高校应用数学学报
2025, 40(1): 109-119

加权Lebesgue空间中超齐次核积分算子的最佳搭配参数

洪　勇*, 赵　茜

(广州华商学院 人工智能学院, 广东广州 511300)

摘　要: 文中引入了超齐次函数的概念, 以超齐次核统一了齐次核、广义齐次核及若干非齐次核, 然后利用权函数方法, 讨论具有超齐次核积分算子在加权Lebesgue空间中的有界性及算子范数问题, 得到了该类积分算子最佳搭配参数的充分必要条件和算子范数的计算公式, 统一了之前的诸多结果.

关键词: 超齐次核; 积分算子; Hilbert型积分不等式; 加权Lebesgue空间; 最佳搭配参数; 有界算子; 算子范数

中图分类号: O177

文献标识码: A　　**文章编号**: 1000-4424(2025)01-0109-11

§1　引言与超齐次函数

Hardy[1]通过引入一对共轭参数(p, q), 给出了Hilbert积分不等式的推广形式: 若$\frac{1}{p}+\frac{1}{q}=1\,(p>1)$, $f\in L_p(0,+\infty)$, $g\in L_q(0,+\infty)$, 则

$$\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)g(y)}{x+y}\mathrm{d}x\mathrm{d}y\le\frac{\pi}{\sin(\frac{\pi}{p})}\|f\|_p\|g\|_q, \tag{1}$$

其中的常数因子$\frac{\pi}{\sin(\frac{\pi}{p})}$是最佳值, 利用式(1), 可得到关于积分算子

$$T(f)(y)=\int_0^{+\infty}\frac{1}{x+y}f(x)\mathrm{d}x \tag{2}$$

的等价不等式$\|T(f)\|_p\le\frac{\pi}{\sin(\frac{\pi}{p})}\|f\|_p$, 于是可知算子$T$是Lebesgue空间$L_p(0,+\infty)$中的有界算子, 且$T$的算子范数$\|T\|=\frac{\pi}{\sin(\frac{\pi}{p})}$.

为了进一步的推广研究, $L_p(0,+\infty)$被推广为带幂权的加权Lebesgue空间: 设$r>1$, $\alpha\in\mathbf{R}$, 定义

$$L_r^{\alpha}(0,+\infty)=\left\{f(x):\|f\|_{r,\alpha}=\left(\int_0^{+\infty}x^{\alpha}|f(x)|^r\mathrm{d}x\right)^{\frac{1}{r}}<+\infty\right\}.$$

收稿日期: 2023-05-15　　修回日期: 2025-02-19
*通讯作者, E-mail: hongyongggdcc@yeah.net
基金项目: 广州华商学院特色科研项目(2024HSTS08); 广东省基础与应用基础研究基金(2022A1515012429)

考虑到式(2)中算子T的核$\frac{1}{x+y}$是-1阶齐次函数, 文[2-3]考虑了$-\lambda$阶齐次核$\frac{1}{x^\lambda+y^\lambda}$及$\frac{1}{(x+y)^\lambda}$的情况, 推广了相关结果, 之后人们考虑了广义齐次核$\frac{1}{x^{\lambda_1}+y^{\lambda_2}}$的情况, 得到更具普遍意义的结论[4].

在讨论齐次核和广义齐次核积分算子的同时, 学者们也常讨论如$\frac{1}{1+xy}$及$\frac{|\ln(xy)|}{(xy)^\lambda+1}$等非齐次核的情形[5-6].

上述问题的讨论, 都有一个共同的特征, 就是只针对一个具体的特定核, 技巧性地引入两个适当的搭配参数, 利用权函数方法得到相应的Hilbert型不等式, 然后证明常数因子最佳, 最后得到相应的算子范数. 为了对抽象的齐次核和广义齐次核找到最佳搭配参数的规律, 2008年, 文[7]首次讨论了齐次核情形的最佳搭配参数问题, 得到最佳搭配参数的充分条件, 但当时并未证明该条件是否必要, 2016年文[8]获得了充分必要条件, 并将齐次核推广到广义齐次核情形. 目前Hilbert型不等式及其算子问题已形成了比较完整的理论体系(见[9-16]).

为了对齐次核, 广义齐次核和非齐次核$G(x^{\lambda_1}y^{\lambda_2})$统一进行研究, 本文引入超齐次函数概念, 设有四个参数σ_1,σ_2,τ_1及τ_2, 使函数$K(x,y)$满足: $\forall t>0$, 有

$$K(tx,y)=t^{\sigma_1}K(x,t^{\tau_1}y),\quad K(x,ty)=t^{\sigma_2}K(t^{\tau_2}x,y),$$

则称$K(x,y)$是具有参数$\{\sigma_1,\sigma_2,\tau_1,\tau_2\}$的超齐次函数.

若$K(x,y)$是λ阶齐次函数, 则$K(x,y)$是具有参数$\{\lambda,\lambda,-1,-1\}$的超齐次函数, 可见超齐次函数是齐次函数的一种自然推广.

若$G(u,v)$是λ阶齐次函数, 则广义齐次函数$K_2(x,y)=G(x^{\lambda_1},y^{\lambda_2})$是具有参数

$$\{\lambda\lambda_1,\lambda\lambda_2,-\frac{\lambda_1}{\lambda_2},-\frac{\lambda_2}{\lambda_1}\}$$

的超齐次函数.

对任何实函数$G(u)$, 非齐次函数$K_3(x,y)=G(x^{\lambda_1}y^{\lambda_2})$是具有参数$\{0,0,\frac{\lambda_1}{\lambda_2},\frac{\lambda_2}{\lambda_1}\}$的超齐次函数.

综上所述, 超齐次函数概念高度统一了常见的Hilbert型不等式及相应算子研究中的核函数, 能使笔者站在更高更广的角度抽象地讨论问题, 获得更具普遍意义的结果.

对于具有参数$\{\sigma_1,\sigma_2,\tau_1,\tau_2\}$的超齐次函数$K(x,y)$, 由于

$$K(tx,y)=t^{\sigma_1}K(x,t^{\tau_1}y)=t^{\sigma_1+\tau_1\sigma_2}K(t^{\tau_1\tau_2}x,y),$$

可见在一般情况下都有$\tau_1\tau_2=1$, $\sigma_1+\tau_1\sigma_2=0$, 因此本文的讨论均在$\tau_1\tau_2=1$及$\sigma_1+\tau_1\sigma_2=0$的条件下进行.

