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加双权的双k-正则函数的Cauchy积分公式

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摘 要: 该文首先给出了Clifford分析中加双权的双k-正则函数的定义, 其次给出了加双权的双k-正则函数的核函数并研究了它的性质, 然后证明了加双权的双k-正则函数的Cauchy-Pompeiu公式、Cauchy积分公式、Cauchy积分定理和平均值定理.

关键词: 加双权的双k-正则函数; Cauchy-Pompeiu公式; Cauchy积分公式; Cauchy积分定理

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§1 引 言

Clifford代数[1]具有可结合不可交换的代数结构, 它起源于20世纪初的几何代数, 并在物理上有许多应用, 例如流体力学、量子力学和弹性力学等.

Clifford分析是上世纪70年代兴起的一个活跃的数学分支, 它是现代理论数学的核心工具之一, 也是复分析的重要推广. Cauchy积分公式是Clifford分析中的重要公式之一, 是研究Plemelj公式、边值问题和T算子性质等问题的基础, 因此研究加双权的双k-正则函数的Cauchy积分公式具有一定的理论和应用价值.

许多专家和学者做了大量的工作, 1976年, Brackx等人[2]引入了实四元数的k-正则函数并给出了它的Cauchy积分公式. 1982年, Brackx等人[3]建立了Clifford分析的理论基础. 2002年, Malonek等人[4]给出了加α-权的Dirac算子. 2006年, 黄沙等人[5]研究了Clifford分析中正则函数的性质和边值问题. 2008年, 王海燕[6]给出了双正则函数的Cauchy积分公式. 2010年, 李小伶等人[7]给出了k-正则函数的Cauchy积分公式及Plemelj公式. 2012年, 乔玉英等人[8]给出了双k-正则函数的Cauchy积分公式. 2017年, García等人[9]给出了次正则函数的Cauchy积分公式. 2018年, 杨贺菊等人[10]给出了加α-权的k-正则函数的Cauchy积分公式. 2020年, Dinh等人[11-12]讨论了Weinstein k-正则函数的积分表示和广义(k_i)正则函数的Cauchy表达式; Blaya等

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人[13]给出了多次正则函数的Cauchy积分公式. 2022年, Peláez等人[14]给出了高阶Dirac方程解的积分表示.

在以上工作的基础上, 本文主要研究了加双权的双k-正则函数的核函数的性质, 以及加双权的双k-正则函数的Cauchy-Pompeiu公式、Cauchy积分公式、Cauchy积分定理和平均值定理, 推广了文献[10]的结果.

§2 预备知识

设$\mathcal{A}$是以$e_0, e_1, e_2, \cdots, e_n; e_1e_2, \cdots, e_{n-1}e_n; e_1e_2\cdots e_n$为基的$2^n$维Clifford代数. $e_i^2=-1$ $(i=1,2,\cdots,n)$, $e_ie_j+e_je_i=-2\delta_{ij}(i,j=1,\cdots,n)$, δ_{ij}是Kronecker符号. $\mathcal{A}$中任一元素都可表示为$a=\sum\limits_A a_Ae_A$, 其中$A=\phi$或者$A=\{\alpha_1,\alpha_2,\cdots,\alpha_h\}\subset\{1,2,\cdots,n\}(1\le\alpha_1<\alpha_2<\cdots<\alpha_h\le n,\ a_A\in\mathbf{R})$, $e_A=e_{\alpha_1}e_{\alpha_2}\cdots e_{\alpha_h}$. 称$x=\sum\limits_{i=1}^n x_ie_i(x_i\in\mathbf{R}, i=1,2,\cdots,n)$为$\mathbf{R}^n$中的向量, 且有$|x|^2=-x^2$, $x^{-1}=-\dfrac{x}{|x|^2}$ $(x\ne 0)$.

设$\Omega_1\times\Omega_2\subset\mathbf{R}^n\times\mathbf{R}^m$是非空连通开集. 记$\Omega_1^*\times\Omega_2^*=\{(u,v)|u+x\in\Omega_1, x\in\overline{\Omega_1}; v+y\in\Omega_2, y\in\overline{\Omega_2}\}$. 函数$f(x,y):\Omega_1\times\Omega_2\to\mathcal{A}$可表示为$f(x,y)=\sum\limits_A f_A(x,y)e_A$, 其中$f_A(x,y)$是实值函数. 函数$f(x,y)$在$\Omega_1\times\Omega_2$上是连续的是指$f(x,y)$的每个分量在$\Omega_1\times\Omega_2$上是连续的.

令$F^{(r)}_{\Omega_1\times\Omega_2}=\{f|f:\Omega_1\times\Omega_2\to\mathcal{A}, f(x,y)=\sum\limits_A f_A(x,y)e_A$, 其中$f_A(x,y)$在$\Omega_1\times\Omega_2$上是$r$次连续可微的, $r\in\mathbf{N}$, $\mathbf{N}$是正整数集$\}$.

对于任意的$f\in F^{(1)}_{\Omega_1\times\Omega_2}$, 定义加$\alpha$-权和加$\beta$-权的Dirac算子为
$$D_x^\alpha f(x,y)=|x|^{-\alpha}x(D_xf(x,y)),\ \ f(x,y)D_y^\beta=(f(x,y)D_y)y|y|^{-\beta},$$
其中$D_xf(x,y)=\sum\limits_{j=1}^n e_j\dfrac{\partial f(x,y)}{\partial x_j}$, $f(x,y)D_y=\sum\limits_{j=1}^n\dfrac{\partial f(x,y)}{\partial y_j}e_j, \alpha,\beta\in\mathbf{R}\backslash\{0\}$.

在本文中, 设Ω为$\mathbf{R}^n$中的非空连通开集.

定义2.1[5] 若$f\in F^{(1)}_\Omega$且在Ω中满足$Df(x)=0\big(f(x)D=0\big)$, 则称f为Ω中的左(右)正则函数, 通常称左正则函数为正则函数.

定义2.2[6] 若$f(x,y)\in F^{(1)}_{\Omega_1\times\Omega_2}$且满足$\begin{cases} D_xf(x,y)=0,\\ f(x,y)D_y=0,\end{cases}$ 则称$f(x,y)$为$\Omega_1\times\Omega_2$中的双正则函数.

定义2.3[10] 若$f\in F^{(k)}_\Omega$且在Ω中满足$(D^\alpha)^kf(x)=0\big(f(x)(D^\alpha)^k=0\big)$, 其中$k\in\mathbf{N}$, 则称$f$为$\Omega$中的加$\alpha$-权的左(右)$k$-正则函数, 通常称加$\alpha$-权的左$k$-正则函数为加$\alpha$-权的$k$-正则函数.

定义2.4 若$f(x,y)\in F^{(k)}_{\Omega_1\times\Omega_2}$且对任意$(x,y)\in\Omega_1\times\Omega_2$, 满足$\begin{cases}(D_x^\alpha)^kf(x,y)=0,\\ f(x,y)(D_y^\beta)^k=0,\end{cases}$ 则称f为$\Omega_1\times\Omega_2$中的加双权的双k-正则函数.

