-08 | 8.9602e-11 | 1.6800e-13 |
| 90 | 6.8697e-09 | 1.1121e-11 | 2.7062e-14 |
| 100 | 1.9941e-09 | 1.0768e-12 | 8.5090e-16 |

由例1、例2和例3可以看出, 当$\omega$不变时, 随着节点个数$n$的增大, 绝对误差减少; 当$n$不变时, $\omega$越大, 绝对误差也越小; 且修正的最速下降法产生的误差阶为$\omega^{-2n-3/2}$.

**结束语** 高振荡积分在众多领域里有着重要的应用. 本文结合Hankel函数的渐近性和改变积分路径来近似计算了一类带对数函数的具有一般振荡子的广义高振荡积分, 得到了一个修正

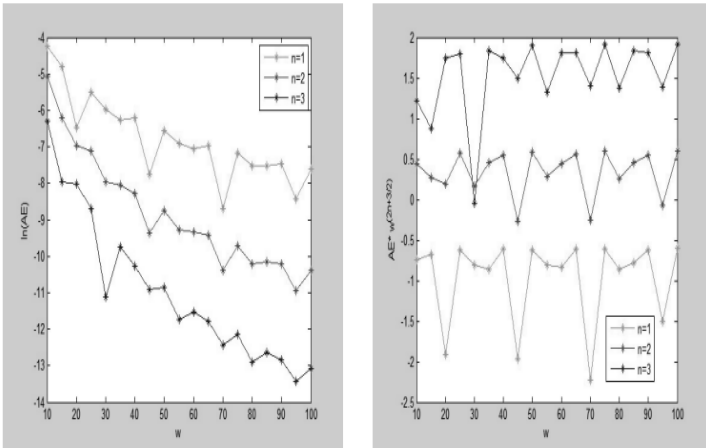


图3　例3中当$n$取1, 2, 3, $\omega$取不同值时, 左图为最速下降法计算积分产生的绝对误差用$\ln(AE)$表示; 右图表示$AE \times \omega^{2n+3/2}$, 其中$AE$为绝对误差

表3　例3中当$n$取1, 2, 3, $\omega$取不同值时, 最速下降法计算积分的绝对误差估计

| $\omega$ | $n=1$ | $n=2$ | $n=3$ |
| --- | --- | --- | --- |
| 15 | 1.6220e-05 | 6.3651e-07 | 1.1190e-08 |
| 35 | 5.5017e-07 | 9.0595e-09 | 1.8161e-10 |
| 55 | 1.2541e-07 | 5.1073e-10 | 1.8685e-12 |
| 75 | 6.7439e-08 | 1.9138e-10 | 7.1276e-13 |
| 100 | 2.5265e-08 | 3.9875e-11 | 8.2537e-14 |

的最速下降法, 并在理论上得到了算法的误差界, 最后通过数值实例验证了结论的可行性.

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## On generalized integration algorithm for a class of highly oscillatory Bessel kernel functions

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**Abstract**: This paper combines the asymptotic properties of Hankel functions and the Gauss Laguerre quadrature method to calculate a class of highly oscillatory infinite integrals with logarithmic function and general oscillators. Firstly, the integration is transformed into two line integrals on the complex plane by changing the integration path. Then, high-precision approximations are obtained using the Gauss Laguerre quadrature formula. Finally, the corresponding error orders are given, and the feasibility of the conclusion is verified with numerical examples.

**Keywords**: Hankel transform; highly oscillatory; Gauss Laguerre quadrature; integration path
**MR Subject Classification**: 39A05; 34B10