为了避免不必要的重复, 本文中始终记

$$A(K,f,g)=\int_0^{+\infty}\int_0^{+\infty}K(x,y)f(x)g(y)\mathrm{d}x\mathrm{d}y.$$

$$W_1(s)=\int_0^{+\infty}K(1,t)t^s\mathrm{d}t,\quad W_2(s)=\int_0^{+\infty}K(t,1)t^s\mathrm{d}t.$$

§2 预备引理

引理1 设$K(x,y)$是具有参数$\{\sigma_1,\sigma_2,\tau_1,\tau_2\}$的超齐次函数, $\tau_1\tau_2\neq 0$, 那么

(i) 若$\tau_1 b-a=\tau_1-\sigma_1-1$, 则

$$W_2(-a)=\int_0^{+\infty}K(t,1)t^{-a}\mathrm{d}t=\frac{1}{|\tau_1|}\int_0^{+\infty}K(1,t)t^{-b}\mathrm{d}t=\frac{1}{|\tau_1|}W_1(-b);$$

(ii) 若$\tau_2 a-b=\tau_2-\sigma_2-1$, 则
$$W_1(-b)=\int_0^{+\infty}K(1,t)t^{-b}\mathrm{d}t=\frac{1}{|\tau_2|}\int_0^{+\infty}K(t,1)t^{-a}\mathrm{d}t=\frac{1}{|\tau_2|}W_2(-a);$$
(iii)
$$\omega_1(x,b)=\int_0^{+\infty}K(x,y)y^{-b}\mathrm{d}y=x^{\sigma_1+\tau_1(b-1)}W_1(-b),$$
$$\omega_2(y,a)=\int_0^{+\infty}K(x,y)x^{-a}\mathrm{d}x=y^{\sigma_2+\tau_2(a-1)}W_2(-a).$$
证　(i) 因为$\tau_1 b-a=\tau_1-\sigma_1-1$, 故$\frac{1}{\tau_1}(\sigma_1-a)+\frac{1}{\tau_1}-1=-b$, 从而
$$W_2(-a)=\int_0^{+\infty}K(1,t^{\tau_1})t^{\sigma_1-a}\mathrm{d}t=\frac{1}{|\tau_1|}\int_0^{+\infty}K(1,u)u^{\frac{1}{\tau_1}(\sigma_1-a)+\frac{1}{\tau_1}-1}\mathrm{d}u=$$
$$\frac{1}{|\tau_1|}\int_0^{+\infty}K(1,u)u^{-b}\mathrm{d}u=\frac{1}{|\tau_1|}W_1(-b).$$
(ii) 同理可证$W_1(-b)=\frac{1}{|\tau_2|}W_2(-a)$.
(iii)
$$\omega_1(x,b)=x^{\sigma_1}\int_0^{+\infty}K(1,x^{\tau_1}y)y^{-b}\mathrm{d}y=$$
$$x^{\sigma_1+\tau_1(b-1)}\int_0^{+\infty}K(1,t)t^{-b}\mathrm{d}t=x^{\sigma_1+\tau_1(b-1)}W_1(-b).$$
同理可证$\omega_2(y,a)=y^{\sigma_2+\tau_2(a-1)}W_2(-a)$.

引理2　$\tau_1 b-a=\tau_1-\sigma_1-1$与$\tau_2 a-b=\tau_2-\sigma_2-1$等价的充分必要条件是
$$\tau_1\tau_2=1,\ \sigma_1+\tau_1\sigma_2=0.$$
证　$\tau_1 b-a=\tau_1-\sigma_1-1$与$\tau_2 a-b=\tau_2-\sigma_2-1$等价相当于二元线性方程组
$$\begin{cases}x_1-\tau_1 x_2=-\tau_1+\sigma_1+1\\ \tau_2 x_1-x_2=\tau_2-\sigma_2-1\end{cases}$$
有无穷多解, 即方程组增广矩阵的秩满足
$$1=\mathrm{Rank}\begin{pmatrix}1&-\tau_1&-\tau_1+\sigma_1+1\\ \tau_2&-1&\tau_2-\sigma_2-1\end{pmatrix}=\mathrm{Rank}\begin{pmatrix}1&-\tau_1&\sigma_1\\ \tau_2&-1&-\sigma_2\end{pmatrix}=$$
$$\mathrm{Rank}\begin{pmatrix}1&-\tau_1&\sigma_1\\ 0&\tau_1\tau_2-1&-\tau_2\sigma_1-\sigma_2\end{pmatrix},$$
这等价于$\tau_1\tau_2=1$, $\tau_2\sigma_1+\sigma_2=0$, 即$\tau_1\tau_2=1$, $\sigma_1+\tau_1\sigma_2=0$.

引理3　设$\frac{1}{p}+\frac{1}{q}=1\,(p>1)$, $K(x,y)$是具有参数$\{\sigma_1,\sigma_2,\tau_1,\tau_2\}$的超齐次函数, $\tau_1\tau_2=1$, $\sigma_1+\tau_1\sigma_2=0$, $\tau_1 b-a=\tau_1-\sigma_1-1$, 则
$$W_1^{\frac{1}{p}}(-b)W_2^{\frac{1}{q}}(-a)=\left(\frac{1}{|\tau_1|}\right)^{\frac{1}{q}}W_1(-b)=\left(\frac{1}{|\tau_2|}\right)^{\frac{1}{p}}W_2(-a).$$
证　根据引理2, 可知$\tau_1 b-a=\tau_1-\sigma_1-1$与$\tau_2 a-b=\tau_2-\sigma_2-1$同时成立, 从而由引理1可得引理3的结论.

§3　超齐次核的Hilbert型积分不等式的最佳搭配参数

定理1　设$\frac{1}{p}+\frac{1}{q}=1\,(p>1)$, $a,b\in\mathbf{R}$, $\tau_1\tau_2\neq 0$, $K(x,y)$是具有参数$\{\sigma_1,\sigma_2,\tau_1,\tau_2\}$的超

齐次非负可测函数, $0 < W_1(-b) < +\infty$, $0 < W_2(-a) < +\infty$, 存在$\delta_0 > 0$使$W_1(-b \pm \delta_0) < +\infty$或$W_2(-a \pm \delta_0) < +\infty$.