引理2.1[5] 若$f,g\in F^{(1)}_\Omega$, 则
$$D(f(x)g(x))=(Df(x))g(x)+\sum_{j=1}^n e_jf(x)\frac{\partial g(x)}{\partial x_j},$$
$$(f(x)g(x))D=\sum_{j=1}^n\frac{\partial f(x)}{\partial x_j}g(x)e_j+f(x)(g(x)D).$$

引理2.2[5] 设Γ是满足$\overline{\Gamma}\subset\Omega$的任意$n$维链, $f,g\in F_{\Omega}^{(k)},k\geq 1$, 则

$$\int_{\partial\Gamma}f(x)\mathrm{d}\sigma_x g(x)=\int_{\Gamma}[(f(x)D)g(x)+f(x)(Dg(x))]\mathrm{d}x^n,$$

其中$\mathrm{d}\widehat{x_i}=\mathrm{d}x_1\wedge\cdots\wedge\mathrm{d}x_{i-1}\wedge\mathrm{d}x_{i+1}\wedge\cdots\wedge\mathrm{d}x_n,i=1,2,\cdots,n$, $\mathrm{d}\sigma_x=\sum\limits_{i=1}^{n}(-1)^{i-1}e_i\mathrm{d}\widehat{x_i}$, $\mathrm{d}x^n=\mathrm{d}x_1\wedge\cdots\wedge\mathrm{d}x_n$.

本文中, 令$H_{i,j}(x,y)=\dfrac{A_iB_j}{|x|^{n-i\alpha}|y|^{m-j\beta}},A_i=\dfrac{(-1)^{i-1}}{\omega_n\alpha^{i-1}(i-1)!},B_j=\dfrac{(-1)^{j-1}}{\omega_m\beta^{j-1}(j-1)!},i,j\geq 1,i,j\in\mathbf{N},\alpha,\beta\in\mathbf{R}\backslash\{0\},\omega_n$和$\omega_m$分别是$\mathbf{R}^n$和$\mathbf{R}^m$中单位球的表面积.

引理2.3[10] 当$k>1$时, 则

$$D_x(H_{k,k}(x,y)|x|^{-\alpha}x)=(H_{k,k}(x,y)(x)|x|^{-\alpha}x)D_x=H_{k-1,k}(x,y).$$

引理2.4[10] 若$f(x)\in F_{\Omega}^{(r)},r\geq k$, 则对于任意的$x_0\in\overline{\Omega}$, $(D_x^{\alpha})^if(x+x_0)$在$\overline{\Omega^*}$上是有界的, 其中$0<\alpha<\frac{1}{k}$.

引理2.5[10] 若$f\in F_{\Omega}^{(k)}$为Ω中的正则函数, 则

$$(D^{\alpha})^k(|x|^{k\alpha}f(x))=(-1)^kk!\alpha^kf(x).$$

命题2.1 若$f(x,y)\in F_{\Omega_1\times\Omega_2}^{(i+j)}$为$\Omega_1\times\Omega_2$中的双正则函数, 其中$i,j\in\mathbf{N}$, 则对于任意$(x,y)\in\Omega_1\times\Omega_2$, 有

$$(D_x^{\alpha})^i(|x|^{i\alpha}f(x,y)|y|^{j\beta})(D_y^{\beta})^j=(-1)^{i+j}i!j!\alpha^i\beta^jf(x,y).$$

证 由引理2.1和引理2.5可得$(D_x^{\alpha})^i(|x|^{i\alpha}f(x,y)|y|^{j\beta})=(-1)^ii!\alpha^if(x,y)|y|^{j\beta}$,

$$(D_x^{\alpha})^i(|x|^{i\alpha}f(x,y)|y|^{j\beta})D_y^{\beta}=$$
$$(-1)^ii!\alpha^i\Big[\sum_{k=1}^{m}\frac{\partial f(x,y)}{\partial y_k}|y|^{j\beta}e_k+f(x,y)(|y|^{j\beta}D_y)\Big]|y|^{-\beta}y=$$
$$(-1)^{i+1}i!j\alpha^i\beta f(x,y)|y|^{(j-1)\beta},$$
$$(D_x^{\alpha})^i(|x|^{i\alpha}f(x,y)|y|^{j\beta})(D_y^{\beta})^2=(-1)^{i+1}i!j\alpha^i\beta(f(x,y)|y|^{(j-1)\beta})D_y^{\beta}=$$
$$(-1)^{i+1}i!j\alpha^i\beta\Big[\sum_{k=1}^{m}\frac{\partial f(x,y)}{\partial y_k}|y|^{(j-1)\beta}e_k+f(x,y)(|y|^{(j-1)\beta}D_y)\Big]|y|^{-\beta}y=$$
$$(-1)^{i+2}i!j(j-1)\alpha^i\beta^2f(x,y)|y|^{(j-2)\beta},$$
$$\cdots,$$

故$(D_x^{\alpha})^i(|x|^{i\alpha}f(x,y)|y|^{j\beta})(D_y^{\beta})^j=(-1)^{i+j}i!j!\alpha^i\beta^jf(x,y)$.

引理2.6[10] 若$f\in F_{\Omega}^{(k)}$是Ω中正则函数, 则$|x|^{(k-j)\alpha}f$是Ω中加α-权的k-正则函数, 其中$j\leq k,k,j\in\mathbf{N}$.

命题2.2 若$f(x,y)$是$\Omega_1\times\Omega_2$中双正则函数, 则$|x|^{(k-i)\alpha}f(x,y)|y|^{(k-j)\beta}$是$\Omega_1\times\Omega_2$中加双权的双$k$-正则函数, 其中$1\leq i,j\leq k$.

证 由命题2.1和引理2.6可得, $(D_x^{\alpha})^k(|x|^{(k-i)\alpha}f(x,y)|y|^{(k-j)\beta})=0$, 因此只需证明

$$(|x|^{(k-i)\alpha}f(x,y)|y|^{(k-j)\beta})(D_y^{\beta})^k=0.$$

因为

$$(|x|^{(k-i)\alpha}f(x,y)|y|^{(k-j)\beta})(D_y^{\beta})^k=(|x|^{(k-i)\alpha}f(x,y)|y|^{(k-j)\beta}(D_y^{\beta})^{k-j})(D_y^{\beta})^j=$$
$$[(-1)^{k-j}(k-j)!\beta^{k-j}|x|^{(k-i)\alpha}f(x,y)](D_y^{\beta})^j=$$
$$[(-1)^{k-j}(k-j)!\beta^{k-j}|x|^{(k-i)\alpha}f(x,y)D_y]|y|^{-\beta}y(D_y^{\beta})^{j-1}=0.$$

所以$|x|^{(k-i)\alpha}f(x,y)|y|^{(k-j)\beta}$是一个加双权的双$k$-正则函数.

引理2.7[10] $H_k(x)|x|^{-j\alpha}x$为Ω中加α-权的k-正则函数, 其中$j\le k, k, j\in\mathbf{N}$.

引理2.8[10](Cauchy-Pompeiu公式) 若$f\in F_{\overline{\Omega}}^{(k)}, r\ge k, n\ge k$, $0<\alpha<\dfrac{1}{k}$, 则对任意$x_0\in\Omega$都有

$$f(x_0)=\sum_{j=1}^{k}(-1)^j\int_{\partial\Omega^*}H_j(x)|x|^{-\alpha}x\mathrm{d}\sigma_x((D^\alpha)^{j-1}f(x+x_0))-$$
$$(-1)^k\int_{\Omega^*}H_k(x)((D^\alpha)^kf(x+x_0))\mathrm{d}x.$$

引理2.9[10](Cauchy定理) 若$f\in F_{\overline{\Omega}}^{(k)}, r\ge k, n\ge k, 0<\alpha<\dfrac{1}{k}$, $f(x+x_0)$为Ω^*中加α-权的k-正则函数, 则有

$$\sum_{j=1}^{k}(-1)^j\int_{\partial\Omega^*}H_j(x)|x|^{-\alpha}x\mathrm{d}\sigma_x((D^\alpha)^{j-1}f(x+x_0))=\begin{cases}f(x_0), & x_0\in\Omega;\\ 0, & x_0\in\mathbf{R}^n\backslash\overline{\Omega}.\end{cases}$$

§3 加双权的双k-正则函数的核函数的一些性质

定理3.1 当$i,j>1$时, 有

$$D_x(x|x|^{-\alpha}H_{i,j}(x,y)|y|^{-\beta}y)D_y=H_{i-1,j-1}(x,y).$$

证 由引理2.3可得

$$D_x(x|x|^{-\alpha}H_{i,j}(x,y)|y|^{-\beta}y)=[D_x(x|x|^{-\alpha}H_{i,j}(x,y))]|y|^{-\beta}y=H_{i-1,j}(x,y)|y|^{-\beta}y,$$
$$D_x(x|x|^{-\alpha}H_{i,j}(x,y)|y|^{-\beta}y)D_y=(H_{i-1,j}(x,y)|y|^{-\beta}y)D_y=H_{i-1,j-1}(x,y).$$

注 当$i=1(j=1)$, $x|x|^{-\alpha}H_{i,j}(x,y)|y|^{-\beta}y$是$\Omega_1\times\Omega_2$中的关于$x(y)$的左(右)的正则函数; 当$i=j=1$时, $x|x|^{-\alpha}H_{i,j}(x,y)|y|^{-\beta}y$是$\Omega_1\times\Omega_2$中双正则函数.

定理3.2 $x|x|^{-i\alpha}H_{k,k}(x,y)|y|^{-j\beta}y$是$\Omega_1\times\Omega_2$中加双权的双$k$-正则函数, 其中$1\le i,j\le k$.

证

$$x|x|^{-i\alpha}H_{k,k}(x,y)|y|^{-j\beta}y=\frac{A_k}{A_1}|x|^{(k-i)\alpha}(x|x|^{-\alpha}H_{1,1}(x,y)|y|^{-\beta}y)|y|^{(k-j)\beta}\frac{B_k}{B_1}.$$

由定理3.1可知

$$x|x|^{-\alpha}H_{1,1}(x,y)|y|^{-\beta}y$$

是$\Omega_1\times\Omega_2$中双正则函数, 再由命题2.2可得

$$x|x|^{-i\alpha}H_{k,k}(x,y)|y|^{-j\beta}y$$

是$\Omega_1\times\Omega_2$中加双权的双k-正则函数, 其中$1\le i,j\le k$.

定理3.3 若$f(x,y)\in F^{(r)}_{\overline{\Omega_1}\times\overline{\Omega_2}}, r\ge i+j, 0<\alpha<\dfrac{1}{k_1}, 0<\beta<\dfrac{1}{k_2}, 0\le i\le k_1, 0\le j\le k_2$, 则对任意的$(x,y)\in\overline{\Omega}_1\times\overline{\Omega}_2$, $(D_x^\alpha)^if(u+x,v+y)(D_y^\beta)^j$在$\overline{\Omega_1^*}\times\overline{\Omega_2^*}$是有界的.

证 当$i=0(j=0)$, 应用引理2.4, 可以得到这个结论.

若$i,j>0$, 令

$$\begin{aligned}f_1(x,y)=&D_xf(u+x,v+y);\\f_2(x,y)=&\alpha f_1(x,y)+D_x(xf_1(x,y));\\f_3(x,y)=&2\alpha f_2(x,y)+D_x(xf_2(x,y));\\&\cdots;\\f_i(x,y)=&(i-1)\alpha f_{i-1}(x,y)+D_x(xf_{i-1}(x,y));\\f_{i+1}(x,y)=&f_i(x,y)D_y;\\f_{i+2}(x,y)=&\beta f_{i+1}(x,y)+(f_{i+1}(x,y)y)D_y;\\&\cdots;\\f_{i+j}(x,y)=&(j-1)\beta f_{i+j-1}(x,y)+(f_{i+j-1}(x,y)y)D_y.\end{aligned}$$

显然$f_1(x,y),f_2(x,y),\cdots,f_{i+j}(x,y)$在$\overline{\Omega_1^*}\times\overline{\Omega_2^*}$是有界的. 根据文献[10]可得

$$(D_x^\alpha)^if(u+x,v+y)=|x|^{-i\alpha}xf_i(x,y).$$

$$(D_x^\alpha)^if(u+x,v+y)D_y^\beta=|x|^{-i\alpha}x(f_i(x,y)D_y|y|^{-\beta}y)=|x|^{-i\alpha}xf_{i+1}(x,y)|y|^{-\beta}y;$$

$$\begin{aligned}&(D_x^\alpha)^if(u+x,v+y)(D_y^\beta)^2=|x|^{-i\alpha}x(f_{i+1}(x,y)|y|^{-\beta}y)D_y|y|^{-\beta}y=\\&|x|^{-i\alpha}x[f_{i+1}(x,y)y(|y|^{-\beta}D_y)+|y|^{-\beta}(f_{i+1}(x,y)y)D_y]|y|^{-\beta}y=\\&|x|^{-i\alpha}xf_{i+2}(x,y)|y|^{-2\beta}y;\\&\quad\cdots;\\&\quad(D_x^\alpha)^if(u+x,v+y)(D_y^\beta)^j=|x|^{-i\alpha}x(f_{i+j-1}(x,y)|y|^{-(j-1)\beta}y)D_y|y|^{-\beta}y=\\&\quad|x|^{-i\alpha}x[f_{i+j-1}(x,y)y(|y|^{-(j-1)\beta}D_y)+|y|^{-(j-1)\beta}(f_{i+j-1}(x,y)y)D_y]|y|^{-\beta}y=\\&\quad|x|^{-i\alpha}xf_{i+j}(x,y)|y|^{-j\beta}y.\end{aligned}$$

因为$0<\alpha<\dfrac{1}{k_1},0<\beta<\dfrac{1}{k_2}$, 因此$i\alpha<1$且$j\beta<1$, 则对于任意的$(x,y)\in\overline{\Omega}_1\times\overline{\Omega}_2$, 都有$(D_x^\alpha)^if(u+x,v+y)(D_y^\beta)^j$在$\overline{\Omega_1^*}\times\overline{\Omega_2^*}$上是有界的.

§4 加双权的双k-正则函数的Cauchy积分公式

定理4.1 (Cauchy-Pompeiu公式) 若$f(x,y)\in F^{(r)}_{\overline{\Omega_1}\times\overline{\Omega_2}}$, 其中$r\geq k_1+k_2$, $n\geq k_1,m\geq k_2$, $0<\alpha<\dfrac{1}{k_1},0<\beta<\dfrac{1}{k_2}$, 则对于任意$(x,y)\in\Omega_1\times\Omega_2$, 都有

$$\begin{aligned}f(x,y)=&\sum_{i=1}^{k_1}\sum_{j=1}^{k_2}(-1)^{i+j}\int_{\partial\Omega_1^*\times\partial\Omega_2^*}|u|^{-\alpha}uH_{i,j}(u,v)\mathrm{d}\sigma_u\big((D_u^\alpha)^{i-1}f(u+x,v+y)(D_v^\beta)^{j-1}\big)\mathrm{d}\sigma_v|v|^{-\beta}v-\\&\sum_{i=1}^{k_1}(-1)^{i+k_2}\int_{\partial\Omega_1^*\times\Omega_2^*}|u|^{-\alpha}uH_{i,k_2}(u,v)\mathrm{d}\sigma_u\big((D_u^\alpha)^{i-1}f(u+x,v+y)(D_v^\beta)^{k_2}\big)\mathrm{d}v-\\&\sum_{j=1}^{k_2}(-1)^{j+k_1}\int_{\Omega_1^*\times\partial\Omega_2^*}H_{k_1,j}(u,v)\mathrm{d}u\big((D_u^\alpha)^{k_1}f(u+x,v+y)(D_v^\beta)^{j-1}\big)\mathrm{d}\sigma_v|v|^{-\beta}v+\\&(-1)^{k_1+k_2}\int_{\Omega_1^*\times\Omega_2^*}H_{k_1,k_2}(u,v)\big((D_u^\alpha)^{k_1}f(u+x,v+y)(D_v^\beta)^{k_2}\big)\mathrm{d}u\mathrm{d}v.\end{aligned}$$