(i) 记$\alpha = a(p-1)+\sigma_1+\tau_1(b-1)$, $\beta = b(q-1)+\sigma_2+\tau_2(a-1)$, 则有Hilbert型积分不等式

$$A(K,f,g) \le W_1^{\frac{1}{p}}(-b)W_2^{\frac{1}{q}}(-a)\|f\|_{p,\alpha}\|g\|_{q,\beta}, \tag{3}$$

其中$f \in L_p^{\alpha}(0,+\infty)$, $g \in L_q^{\beta}(0,+\infty)$. 当$\tau_1\tau_2 = 1$, $\sigma_1+\tau_1\sigma_2 = 0$, $\tau_1 b - a = \tau_1 - \sigma_1 - 1$时, 式(3)化为

$$A(K,f,g) \le \left(\frac{1}{|\tau_1|}\right)^{\frac{1}{q}} W_1(-b)\|f\|_{p,ap-1}\|g\|_{q,bq-1} = \left(\frac{1}{|\tau_2|}\right)^{\frac{1}{p}} W_2(-a)\|f\|_{p,ap-1}\|g\|_{q,bq-1}. \tag{4}$$

(ii) 若$\tau_1\tau_2 = 1$, $\sigma_1+\tau_1\sigma_2 = 0$, 则当且仅当$\tau_1 b - a = \tau_1 - \sigma_1 - 1$, 式(3)的常数$W_1^{\frac{1}{p}}(-b)W_2^{\frac{1}{q}}(-a)$因子是最佳值.

证 不妨设$W_1(-b \pm \delta_0) < +\infty$.

(i) 根据Hölder不等式及引理1, 有

$$\begin{aligned} A(K,f,g) &\le \int_0^{+\infty}\int_0^{+\infty}\left(\frac{x^{\frac{a}{q}}}{y^{\frac{b}{p}}}|f(x)|\right)\left(\frac{y^{\frac{b}{p}}}{x^{\frac{a}{q}}}|g(y)|\right)K(x,y)\mathrm{d}x\mathrm{d}y \le \\ &\left(\int_0^{+\infty}\int_0^{+\infty}\frac{x^{a(p-1)}}{y^b}|f(x)|^pK(x,y)\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{p}} \times \\ &\left(\int_0^{+\infty}\int_0^{+\infty}\frac{y^{b(q-1)}}{x^a}|g(y)|^qK(x,y)\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{q}} = \\ &\left(\int_0^{+\infty}x^{a(p-1)}|f(x)|^p\omega_1(x,b)\mathrm{d}x\right)^{\frac{1}{p}}\left(\int_0^{+\infty}y^{b(q-1)}|g(y)|^q\omega_2(y,a)\mathrm{d}y\right)^{\frac{1}{q}} \le \\ &W_1^{\frac{1}{p}}(-b)W_2^{\frac{1}{q}}(-a)\left(\int_0^{+\infty}x^{a(p-1)+\sigma_1+\tau_1(b-1)}|f(x)|^p\mathrm{d}x\right)^{\frac{1}{p}} \times \\ &\left(\int_0^{+\infty}y^{b(q-1)+\sigma_2+\tau_2(a-1)}|g(y)|^q\mathrm{d}y\right)^{\frac{1}{q}} = \\ &W_1^{\frac{1}{p}}(-b)W_2^{\frac{1}{q}}(-a)\|f\|_{p,\alpha}\|g\|_{q,\beta}. \end{aligned}$$

故式(3)成立.

若$\tau_1\tau_2 = 1$, $\sigma_1+\tau_1\sigma_2 = 0$, $\tau_1 b - a = \tau_1 - \sigma_1 - 1$, 根据引理2, 也有$\tau_2 a - b = \tau_2 - \sigma_2 - 1$, 再根据引理3, $W_1^{\frac{1}{p}}(-b)W_2^{\frac{1}{q}}(-a) = (\frac{1}{|\tau_1|})^{\frac{1}{q}}W_1(-b) = (\frac{1}{|\tau_2|})^{\frac{1}{p}}W_2(-a)$, 同时经简单计算可知$\alpha = ap-1$, $\beta = bq-1$, 于是式(3)化为式(4).

(ii) 充分性 设$\tau_1 b - a = \tau_1 - \sigma_1 - 1$, 根据(i)可知式(3)已化为式(4), 若式(4)的常数因子不是最佳的, 则存在常数$M_0 > 0$使

$$M_0 < \left(\frac{1}{|\tau_1|}\right)^{\frac{1}{q}}W_1(-b), \quad A(K,f,g) \le M_0\|f\|_{p,ap-1}\|g\|_{q,bq-1}.$$

若$\tau_1 < 0$, 取$\varepsilon > 0$及$\delta > 0$充分小, 令

$$f(x) = \begin{cases} x^{-(ap-\tau_1\varepsilon)/p}, & x \geq 1, \\ 0, & 0 < x < 1, \end{cases}$$

$$g(y) = \begin{cases} y^{-(bq+\varepsilon)/q}, & x \geq \delta, \\ 0, & 0 < x < \delta, \end{cases}$$

则有

$$M_0\|f\|_{p,ap-1}\|g\|_{q,bq-1} = M_0\left(\int_1^{+\infty} x^{-1+\tau_1\varepsilon}\mathrm{d}x\right)^{\frac{1}{p}}\left(\int_\delta^{+\infty} y^{-1-\varepsilon}\mathrm{d}y\right)^{\frac{1}{q}} = \frac{M_0}{\varepsilon}\left(\frac{1}{|\tau_1|}\right)^{\frac{1}{p}}\delta^{-\frac{\varepsilon}{q}},$$

$$A(K,f,g) = \int_1^{+\infty} x^{-a+\frac{\tau_1\varepsilon}{p}}\left(\int_\delta^{+\infty} K(x,y)y^{-b-\frac{\varepsilon}{q}}\mathrm{d}y\right)\mathrm{d}x = \int_1^{+\infty} x^{\sigma_1-a+\frac{\tau_1\varepsilon}{p}}\left(\int_\delta^{+\infty} K(1,x^{\tau_1}y)y^{-b-\frac{\varepsilon}{q}}\mathrm{d}y\right)\mathrm{d}x = \int_1^{+\infty} x^{\sigma_1-a+\frac{\tau_1\varepsilon}{p}+\tau_1(b-1)+\frac{\tau_1\varepsilon}{q}}\left(\int_{\delta x^{\tau_1}}^{+\infty} K(1,t)t^{-b-\frac{\varepsilon}{q}}\mathrm{d}t\right)\mathrm{d}x \geq \int_1^{+\infty} x^{-1+\tau_1\varepsilon}\left(\int_\delta^{+\infty} K(1,t)t^{-b-\frac{\varepsilon}{q}}\mathrm{d}t\right)\mathrm{d}x = \frac{1}{|\tau_1|\varepsilon}\int_\delta^{+\infty} K(1,t)t^{-b-\frac{\varepsilon}{q}}\mathrm{d}t,$$