证 令

$$I_1=\sum_{i=1}^{k_1}\sum_{j=1}^{k_2}(-1)^{i+j}\int_{\partial\Omega_1^*\times\partial\Omega_2^*}|u|^{-\alpha}uH_{i,j}(u,v)\mathrm{d}\sigma_u\big((D_u^\alpha)^{i-1}f(u+x,v+y)(D_v^\beta)^{j-1}\big)\mathrm{d}\sigma_v|v|^{-\beta}v,$$

$$I_2=\sum_{i=1}^{k_1}(-1)^{i+k_2}\int_{\partial\Omega_1^*\times\Omega_2^*}|u|^{-\alpha}uH_{i,k_2}(u,v)\mathrm{d}\sigma_u\big((D_u^\alpha)^{i-1}f(u+x,v+y)(D_v^\beta)^{k_2}\big)\mathrm{d}v,$$

$$I_3=\sum_{j=1}^{k_2}(-1)^{j+k_1}\int_{\Omega_1^*\times\partial\Omega_2^*}H_{k_1,j}(u,v)\mathrm{d}u\big((D_u^\alpha)^{k_1}f(u+x,v+y)(D_v^\beta)^{j-1}\big)\mathrm{d}\sigma_v|v|^{-\beta}v,$$

$$I_4=(-1)^{k_1+k_2}\int_{\Omega_1^*\times\Omega_2^*}H_{k_1,k_2}(u,v)\big((D_u^\alpha)^{k_1}f(u+x,v+y)(D_v^\beta)^{k_2}\big)\mathrm{d}u\mathrm{d}v.$$

对任意$(x,y)\in\Omega_1\times\Omega_2$, 由于$(0+x,0+y)\in\Omega_1\times\Omega_2$, 因此$(0,0)\in\Omega_1^*\times\Omega_2^*$. 取$\delta_1,\delta_2>0$, 令$B_{\delta_1}=\{u:|u|<\delta_1\}$, $B_{\delta_2}=\{v:|v|<\delta_2\}$, 并满足$\overline{B_{\delta_1}}\subset\Omega_1^*,\overline{B_{\delta_2}}\subset\Omega_2^*$. 则由引理2.2和引理2.3得

$$I_4=\lim_{\substack{\delta_1\to0\\\delta_2\to0}}(-1)^{k_1+k_2}\int_{\{\Omega_1^*\backslash B_{\delta_1}\}\times\{\Omega_2^*\backslash B_{\delta_2}\}}H_{k_1,k_2}(u,v)((D_u^\alpha)^{k_1}f(u+x,v+y)(D_v^\beta)^{k_2})\mathrm{d}v\mathrm{d}u=$$

$$\lim_{\substack{\delta_1\to0\\\delta_2\to0}}(-1)^{k_1+k_2}\int_{\Omega_2^*\backslash B_{\delta_2}}\{\int_{\Omega_1^*\backslash B_{\delta_1}}H_{k_1,k_2}(u,v)|u|^{-\alpha}uD_u((D_u^\alpha)^{k_1-1}f(u+x,v+y)(D_v^\beta)^{k_2})\mathrm{d}u\}\mathrm{d}v=$$

$$\lim_{\substack{\delta_1\to0\\\delta_2\to0}}(-1)^{k_1+k_2}\int_{\Omega_2^*\backslash B_{\delta_2}}\{\int_{\partial\Omega_1^*}|u|^{-\alpha}uH_{k_1,k_2}(u,v)\mathrm{d}\sigma_u((D_u^\alpha)^{k_1-1}f(u+x,v+y)(D_v^\beta)^{k_2})\}\mathrm{d}v-$$

$$\lim_{\substack{\delta_1\to0\\\delta_2\to0}}(-1)^{k_1+k_2}\int_{\Omega_2^*\backslash B_{\delta_2}}\{\int_{\partial B_{\delta_1}}|u|^{-\alpha}uH_{k_1,k_2}(u,v)\mathrm{d}\sigma_u((D_u^\alpha)^{k_1-1}f(u+x,v+y)(D_v^\beta)^{k_2})\}\mathrm{d}v-$$

$$\lim_{\substack{\delta_1\to0\\\delta_2\to0}}(-1)^{k_1+k_2}\int_{\Omega_2^*\backslash B_{\delta_2}}\{\int_{\Omega_1^*\backslash B_{\delta_1}}H_{k_1-1,k_2}(u,v)((D_u^\alpha)^{k_1-1}f(u+x,v+y)(D_v^\beta)^{k_2})\mathrm{d}u\}\mathrm{d}v=$$

$$I_{11}+I_{12}+I_{13};$$

$$I_{13}=$$

$$\lim_{\substack{\delta_1\to0\\\delta_2\to0}}(-1)^{(k_1-1)+k_2}\int_{\Omega_2^*\backslash B_{\delta_2}}\{\int_{\partial\Omega_1^*}|u|^{-\alpha}uH_{k_1-1,k_2}(u,v)\mathrm{d}\sigma_u((D_u^\alpha)^{k_1-2}f(u+x,v+y)(D_v^\beta)^{k_2})\}\mathrm{d}v-$$

$$\lim_{\substack{\delta_1\to0\\\delta_2\to0}}(-1)^{(k_1-1)+k_2}\int_{\Omega_2^*\backslash B_{\delta_2}}\{\int_{\partial B_{\delta_1}}|u|^{-\alpha}uH_{k_1-1,k_2}(u,v)\mathrm{d}\sigma_u((D_u^\alpha)^{k_1-2}f(u+x,v+y)(D_v^\beta)^{k_2})\}\mathrm{d}v-$$

$$\lim_{\substack{\delta_1\to0\\\delta_2\to0}}(-1)^{(k_1-1)+k_2}\int_{\Omega_2^*\backslash B_{\delta_2}}\{\int_{\Omega_1^*\backslash B_{\delta_1}}H_{k_1-2,k_2}(u,v)((D_u^\alpha)^{k_1-2}f(u+x,v+y)(D_v^\beta)^{k_2})\mathrm{d}u\}\mathrm{d}v=$$

$$I_{21}+I_{22}+I_{23};$$

$\cdots$;

$$I_{(k_1-1)3}=\lim_{\substack{\delta_1\to0\\\delta_2\to0}}(-1)^{1+k_2}\int_{\Omega_2^*\backslash B_{\delta_2}}\left\{\int_{\partial\Omega_1^*}|u|^{-\alpha}uH_{1,k_2}(u,v)\mathrm{d}\sigma_uf(u+x,v+y)(D_v^\beta)^{k_2}\right\}\mathrm{d}v-$$

$$\lim_{\substack{\delta_1\to0\\\delta_2\to0}}(-1)^{1+k_2}\int_{\Omega_2^*\backslash B_{\delta_2}}\left\{\int_{\partial B_{\delta_1}}|u|^{-\alpha}uH_{1,k_2}(u,v)\mathrm{d}\sigma_uf(u+x,v+y)(D_v^\beta)^{k_2}\right\}\mathrm{d}v-$$

$$\lim_{\substack{\delta_1\to0\\\delta_2\to0}}(-1)^{1+k_2}\int_{\Omega_2^*\backslash B_{\delta_2}}\left\{\int_{\Omega_1^*\backslash B_{\delta_1}}(|u|^{-\alpha}uH_{1,k_2}(u,v)D_u)f(u+x,v+y)(D_v^\beta)^{k_2}\right\}\mathrm{d}v=$$

$$I_{k_11}+I_{k_12}+I_{k_13}.$$

由定理3.1得$I_{k_13}=0$, 因此

$$I_4=$$

$$\lim_{\substack{\delta_1\to 0\\ \delta_2\to 0}}\sum_{i=1}^{k_1}(-1)^i(-1)^{k_2}\int_{\Omega_2^*\setminus B_{\delta_2}}\left\{\int_{\partial\Omega_1^*}|u|^{-\alpha}uH_{i,k_2}(u,v)\mathrm{d}\sigma_u\big((D_u^\alpha)^{i-1}f(u+x,v+y)(D_v^\beta)^{k_2}\big)\right\}\mathrm{d}v-$$

$$\lim_{\substack{\delta_1\to 0\\ \delta_2\to 0}}\sum_{i=1}^{k_1}(-1)^i(-1)^{k_2}\int_{\Omega_2^*\setminus B_{\delta_2}}\left\{\int_{\partial B_{\delta_1}}|u|^{-\alpha}uH_{i,k_2}(u,v)\mathrm{d}\sigma_u\big((D_u^\alpha)^{i-1}f(u+x,v+y)(D_v^\beta)^{k_2}\big)\right\}\mathrm{d}v=$$

$$I_5-I_6.$$

$$\lim_{\substack{\delta_1\to 0\\ \delta_2\to 0}}(-1)^{k_2}\int_{\Omega_2^*\setminus B_{\delta_2}}\left\{\int_{\partial\Omega_1^*}|u|^{-\alpha}uH_{i,k_2}(u,v)\mathrm{d}\sigma_u\big((D_u^\alpha)^{i-1}f(u+x,v+y)(D_v^\beta)^{k_2}\big)\right\}\mathrm{d}v=$$

$$\lim_{\substack{\delta_1\to 0\\ \delta_2\to 0}}(-1)^{k_2}\int_{\Omega_2^*\setminus B_{\delta_2}}\left\{\int_{\partial\Omega_1^*}|u|^{-\alpha}u\mathrm{d}\sigma_u(D_u^\alpha)^{i-1}f(u+x,v+y)\right\}(D_v^\beta)^{k_2}H_{i,k_2}(u,v)\mathrm{d}v=$$

$$\lim_{\substack{\delta_1\to 0\\ \delta_2\to 0}}(-1)^{k_2}\int_{\Omega_2^*\setminus B_{\delta_2}}\left\{\left\{\int_{\partial\Omega_1^*}|u|^{-\alpha}u\mathrm{d}\sigma_u(D_u^\alpha)^{i-1}f(u+x,v+y)\right\}(D_v^\beta)^{k_2-1}D_v\right\}\times$$
$$|v|^{-\beta}vH_{i,k_2}(u,v)\mathrm{d}v=$$

$$\lim_{\substack{\delta_1\to 0\\ \delta_2\to 0}}(-1)^{k_2}\int_{\partial\Omega_2^*}\left\{\int_{\partial\Omega_1^*}|u|^{-\alpha}u\mathrm{d}\sigma_u(D_u^\alpha)^{i-1}f(u+x,v+y)\right\}(D_v^\beta)^{k_2-1}\mathrm{d}\sigma_v|v|^{-\beta}vH_{i,k_2}(u,v)-$$

$$\lim_{\substack{\delta_1\to 0\\ \delta_2\to 0}}(-1)^{k_2}\int_{\partial B_{\delta_2}}\left\{\int_{\partial\Omega_1^*}|u|^{-\alpha}u\mathrm{d}\sigma_u(D_u^\alpha)^{i-1}f(u+x,v+y)\right\}(D_v^\beta)^{k_2-1}\mathrm{d}\sigma_v|v|^{-\beta}vH_{i,k_2}(u,v)-$$

$$\lim_{\substack{\delta_1\to 0\\ \delta_2\to 0}}(-1)^{k_2}\int_{\Omega_2^*\setminus B_{\delta_2}}\left\{\int_{\partial\Omega_1^*}|u|^{-\alpha}u\mathrm{d}\sigma_u(D_u^\alpha)^{i-1}f(u+x,v+y)\right\}(D_v^\beta)^{k_2-1}H_{i,k_2-1}(u,v)\mathrm{d}v=$$

$$I_{14}+I_{15}+I_{16};$$

$$I_{16}=\lim_{\substack{\delta_1\to 0\\ \delta_2\to 0}}(-1)^{k_2-1}\int_{\partial\Omega_2^*}\left\{\int_{\partial\Omega_1^*}|u|^{-\alpha}u\mathrm{d}\sigma_u(D_u^\alpha)^{i-1}f(u+x,v+y)\right\}(D_v^\beta)^{k_2-2}\mathrm{d}\sigma_v|v|^{-\beta}vH_{i,k_2-1}(u,v)-$$

$$\lim_{\substack{\delta_1\to 0\\ \delta_2\to 0}}(-1)^{k_2-1}\int_{\partial B_{\delta_2}}\left\{\int_{\partial\Omega_1^*}|u|^{-\alpha}u\mathrm{d}\sigma_u(D_u^\alpha)^{i-1}f(u+x,v+y)\right\}(D_v^\beta)^{k_2-2}\mathrm{d}\sigma_v|v|^{-\beta}vH_{i,k_2-1}(u,v)-$$

$$\lim_{\substack{\delta_1\to 0\\ \delta_2\to 0}}(-1)^{k_2-1}\int_{\Omega_2^*\setminus B_{\delta_2}}\left\{\int_{\partial\Omega_1^*}|u|^{-\alpha}u\mathrm{d}\sigma_u(D_u^\alpha)^{i-1}f(u+x,v+y)\right\}(D_v^\beta)^{k_2-2}H_{i,k_2-2}(u,v)\mathrm{d}v=$$

$$I_{24}+I_{25}+I_{26};$$

$\cdots;$

$$I_{(k_2-1)6}=(-1)^1\lim_{\substack{\delta_1\to 0\\ \delta_2\to 0}}\int_{\Omega_2^*\backslash B_{\delta_2}}\left\{\int_{\partial\Omega_1^*}|u|^{-\alpha}u\mathrm{d}\sigma_u(D_u^{\alpha})^{i-1}f(u+x,v+y)\right\}D_v^{\beta}H_{i,1}(u,v)\mathrm{d}v=$$

$$(-1)^1\lim_{\substack{\delta_1\to 0\\ \delta_2\to 0}}\int_{\partial\Omega_2^*}\left\{\int_{\partial\Omega_1^*}|u|^{-\alpha}u\mathrm{d}\sigma_u(D_u^{\alpha})^{i-1}f(u+x,v+y)\right\}\mathrm{d}\sigma_v|v|^{-\beta}vH_{i,1}(u,v)-$$

$$(-1)^1\lim_{\substack{\delta_1\to 0\\ \delta_2\to 0}}\int_{\partial B_{\delta_2}}\left\{\int_{\partial\Omega_1^*}|u|^{-\alpha}u\mathrm{d}\sigma_u(D_u^{\alpha})^{i-1}f(u+x,v+y)\right\}\mathrm{d}\sigma_v|v|^{-\beta}vH_{i,1}(u,v)-$$