于是得到

$$\frac{1}{|\tau_1|}\int_\delta^{+\infty} K(1,t)t^{-b-\frac{\varepsilon}{q}}\mathrm{d}t \leq M_0\left(\frac{1}{|\tau_1|}\right)^{\frac{1}{p}}\delta^{-\frac{\varepsilon}{q}}, \tag{5}$$

令

$$F_1(t) = \begin{cases} K(1,t)t^{-b-\delta_0}, & 0 < t \leq 1, \\ K(1,t)t^{-b}, & t > 1, \end{cases}$$

则$K(1,t)t^{-b-\frac{\varepsilon}{q}} \leq F_1(t)$, 且

$$\int_\delta^{+\infty} F_1(t)\mathrm{d}t = \int_\delta^1 K(1,t)t^{-b-\delta_0}\mathrm{d}t + \int_1^{+\infty} K(1,t)t^{-b}\mathrm{d}t \leq W_1(-b-\delta_0) + W_1(-b) < +\infty,$$

于是根据Lebesgue控制收敛定理, 有

$$\lim_{\varepsilon\to 0^+}\int_\delta^{+\infty} K(1,t)t^{-b-\frac{\varepsilon}{q}}\mathrm{d}t = \int_\delta^{+\infty} K(1,t)t^{-b}\mathrm{d}t,$$

在式(5)中令$\varepsilon \to 0^+$, 得

$$\frac{1}{|\tau_1|}\int_\delta^{+\infty} K(1,t)t^{-b}\mathrm{d}t \leq M_0\left(\frac{1}{|\tau_1|}\right)^{\frac{1}{p}},$$

再令$\delta \to 0^+$, 得

$$\left(\frac{1}{|\tau_1|}\right)^{\frac{1}{q}} W_1(-b) = \left(\frac{1}{|\tau_1|}\right)^{\frac{1}{q}}\int_0^{+\infty} K(1,t)t^{-b}\mathrm{d}t \leq M_0,$$

这与$M_0 < (\frac{1}{|\tau_1|})^{\frac{1}{q}} W_1(-b)$矛盾, 故此时式(4)中的常数因子是最佳的.

若$\tau_1 > 0$, 取$\varepsilon > 0$充分小, n足够大, 令

$$f(x) = \begin{cases} x^{-(ap+\tau_1\varepsilon)/p}, & x \geq 1, \\ 0, & 0 < x < 1, \end{cases}$$

$$g(y) = \begin{cases} y^{-(bq-\varepsilon)/q}, & 0 < y \leq n, \\ 0, & y > n, \end{cases}$$

则有

$$M_0 \|f\|_{p,ap-1} \|g\|_{q,bq-1} = M_0 \left(\int_1^{+\infty} x^{-1-\tau_1\varepsilon} \mathrm{d}x \right)^{\frac{1}{p}} \left(\int_0^n y^{-1+\varepsilon} \mathrm{d}y \right)^{\frac{1}{q}} = \frac{M_0}{\varepsilon} \left(\frac{1}{\tau_1} \right)^{\frac{1}{p}} n^{\frac{\varepsilon}{q}},$$

$$A(K,f,g) = \int_1^{+\infty} x^{-a-\frac{\tau_1\varepsilon}{p}} \left(\int_0^n K(x,y) y^{-b+\frac{\varepsilon}{q}} \mathrm{d}y \right) \mathrm{d}x = \int_1^{+\infty} x^{\sigma_1-a-\frac{\tau_1\varepsilon}{p}} \left(\int_0^n K(1,x^{\tau_1}y) y^{-b+\frac{\varepsilon}{q}} \mathrm{d}y \right) \mathrm{d}x = \int_1^{+\infty} x^{-1-\tau_1\varepsilon} \left(\int_0^{nx^{\tau_1}} K(1,t) t^{-b+\frac{\varepsilon}{q}} \mathrm{d}t \right) \mathrm{d}x \geq \int_1^{+\infty} x^{-1-\tau_1\varepsilon} \left(\int_0^n K(1,t) t^{-b+\frac{\varepsilon}{q}} \mathrm{d}t \right) \mathrm{d}x = \frac{1}{\tau_1\varepsilon} \int_0^n K(1,t) t^{-b+\frac{\varepsilon}{q}} \mathrm{d}t,$$

于是

$$\frac{1}{\tau_1} \int_0^n K(1,t) t^{-b+\frac{\varepsilon}{q}} \mathrm{d}t \leq M_0 \left(\frac{1}{\tau_1} \right)^{\frac{1}{p}} n^{\frac{\varepsilon}{q}}, \tag{6}$$

令

$$F_2(t) = \begin{cases} K(1,t) t^{-b}, & 0 < t \leq 1, \\ K(1,t) t^{-b+\delta_0}, & t > 1. \end{cases}$$

则$K(1,t)t^{-b+\frac{\varepsilon}{q}} \leq F_2(t)$, 且

$$\int_0^n F_1(t) \mathrm{d}t = \int_0^1 K(1,t) t^{-b} \mathrm{d}t + \int_1^n K(1,t) t^{-b+\delta_0} \mathrm{d}t \leq W_1(-b) + W_1(-b+\delta_0) < +\infty,$$

根据Lebesgue控制收敛定理, 在式(6)中令$\varepsilon \to 0^+$, 有

$$\frac{1}{\tau_1} \int_0^n K(1,t) t^{-b} \mathrm{d}t \leq M_0 \left(\frac{1}{\tau_1} \right)^{\frac{1}{p}},$$

再令$n \to +\infty$, 得

$$\left(\frac{1}{|\tau_1|} \right)^{\frac{1}{q}} W_1(-b) = \left(\frac{1}{\tau_1} \right)^{\frac{1}{q}} \int_0^{+\infty} K(1,t) t^{-b} \mathrm{d}t \leq M_0,$$

这与$M_0 < (\frac{1}{|\tau_1|})^{\frac{1}{q}} W_1(-b)$矛盾, 故此时式(4)中的常数因子也是最佳的.

必要性 设式(3)中的常数因子$W_1^{\frac{1}{p}}(-b)W_2^{\frac{1}{q}}(-a)$是最佳常值. 因为$\tau_1\tau_2=1$, $\sigma_1+\tau_1\sigma_2=0$, 可知$\sigma_1\sigma_2\neq 0$或$\sigma_1=\sigma_2=0$.

对于$\sigma_1\sigma_2\neq 0$的情形, 有$\tau_1=-\frac{\sigma_1}{\sigma_2}$, $\tau_2=-\frac{\sigma_2}{\sigma_1}$, 于是$\tau_1 b-a=\tau_1-\sigma_1-1$化为$\sigma_1 b+\sigma_2 a=\sigma_1+\sigma_2+\sigma_1\sigma_2$. 记

$$\sigma_1 b+\sigma_2 a-(\sigma_1+\sigma_2+\sigma_1\sigma_2)=c_1,\ a_1=a-\frac{c_1}{\sigma_2 p},\ b_1=b-\frac{c_1}{\sigma_1 q},$$

则经简单的计算可知$\sigma_1 b_1+\sigma_2 a_1=\sigma_1+\sigma_2+\sigma_1\sigma_2$, $\alpha=a_1p-1$, $\beta=b_1q-1$, 且

$$W_2(-a)=\int_0^{+\infty}K(t,1)t^{-a}\mathrm{d}t=\int_0^{+\infty}K(1,t^{-\frac{\sigma_1}{\sigma_2}})t^{\sigma_1-a}\mathrm{d}t=$$
$$\left|\frac{\sigma_2}{\sigma_1}\right|\int_0^{+\infty}K(1,u)u^{-b+\frac{c_1}{\sigma_1}}\mathrm{d}u=\left|\frac{\sigma_2}{\sigma_1}\right|W_1(-b+\frac{c_1}{\sigma_1}),$$

于是式(3)化为等价式

$$A(K,f,g)\leq\left|\frac{\sigma_2}{\sigma_1}\right|^{\frac{1}{q}}W_1^{\frac{1}{p}}(-b)W_1^{\frac{1}{q}}(-b+\frac{c_1}{\sigma_1})\|f\|_{p,a_1p-1}\|g\|_{q,b_1q-1},\tag{7}$$

由于式(3)的常数因子最佳, 故式(7)的最佳常数因子是

$$\left|\frac{\sigma_2}{\sigma_1}\right|^{\frac{1}{q}}W_1^{\frac{1}{p}}(-b)W_1^{\frac{1}{q}}(-b+\frac{c_1}{\sigma_1}).$$

又因为$\sigma_1 b_1+\sigma_2 a_1=\sigma_1+\sigma_2+\sigma_1\sigma_2$, 即$\tau_1 b_1-a_1=\tau_1-\sigma_1-1$, 根据前面充分性的证明, 可知式(7)的最佳常数因子应为

$$\left(\frac{1}{|\tau_1|}\right)^{\frac{1}{q}}W_1(-b_1)=\left|\frac{\sigma_2}{\sigma_1}\right|^{\frac{1}{q}}W_1(-b+\frac{c_1}{\sigma_1 q}),$$

从而可得

$$W_1(-b+\frac{c_1}{\sigma_1 q})=W_1^{\frac{1}{p}}(-b)W_1^{\frac{1}{q}}(-b+\frac{c_1}{\sigma_1}).\tag{8}$$

根据Hölder积分不等式, 有

$$W_1(-b+\frac{c_1}{\sigma_1 q})=\int_0^{+\infty}t^{\frac{c_1}{\sigma_1 q}}K(1,t)t^{-b}\mathrm{d}t\leq$$
$$\left(\int_0^{+\infty}K(1,t)t^{-b}\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_0^{+\infty}t^{\frac{c_1}{\sigma_1}}K(1,t)t^{-b}\mathrm{d}t\right)^{\frac{1}{q}}=$$
$$W_1^{\frac{1}{p}}(-b)W_1^{\frac{1}{q}}(-b+\frac{c_1}{\sigma_1}),\tag{9}$$

由式(8)知式(9)取等号, 根据Hölder积分不等式取等号的条件, 有$t^{\frac{c_1}{\sigma_1}}=$常数, 故$c_1=0$, 即$\sigma_1 b+\sigma_2 a=\sigma_1+\sigma_2+\sigma_1\sigma_2$, 即$\tau_1 b-a=\tau_1-\sigma_1-1$.

对于$\sigma_1=\sigma_2=0$的情形, $\tau_1 b-a=\tau_1-\sigma_1-1$化为$\tau_1 b-a=\tau_1-1$. 因为$\tau_1\tau_2=1$, 可令$\tau_1=\frac{\lambda_1}{\lambda_2}$, $\tau_2=\frac{\lambda_2}{\lambda_1}$, 于是$\tau_1 b-a=\tau_1-1$又化为了$\lambda_1(b-1)=\lambda_2(a-1)$. 记

$$\lambda_1(b-1)-\lambda_2(a-1)=c_2,\ a_2=a+\frac{c_2}{\lambda_2 p},\ b_2=b-\frac{c_2}{\lambda_1 q},$$

则经简单计算, 可得$\lambda_1(b_2-1)=\lambda_2(a_2-1)$, $\alpha=a_2p-1$, $\beta=b_2q-1$, 且

$$W_2(-a)=\int_0^{+\infty}K(t,1)t^{-a}\mathrm{d}t=\int_0^{+\infty}K(1,t^{\frac{\lambda_1}{\lambda_2}})t^{-a}\mathrm{d}t=$$
$$\left|\frac{\lambda_2}{\lambda_1}\right|\int_0^{+\infty}K(1,u)u^{-b+\frac{c_2}{\lambda_1}}\mathrm{d}u=\left|\frac{\lambda_2}{\lambda_1}\right|W_1(-b+\frac{c_2}{\lambda_1}),$$