$$(-1)^1\lim_{\substack{\delta_1\to 0\\ \delta_2\to 0}}\int_{\Omega_2^*\backslash B_{\delta_2}}\left\{\int_{\partial\Omega_1^*}|u|^{-\alpha}u\mathrm{d}\sigma_u(D_u^{\alpha})^{i-1}f(u+x,v+y)\right\}D_v(|v|^{-\beta}vH_{i,1}(u,v))\mathrm{d}v=$$

$$I_{k_24}+I_{k_25}+I_{k_26}.$$

由定理3.1得$I_{k_26}=0$, 因此

$$I_5=\lim_{\substack{\delta_1\to 0\\ \delta_2\to 0}}\sum_{i=1}^{k_1}\sum_{j=1}^{k_2}(-1)^{i+j}\int_{\partial\Omega_1^*\times\partial\Omega_2^*}|u|^{-\alpha}uH_{i,j}(u,v)\mathrm{d}\sigma_u\big((D_u^{\alpha})^{i-1}f(u+x,v+y)(D_v^{\beta})^{j-1}\big)\mathrm{d}\sigma_v|v|^{-\beta}v-$$

$$\lim_{\substack{\delta_1\to 0\\ \delta_2\to 0}}\sum_{i=1}^{k_1}\sum_{j=1}^{k_2}(-1)^{i+j}\int_{\partial\Omega_1^*\times\partial B_{\delta_2}}|u|^{-\alpha}uH_{i,j}(u,v)\mathrm{d}\sigma_u\big((D_u^{\alpha})^{i-1}f(u+x,v+y)(D_v^{\beta})^{j-1}\big)\mathrm{d}\sigma_v|v|^{-\beta}v.$$

类似于上述证明可得

$$I_6=\lim_{\substack{\delta_1\to 0\\ \delta_2\to 0}}\sum_{i=1}^{k_1}\sum_{j=1}^{k_2}(-1)^{i+j}\int_{\partial B_{\delta_1}\times\partial\Omega_2^*}|u|^{-\alpha}uH_{i,j}(u,v)\mathrm{d}\sigma_u\big((D_u^{\alpha})^{i-1}f(u+x,v+y)(D_v^{\beta})^{j-1}\big)\mathrm{d}\sigma_v|v|^{-\beta}v-$$

$$\lim_{\substack{\delta_1\to 0\\ \delta_2\to 0}}\sum_{i=1}^{k_1}\sum_{j=1}^{k_2}(-1)^{i+j}\int_{\partial B_{\delta_1}\times\partial B_{\delta_2}}|u|^{-\alpha}uH_{i,j}(u,v)\mathrm{d}\sigma_u\big((D_u^{\alpha})^{i-1}f(u+x,v+y)(D_v^{\beta})^{j-1}\big)\mathrm{d}\sigma_v|v|^{-\beta}v;$$

$$I_2=\lim_{\substack{\delta_1\to 0\\ \delta_2\to 0}}\sum_{i=1}^{k_1}\sum_{j=1}^{k_2}(-1)^{i+j}\int_{\partial\Omega_1^*\times\partial\Omega_2^*}|u|^{-\alpha}uH_{i,j}(u,v)\mathrm{d}\sigma_u\big((D_u^{\alpha})^{i-1}f(u+x,v+y)(D_v^{\beta})^{j-1}\big)\mathrm{d}\sigma_v|v|^{-\beta}v-$$

$$\lim_{\substack{\delta_1\to 0\\ \delta_2\to 0}}\sum_{i=1}^{k_1}\sum_{j=1}^{k_2}(-1)^{i+j}\int_{\partial\Omega_1^*\times\partial B_{\delta_2}}|u|^{-\alpha}uH_{i,j}(u,v)\mathrm{d}\sigma_u\big((D_u^{\alpha})^{i-1}f(u+x,v+y)(D_v^{\beta})^{j-1}\big)\mathrm{d}\sigma_v|v|^{-\beta}v;$$

$$I_3=\lim_{\substack{\delta_1\to 0\\ \delta_2\to 0}}\sum_{i=1}^{k_1}\sum_{j=1}^{k_2}(-1)^{i+j}\int_{\partial\Omega_1^*\times\partial\Omega_2^*}|u|^{-\alpha}uH_{i,j}(u,v)\mathrm{d}\sigma_u\big((D_u^{\alpha})^{i-1}f(u+x,v+y)(D_v^{\beta})^{j-1}\big)\mathrm{d}\sigma_v|v|^{-\beta}v-$$

$$\lim_{\substack{\delta_1\to 0\\ \delta_2\to 0}}\sum_{i=1}^{k_1}\sum_{j=1}^{k_2}(-1)^{i+j}\int_{\partial B_{\delta_1}\times\partial\Omega_2^*}|u|^{-\alpha}uH_{i,j}(u,v)\mathrm{d}\sigma_u\big((D_u^{\alpha})^{i-1}f(u+x,v+y)(D_v^{\beta})^{j-1}\big)\mathrm{d}\sigma_v|v|^{-\beta}v.$$

记$\theta(x,y)=I_1-I_2-I_3+I_4$, 则

$$\theta(x,y)=$$

$$\lim_{\substack{\delta_1\to0\\\delta_2\to0}}\sum_{i=1}^{k_1}\sum_{j=1}^{k_2}(-1)^{i+j}\int_{\partial B_{\delta_1}\times\partial B_{\delta_2}}|u|^{-\alpha}uH_{i,j}(u,v)\mathrm{d}\sigma_u\big((D_u^\alpha)^{i-1}f(u+x,v+y)(D_v^\beta)^{j-1}\big)\mathrm{d}\sigma_v\times |v|^{-\beta}v=$$

$$\lim_{\substack{\delta_1\to0\\\delta_2\to0}}\sum_{i=2}^{k_1}\sum_{j=2}^{k_2}(-1)^{i+j}\int_{\partial B_{\delta_1}\times\partial B_{\delta_2}}|u|^{-\alpha}uH_{i,j}(u,v)\mathrm{d}\sigma_u\big((D_u^\alpha)^{i-1}f(u+x,v+y)(D_v^\beta)^{j-1}\big)\mathrm{d}\sigma_v\times |v|^{-\beta}v+$$

$$\lim_{\substack{\delta_1\to0\\\delta_2\to0}}\sum_{j=2}^{k_2}(-1)^{j+1}\int_{\partial B_{\delta_1}\times\partial B_{\delta_2}}|u|^{-\alpha}uH_{1,j}(u,v)\mathrm{d}\sigma_u f(u+x,v+y)(D_v^\beta)^{j-1}\mathrm{d}\sigma_v|v|^{-\beta}v+$$

$$\lim_{\substack{\delta_1\to0\\\delta_2\to0}}\sum_{i=2}^{k_1}(-1)^{i+1}\int_{\partial B_{\delta_1}\times\partial B_{\delta_2}}|u|^{-\alpha}uH_{i,1}(u,v)\mathrm{d}\sigma_u(D_u^\alpha)^{i-1}f(u+x,v+y)\mathrm{d}\sigma_v|v|^{-\beta}v+$$

$$\lim_{\substack{\delta_1\to0\\\delta_2\to0}}\int_{\partial B_{\delta_1}\times\partial B_{\delta_2}}|u|^{-\alpha}uH_{1,1}(u,v)\mathrm{d}\sigma_u f(u+x,v+y)\mathrm{d}\sigma_v|v|^{-\beta}v=$$

$$\theta_1(x,y)+\theta_2(x,y)+\theta_3(x,y)+\theta_4(x,y).$$

$$\theta_4(x,y)=\lim_{\substack{\delta_1\to0\\\delta_2\to0}}\int_{\partial B_{\delta_1}\times\partial B_{\delta_2}}\frac{1}{\omega_n|u|^{n-\alpha}\omega_m|v|^{m-\beta}}|u|^{-\alpha}u\mathrm{d}\sigma_u f(u+x,v+y)\mathrm{d}\sigma_v|v|^{-\beta}v=$$