于是式(3)等价地化为

$$A(K,f,g)=\left|\frac{\lambda_2}{\lambda_1}\right|^{\frac{1}{q}}W_1^{\frac{1}{p}}(-b)W_1^{\frac{1}{q}}(-b+\frac{c_2}{\lambda_1})\|f\|_{p,a_2p-1}\|g\|_{q,b_2q-1}, \tag{10}$$

由于式(3)的常数因子最佳, 故式(10)的最佳常数因子是

$$\left|\frac{\lambda_2}{\lambda_1}\right|^{\frac{1}{q}}W_1^{\frac{1}{p}}(-b)W_1^{\frac{1}{q}}(-b+\frac{c_2}{\lambda_1}).$$

又因为$\lambda_1(b_2-1)=\lambda_2(a_2-1)$, 即$\tau_1b_2-a_2=\tau_1-1$, 根据上面充分性的证明, 可知式(10)的最佳常数因子应为

$$\left(\frac{1}{|\tau_1|}\right)^{\frac{1}{q}}W_1(-b_2)=\left|\frac{\lambda_2}{\lambda_1}\right|^{\frac{1}{q}}W_1(-b+\frac{c_2}{\lambda_1q}),$$

从而可得

$$W_1(-b+\frac{c_2}{\lambda_1q})=W_1^{\frac{1}{p}}(-b)W_1^{\frac{1}{q}}(-b+\frac{c_2}{\lambda_1}).$$

与前面证明相似, 利用Hölder积分不等式取等号的条件, 也可得到$t^{\frac{c_2}{\lambda_1}}=$常数, 故$c_2=0$, 即$\lambda_1(b-1)=\lambda_2(a-1)$, 所以$\tau_1b-a=\tau_1-1$.

§4 加权lebesgue空间中积分算子的最佳搭配参数

设$K(x,y)\geq0$, 在加权Lebesgue空间中定义积分算子

$$T(f)(y)=\int_0^{+\infty}K(x,y)f(x)\mathrm{d}x,\quad f\in L_p^{\alpha}(0,+\infty). \tag{11}$$

讨论算子T的有界性及算子范数是算子理论的一个重要问题.

若$K(x,y)$是具有参数$\{\sigma_1,\sigma_2,\tau_1,\tau_2\}$超齐次函数, $\frac{1}{p}+\frac{1}{q}=1\,(p>1)$, 根据Hilbert不等式的基本理论, Hilbert型不等式(3)等价于算子不等式

$$\|T(f)\|_{p,\beta(1-p)}\leq W_1^{\frac{1}{p}}(-b)W_2^{\frac{1}{q}}(-a)\|f\|_{p,\alpha}. \tag{12}$$

若T的算子范数$\|T\|=W_1^{\frac{1}{p}}(-b)W_2^{\frac{1}{q}}(-a)$, 则称搭配参数$a,b$为最佳搭配参数, a,b与式(12)中涉及的参数$p,q,\sigma_1,\sigma_2,\tau_1$及$\tau_2$应满足什么条件才能成为最佳搭配参数, 这是本文要解决的问题.

由于式(3)与式(12)等价, 于是根据定理1, 可得下面的定理2.

定理2 设$\frac{1}{p}+\frac{1}{q}=1\,(p>1)$, $a,b\in\mathbf{R}$, $\tau_1\tau_2\neq0$, $K(x,y)$是具有参数$\{\sigma_1,\sigma_2,\tau_1,\tau_2\}$的超齐次非负可测函数, $0<W_1(-b)<+\infty$, $0<W_2(-a)<+\infty$, 存在$\delta_0>0$使$W_1(-b\pm\delta_0)<+\infty$或$W_2(-a\pm\delta_0)<+\infty$, 积分算子$T$由式(11)定义.

(i) 记

$$\alpha=a(p-1)+\sigma_1+\tau_1(b-1),\ \beta=b(q-1)+\sigma_2+\tau_2(a-1),$$

则T是$L_p^{\alpha}(0,+\infty)$到$L_p^{\beta(1-p)}(0,+\infty)$的有界算子, 且$T$的算子范数

$$\|T\|\leq W_1^{\frac{1}{p}}(-b)W_2^{\frac{1}{q}}(-a).$$

(ii) 若$\tau_1\tau_2=1$, $\sigma_1+\tau_1\sigma_2=0$, 则当且仅当$\tau_1 b-a=\tau_1-\sigma_1-1$时, a,b是最佳搭配参数, 即$\|T\|=W_1^{\frac{1}{p}}(-b)W_2^{\frac{1}{q}}(-a)$, 且当$\tau_1 b-a=\tau_1-\sigma_1-1$时, 算子$T: L_p^{ap-1}(0,+\infty)\to L_p^{(bq-1)(1-p)}(0,+\infty)$的算子范数为

$$\|T\|=\left(\frac{1}{|\tau_1|}\right)^{\frac{1}{q}}W_1(-b)=\left(\frac{1}{|\tau_2|}\right)^{\frac{1}{p}}W_2(-a).$$

在定理2中取$a=\frac{1}{p}$, $b=\frac{1}{q}$, 则$\alpha=\frac{1}{q}-\frac{\tau_1}{p}+\sigma_1$, $\beta=\frac{1}{p}-\frac{\tau_2}{q}+\sigma_2$, 且$\tau_1 b-a=\tau_1-\sigma_1-1$化为了$\frac{\tau_1}{p}-\frac{1}{q}=\sigma_1$, 于是根据定理2, 可得下面推论1.

推论1　设$\frac{1}{p}+\frac{1}{q}=1\,(p>1)$, $\tau_1\tau_2\neq 0$, $K(x,y)$是具有参数$\{\sigma_1,\sigma_2,\tau_1,\tau_2\}$的超齐次非负函数, $0<W_1(-\frac{1}{q})<+\infty$, $0<W_2(-\frac{1}{p})<+\infty$, 存在$\delta_0>0$使$W_1(-\frac{1}{q}\pm\delta_0)<+\infty$或$W_2(-\frac{1}{p}\pm\delta_0)<+\infty$, 积分算子$T$由式(11)定义.

(i) 记$\alpha=\frac{1}{q}-\frac{\tau_1}{p}+\sigma_1$, $\beta=\frac{1}{p}-\frac{\tau_2}{q}+\sigma_2$, 则$T$是$L_p^{\alpha}(0,+\infty)$到$L_p^{\beta(1-p)}(0,+\infty)$的有界算子, 且$T$的算子范数

$$\|T\|\leq W_1^{\frac{1}{p}}(-\frac{1}{q})W_2^{\frac{1}{q}}(-\frac{1}{p}).$$

(ii) 若$\tau_1\tau_2=1$, $\sigma_1+\tau_1\sigma_2=0$, 则当且仅当$\frac{\tau_1}{p}-\frac{1}{q}=\sigma_1$时, $\|T\|=W_1^{\frac{1}{p}}(-\frac{1}{q})W_2^{\frac{1}{q}}(-\frac{1}{p})$, 且当$\frac{\tau_1}{p}-\frac{1}{q}=\sigma_1$时, 有

$$\|T\|=\left(\frac{1}{|\tau_1|}\right)^{\frac{1}{q}}W_1(-\frac{1}{q})=\left(\frac{1}{|\tau_2|}\right)^{\frac{1}{p}}W_2(-\frac{1}{p}).$$

推论2　设$\frac{1}{p}+\frac{1}{q}=1\,(p>1)$, $0<\frac{1-a}{\lambda_1}<1$, $0<\frac{1-b}{\lambda_2}<1$, $\frac{1-a}{\lambda_1}+\frac{1-b}{\lambda_2}=1$, 则积分算子

$$T(f)(y)=\int_0^{+\infty}\frac{\ln(\frac{x^{\lambda_1}}{y^{\lambda_2}})}{x^{\lambda_1}-y^{\lambda_2}}f(x)\mathrm{d}x,\quad f\in L_p^{ap-1}(0,+\infty)$$

是$L_p^{ap-1}(0,+\infty)$到$L_p^{(bq-1)(1-p)}(0,+\infty)$的有界算子, 且$T$的算子范数为

$$\|T\|=\frac{1}{|\lambda_1|^{\frac{1}{q}}|\lambda_2|^{\frac{1}{p}}}\left(\frac{\pi}{\sin\frac{\pi}{\lambda_1}(1-a)}\right)^2.$$

证　令

$$K(x,y)=\frac{\ln(\frac{x^{\lambda_1}}{y^{\lambda_2}})}{x^{\lambda_1}-y^{\lambda_2}},\ (x>0,y>0)$$

则$K(x,y)$是具有参数$\{-\lambda_1,-\lambda_2,-\frac{\lambda_1}{\lambda_2},-\frac{\lambda_2}{\lambda_1}\}$的超齐次非负函数. 因为$\sigma_1=-\lambda_1$, $\sigma_2=-\lambda_2$, $\tau_1=-\frac{\lambda_1}{\lambda_2}$, $\tau_2=-\frac{\lambda_2}{\lambda_1}$, 故有$\tau_1\tau_2=1$, $\sigma_1+\tau_1\sigma_2=0$. 又因为$\frac{1-a}{\lambda_1}+\frac{1-b}{\lambda_2}=1$, 从而$\tau_1 b-a=\tau_1-\sigma_1-1$.

由于$0<\frac{1-a}{\lambda_1}<1$, $0<\frac{1-b}{\lambda_2}<1$, $\frac{1-a}{\lambda_1}+\frac{1-b}{\lambda_2}=1$, 根据Beta函数性质, 有

$$W_1(-b)=\int_0^{+\infty}K(1,t)t^{-b}\mathrm{d}t=\int_0^{+\infty}\frac{\ln(t^{-\lambda_2})}{1-t^{\lambda_2}}t^{-b}\mathrm{d}t=$$
$$\frac{1}{|\lambda_2|}\int_0^{+\infty}\frac{\ln u}{u-1}u^{\frac{1-b}{\lambda_2}-1}\mathrm{d}u=\frac{1}{|\lambda_2|}B^2\left(\frac{1-b}{\lambda_2},1-\frac{1-b}{\lambda_2}\right)=$$
$$\frac{1}{|\lambda_2|}B^2\left(\frac{1-a}{\lambda_2},\frac{1-b}{\lambda_2}\right)=\frac{1}{|\lambda_2|}\left(\frac{\pi}{\sin\frac{\pi}{\lambda_1}(1-a)}\right)^2<+\infty,$$

同理也可得

$$W_2(-a)=\frac{1}{|\lambda_1|}\left(\frac{\pi}{\sin\frac{\pi}{\lambda_2}(1-b)}\right)^2=\frac{1}{|\lambda_1|}\left(\frac{\pi}{\sin\frac{\pi}{\lambda_1}(1-a)}\right)^2<+\infty.$$

因为$0 < \frac{1-a}{\lambda_1} < 1$, $0 < \frac{1-b}{\lambda_2} < 1$, 由实数的稠密性, 存在$\delta_0 > 0$使$0 < \frac{1-a}{\lambda_1} \mp \frac{\delta_0}{\lambda_2} < 1$, $0 < \frac{1-b}{\lambda_2} \pm \frac{\delta_0}{\lambda_2} < 1$, 于是类似于前面的计算可得

$$W_1(-b \pm \delta_0) = \frac{1}{|\lambda_2|} B^2\left(\frac{1}{\lambda_2}[1-(b \mp \delta_0)], 1 - \frac{1}{\lambda_2}[1-(b \mp \delta_0)]\right) =$$

$$\frac{1}{|\lambda_2|} B^2\left(\frac{1-b}{\lambda_2} \pm \frac{\delta_0}{\lambda_2}, 1 - \frac{1-b}{\lambda_2} \mp \frac{\delta_0}{\lambda_2}\right) =$$

$$\frac{1}{|\lambda_2|} B^2\left(\frac{1-b}{\lambda_2} \pm \frac{\delta_0}{\lambda_2}, \frac{1-a}{\lambda_1} \mp \frac{\delta_0}{\lambda_2}\right) < +\infty.$$

综上所述, 根据定理2, 知本推论的结论成立.

在推论2中, 取$a = \frac{1}{p}$, $b = \frac{1}{q}$, 则可得下面推论3.