$$\lim_{\substack{\delta_1\to0\\\delta_2\to0}}\int_{\partial B_{\delta_1}\times\partial B_{\delta_2}}\frac{1}{\omega_n|u|^n}u\frac{u}{|u|}\mathrm{d}S_u f(u+x,v+y)\mathrm{d}S_v\frac{1}{\omega_m|v|^m}\frac{v}{|v|}v=$$

$$\lim_{\substack{\delta_1\to0\\\delta_2\to0}}\int_{\partial B_{\delta_1}\times\partial B_{\delta_2}}\big(-\frac{1}{\omega_n\delta_1^{n-1}}\big)\mathrm{d}S_u f(u+x,v+y)\mathrm{d}S_v\big(-\frac{1}{\omega_m\delta_2^{m-1}}\big)=$$

$$\frac{1}{\omega_n\delta_1^{n-1}}\frac{1}{\omega_m\delta_2^{m-1}}\lim_{\substack{\delta_1\to0\\\delta_2\to0}}\int_{\partial B_{\delta_1}\times\partial B_{\delta_2}}\mathrm{d}S_u f(u+x,v+y)\mathrm{d}S_v=$$

$$f(x,y).$$

由定理3.3得$(D_u^\alpha)^i f(u+x,v+y)(D_v^\beta)^j$在$\Omega_1^*\times\Omega_2^*$上是有界的, 则

$$|(D_u^\alpha)^i f(u+x,v+y)(D_v^\beta)^j|\le M,$$

其中$2\le i\le k_1(2\le j\le k_2)$, 因此

$$|\theta_1(x,y)|\le\lim_{\substack{\delta_1\to0\\\delta_2\to0}}M\sum_{i=2}^{k_1}\sum_{j=2}^{k_2}\int_{\partial B_{\delta_1}\times\partial B_{\delta_2}}|H_{i,j}(u,v)||u|^{1-\alpha}|\mathrm{d}\sigma_u||\mathrm{d}\sigma_v||v|^{1-\beta}=$$

$$\lim_{\substack{\delta_1\to0\\\delta_2\to0}}M\sum_{i=2}^{k_1}\sum_{j=2}^{k_2}\int_{\partial B_{\delta_1}\times\partial B_{\delta_2}}|u|^{(i-1)\alpha-n+1}||v|^{(j-1)\beta-m+1}|\mathrm{d}\sigma_u||\mathrm{d}\sigma_v|\le$$

$$\lim_{\substack{\delta_1\to0\\\delta_2\to0}}\sum_{i=2}^{k_1}\sum_{j=2}^{k_2}M_2A_iB_j\delta_1^{(i-1)\alpha}\delta_2^{(j-1)\beta}=0.$$

$$|\theta_2(x,y)|=\left|\lim_{\substack{\delta_1\to 0\\ \delta_2\to 0}}\sum_{j=2}^{k_2}(-1)^j\int_{\partial B_{\delta_1}\times\partial B_{\delta_2}}\frac{B_j}{\omega_n\delta_1^{n-1}|v|^{m-j\beta}}\mathrm{d}S_u(f(u+x,v+y)(D_v^\beta)^{j-1})\mathrm{d}\sigma_v|v|^{-\beta}v\right|=$$

$$\left|\lim_{\substack{\delta_1\to 0\\ \delta_2\to 0}}\sum_{j=2}^{k_2}(-1)^j\int_{\partial B_{\delta_2}}\left\{\int_{\partial B_{\delta_1}}\frac{1}{\omega_n\delta^{n-1}}\mathrm{d}S_u(f(u+x,v+y)(D_v^\beta)^{j-1})\right\}\mathrm{d}\sigma_v\frac{B_j}{|v|^{m-j\beta}}|v|^{-\beta}v\right|=$$

$$\left|\lim_{\substack{\delta_1\to 0\\ \delta_2\to 0}}\sum_{j=2}^{k_2}(-1)^j\int_{\partial B_{\delta_2}}(f(x,v+y)(D_v^\beta)^{j-1})\mathrm{d}\sigma_v\frac{B_j}{|v|^{m-j\beta}}|v|^{-\beta}v\right|\le$$

$$\lim_{\substack{\delta_1\to 0\\ \delta_2\to 0}}M\sum_{j=2}^{k_2}\int_{\partial B_{\delta_2}}|\mathrm{d}\sigma_v|\left|\frac{B_j}{|v|^{m-j\beta}}\right||v|^{1-\beta}=$$

$$\lim_{\substack{\delta_1\to 0\\ \delta_2\to 0}}\sum_{j=2}^{k_2}MB_j\delta_2^{(j-1)\beta}=0.$$

类似于$\theta_2(x,y)$的证明可得$\theta_3(x,y)=0$, 因此$\theta(x,y)=f(x,y)$.

推论4.2 (Cauchy-Pompeiu公式) 若$f(x,y)\in F^{(r)}_{\overline{\Omega_1}\times\overline{\Omega_2}}$, 其中$r\ge 2k$, $n\ge k, m\ge k, 0<\alpha,\beta<\dfrac{1}{k}$, 则对任意$(x,y)\in\Omega_1\times\Omega_2$, 都有

$$f(x,y)=\sum_{i=1}^{k}\sum_{j=1}^{k}(-1)^{i+j}\int_{\partial\Omega_1^*\times\partial\Omega_2^*}|u|^{-\alpha}uH_{i,j}(u,v)\mathrm{d}\sigma_u\big((D_u^\alpha)^{i-1}f(u+x,v+y)(D_v^\beta)^{j-1}\big)\mathrm{d}\sigma_v|v|^{-\beta}v-$$

$$\sum_{i=1}^{k}(-1)^{i+k}\int_{\partial\Omega_1^*\times\Omega_2^*}|u|^{-\alpha}uH_{i,k}(u,v)\mathrm{d}\sigma_u\big((D_u^\alpha)^{i-1}f(u+x,v+y)(D_v^\beta)^{k}\big)\mathrm{d}v-$$

$$\sum_{j=1}^{k}(-1)^{j+k}\int_{\Omega_1^*\times\partial\Omega_2^*}H_{k,j}(u,v)\big((D_u^\alpha)^{k}f(u+x,v+y)(D_v^\beta)^{j-1}\big)\mathrm{d}\sigma_v|v|^{-\beta}v\mathrm{d}u+$$

$$\int_{\Omega_1^*\times\Omega_2^*}H_{k,k}(u,v)\big((D_u^\alpha)^{k}f(u+x,v+y)(D_v^\beta)^{k}\big)\mathrm{d}u\mathrm{d}v.$$

定理4.3 (Cauchy积分公式) 若$f(u+x,v+y)$在$\Omega_1^*\times\Omega_2^*$中是一个加双权的双$k$-正则函数, 且$f(x,y)\in F^{(r)}_{\overline{\Omega_1}\times\overline{\Omega_2}}$, 其中$r\ge 2k$, $n\ge k, m\ge k, 0<\alpha,\beta<\dfrac{1}{k}$, 则对任意$(x,y)\in\Omega_1\times\Omega_2$, 都有

$$f(x,y)=$$
$$\sum_{i=1}^{k}\sum_{j=1}^{k}(-1)^{i+j}\int_{\partial\Omega_1^*\times\partial\Omega_2^*}|u|^{-\alpha}uH_{i,j}(u,v)\mathrm{d}\sigma_u\big((D_u^\alpha)^{i-1}f(u+x,v+y)(D_v^\beta)^{j-1}\big)\mathrm{d}\sigma_v|v|^{-\beta}v.$$

证 若$(x,y)\in\Omega_1\times\Omega_2$, 则$f(u+x,v+y)$在$\Omega_1^*\times\Omega_2^*$中是一个加双权的双$k$-正则函数, 因此$\begin{cases}(D_u^\alpha)^k f(u+x,v+y)=0,\\ f(u+x,v+y)(D_v^\alpha)^k=0,\end{cases}$ 再由推论4.2可得此结论.