推论3 设$\frac{1}{p} + \frac{1}{q} = 1\,(p > 1)$, $\lambda_1 > \frac{1}{q}$, $\lambda_2 > \frac{1}{p}$, $\frac{1}{\lambda_1 q} + \frac{1}{\lambda_2 p} = 1$, 则积分算子

$$T(f)(y) = \int_0^{+\infty} \frac{\ln(\frac{x^{\lambda_1}}{y^{\lambda_2}})}{x^{\lambda_1} - y^{\lambda_2}} f(x)\mathrm{d}x, \quad f \in L_p(0, +\infty)$$

是$L_p(0, +\infty)$中的有界算子, 且T的算子范数为

$$\|T\| = \frac{1}{|\lambda_1|^{\frac{1}{q}} |\lambda_2|^{\frac{1}{p}}} \left(\frac{\pi}{\sin \frac{\pi}{\lambda_1 q}}\right)^2.$$

设$\frac{1}{r} + \frac{1}{s} = 1\,(r > 1)$, 在推论2中取$a = 1 - \frac{\lambda_1}{r}$, $b = 1 - \frac{\lambda_2}{s}$, 则可得推论4.

推论4 设$\frac{1}{p} + \frac{1}{q} = 1\,(p > 1)$, $\frac{1}{r} + \frac{1}{s} = 1\,(r > 1)$, $\alpha = p(\frac{1}{q} - \frac{\lambda_1}{r})$, $\gamma = p(\frac{\lambda_2}{s} - \frac{1}{p})$, 则积分算子

$$T(f)(y) = \int_0^{+\infty} \frac{\ln(\frac{x^{\lambda_1}}{y^{\lambda_2}})}{x^{\lambda_1} - y^{\lambda_2}} f(x)\mathrm{d}x, \quad f \in L_p^{\alpha}(0, +\infty)$$

是$L_p^{\alpha}(0, +\infty)$到$L_p^{\gamma}(0, +\infty)$的有界算子, 且T的算子范数为

$$\|T\| = \frac{1}{|\lambda_1|^{\frac{1}{q}} |\lambda_2|^{\frac{1}{p}}} \left(\frac{\pi}{\sin \frac{\pi}{r}}\right)^2.$$

参考文献:

[1] Hardy G H. Note on a theorem of Hilbert erning series of positive terms[J]. Proceedings of the London Mathematical Society, 1925, 23: 45-48.

[2] Yang Bisheng. On an extension of Hilbert's integral inequality with some parameters[J]. The Australian Journal of Mathematical Analysis and Applications, 2004, 1(1): 1-8.

[3] Yang Bicheng. On Hilbert's integral inequality[J]. Journal of Mathematical Analysis and Applications, 1998, 220: 778-785.

[4] Xu Jingshi. Hardy-Hilbert's inequalities with two parameters[J]. Advances in Mathematics, 2007, 36(2): 189-202.

[5] 洪勇, 吴春阳, 陈强. 一类非齐次核的最佳Hilbert型积分不等式的最佳搭配参数条件[J]. 吉林大学学报(理学版), 2021, 59(2): 207-212.

[6] Rassias M T H, Yang Bicheng, Raigorodskii A. Equivalent properties of two kinds of Hardy-type integral inequalities[J]. Symmetry, 2021, 13(6): 1006, 17pages.

[7] 洪勇. 带对称齐次核的级数算子的范数刻画及其应用[J]. 数学学报, 2008, 51(2): 363-370.

[8] 洪勇, 温雅敏. 齐次核的Hilbert型级数不等式取最佳常数因子的充要条件[J]. 数学年刊A辑, 2016, 37(3): 329-336.

[9] 洪勇, 和炳. Hilbert型不等式的理论与应用[M]. 北京: 科学出版社, 2022.

[10] Yang Bicheng. On best extensions of Hardy-Hilbert's inequality with two parameters[J]. Journal of Inequalities in Pure and Applied Mathematics, 2005, 6(3): 81, 15 pages.

[11] Hong Yong, Huang Qiliang, Yang Bicheng, et al. The necessary and sufficient conditions for the existence of a kind of Hilbert-type multiple integral inequality with the non-homogeneous kernel and its applications[J]. Journal of Inequalities and Applications, 2017, 316: 1-12.

[12] Wang Aaizhen, Yang Bicheng. Equivalent statements of a Hilbert-Type integral inequality with the extended Hurwitz Zeta function in the whole plane[J]. Journal of Mathematical Inequalities, 2020, 14(4): 1039-1054.

[13] Rassias M T H, Yang Bicheng, Raigorodskii A. On a more accurate reverse Hilbert-type inequality in the whole plane[J]. Journal of Mathematics Inequalities, 2020, 14(4): 1359-1374.

[14] Liao Jianquan, Hong Yong, Yang Bicheng. Equivalent conditions of a Hilbert-type multiple integral inequality holding[J]. Journal of Function Spaces, Volume 2020, Article ID 3050952, 6 pages.

[15] He Bing, Hong Yong, Li Zhen. Conditions for the validity of a class of optimal Hilbert type multiple integral inequalities with nonhomogeneous kernels[J]. Journal of Inequalities and Applications, 2021, 64: 1-12.

[16] Wang Aaizhen, Yang Bicheng, Chen Qiang. Equivalent properties of a reverse half-discrete Hilbert's inequality[J]. Journal of Inequalities Applications, 2019, 279: 1-12.

Optimal matching parameters for the super-homogeneous kernel integral operator in weighted Lebesgue space

HONG Yong, ZHAO Qian

(Artificial Intelligence College, Guangzhou Huashang College, Guangzhou 511300, China)

Abstract: The super-homogeneous function is introduced to unify the homogeneous kernel, generalized homogeneous kernel and several non-homogeneous kernels with the super-homogeneous kernel, and then the boundeness of integral operators with super-homogeneous kernel in the weighted Lebesgue space and the operator norm problem are discussed by using the weight function method, and the sufficient necessary conditions for the best matching parameters of this class of integral operators and the formula for the operator norm are obtained, which unifies many previous results.

Keywords: super-homogeneous kernel; integral operator; Hilbert-type integral inequality; weighted Lebesgue space; the best matching parameters; bounded operator; operator norm

MR Subject Classification: 45P05