定理4.4 (Cauchy积分定理) 若$f(u+x,v+y)$在$\Omega_1^*\times\Omega_2^*$中是一个加双权的双$k$-正则函数, 且$f(x,y)\in F^{(r)}_{\overline{\Omega_1}\times\overline{\Omega_2}}$, 其中$r\ge 2k$, $n\ge k, m\ge k, 0<\alpha,\beta<\dfrac{1}{k}$, 则对任意的$(x,y)\in\{\mathbf{R}^n\times\mathbf{R}^m\}\backslash\{\overline{\Omega_1}\times\overline{\Omega_2}\}$, 都有

$$\sum_{i=1}^{k}\sum_{j=1}^{k}(-1)^{i+j}\int_{\partial\Omega_1^*\times\partial\Omega_2^*}|u|^{-\alpha}uH_{i,j}(u,v)\mathrm{d}\sigma_u\big((D_u^\alpha)^{i-1}f(u+x,v+y)(D_v^\beta)^{j-1}\big)\mathrm{d}\sigma_v|v|^{-\beta}v=0.$$

证 根据引理2.2可得

$$(-1)^k(-1)^k\int_{\Omega_1^*\times\Omega_2^*}H_{k,k}(u,v)\big((D_u^\alpha)^k f(u+x,v+y)(D_v^\beta)^k\big)\mathrm{d}v\mathrm{d}u=$$

$$(-1)^k(-1)^k\int_{\Omega_2^*}\left\{\int_{\Omega_1^*}H_{k,k}(u,v)|u|^{-\alpha}uD_u\big((D_u^\alpha)^{k-1}f(u+x,v+y)(D_v^\beta)^k\big)\mathrm{d}u\right\}\mathrm{d}v=$$

$$(-1)^k(-1)^k\int_{\Omega_2^*}\left\{\int_{\partial\Omega_1^*}H_{k,k}(u,v)|u|^{-\alpha}u\mathrm{d}\sigma_u\big((D_u^\alpha)^{k-1}f(u+x,v+y)(D_v^\beta)^k\big)\right\}\mathrm{d}v-$$

$$(-1)^k(-1)^k\int_{\Omega_2^*}\left\{\int_{\Omega_1^*}H_{k-1,k}(u,v)\big((D_u^\alpha)^{k-1}f(u+x,v+y)(D_v^\beta)^k\big)\mathrm{d}u\right\}\mathrm{d}v=$$

$$(-1)^k(-1)^k\int_{\Omega_2^*}\left\{\int_{\partial\Omega_1^*}H_{k,k}(u,v)|u|^{-\alpha}u\mathrm{d}\sigma_u\big((D_u^\alpha)^{k-1}f(u+x,v+y)(D_v^\beta)^k\big)\right\}\mathrm{d}v+$$

$$(-1)^k(-1)^{k-1}\int_{\Omega_2^*}\left\{\int_{\Omega_1^*}H_{k-1,k}(u,v)\big((D_u^\alpha)^{k-1}f(u+x,v+y)(D_v^\beta)^k\big)\mathrm{d}u\right\}\mathrm{d}v=$$

$$(-1)^k(-1)^k\int_{\Omega_2^*}\left\{\int_{\partial\Omega_1^*}H_{k,k}(u,v)|u|^{-\alpha}u\mathrm{d}\sigma_u\big((D_u^\alpha)^{k-1}f(u+x,v+y)(D_v^\beta)^k\big)\right\}\mathrm{d}v+$$

$$(-1)^k(-1)^{k-1}\int_{\Omega_2^*}\left\{\int_{\partial\Omega_1^*}H_{k-1,k}(u,v)|u|^{-\alpha}u\mathrm{d}\sigma_u\big((D_u^\alpha)^{k-2}f(u+x,v+y)(D_v^\beta)^k\big)\right\}\mathrm{d}v+$$

$$(-1)^k(-1)^{k-2}\int_{\Omega_2^*}\left\{\int_{\Omega_1^*}H_{k-2,k}(u,v)\big((D_u^\alpha)^{k-2}f(u+x,v+y)(D_v^\beta)^k\big)\mathrm{d}u\right\}\mathrm{d}v=$$

$$\cdots=$$

$$\sum_{i=1}^k(-1)^i(-1)^k\int_{\Omega_2^*}\left\{\int_{\partial\Omega_1^*}|u|^{-\alpha}u\mathrm{d}\sigma_u(D_u^\alpha)^{i-1}f(u+x,v+y)\right\}(D_v^\beta)^kH_{i,k}(u,v)\mathrm{d}v=$$

$$\sum_{i=1}^k(-1)^i(-1)^k\int_{\Omega_2^*}\left\{\int_{\partial\Omega_1^*}|u|^{-\alpha}u\mathrm{d}\sigma_u(D_u^\alpha)^{i-1}f(u+x,v+y)\right\}(D_v^\beta)^{k-1}D_vH_{i,k}(u,v)|v|^{-\beta}v\mathrm{d}v=$$

$$\sum_{i=1}^k(-1)^i(-1)^k\int_{\partial\Omega_2^*}\left\{\int_{\partial\Omega_1^*}|u|^{-\alpha}u\mathrm{d}\sigma_u(D_u^\alpha)^{i-1}f(u+x,v+y)\right\}(D_v^\beta)^{k-1}\mathrm{d}\sigma_vH_{i,k}(u,v)|v|^{-\beta}v-$$

$$\sum_{i=1}^k(-1)^i(-1)^k\int_{\Omega_2^*}\left\{\int_{\partial\Omega_1^*}|u|^{-\alpha}u\mathrm{d}\sigma_u(D_u^\alpha)^{i-1}f(u+x,v+y)\right\}(D_v^\beta)^{k-1}H_{i,k-1}(u,v)\mathrm{d}v=$$

$$\sum_{i=1}^k(-1)^i(-1)^k\int_{\partial\Omega_2^*}\left\{\int_{\partial\Omega_1^*}|u|^{-\alpha}u\mathrm{d}\sigma_u(D_u^\alpha)^{i-1}f(u+x,v+y)\right\}(D_v^\beta)^{k-1}\mathrm{d}\sigma_vH_{i,k}(u,v)|v|^{-\beta}v+$$

$$\sum_{i=1}^k(-1)^i(-1)^{k-1}\int_{\Omega_2^*}\left\{\int_{\partial\Omega_1^*}|u|^{-\alpha}u\mathrm{d}\sigma_u(D_u^\alpha)^{i-1}f(u+x,v+y)\right\}(D_v^\beta)^{k-1}H_{i,k-1}(u,v)\mathrm{d}v=$$

$$\sum_{i=1}^k(-1)^i(-1)^k\int_{\partial\Omega_2^*}\left\{\int_{\partial\Omega_1^*}|u|^{-\alpha}u\mathrm{d}\sigma_u(D_u^\alpha)^{i-1}f(u+x,v+y)\right\}(D_v^\beta)^{k-1}\mathrm{d}\sigma_vH_{i,k}(u,v)|v|^{-\beta}v+$$

$$\sum_{i=1}^k(-1)^i(-1)^{k-1}\int_{\partial\Omega_2^*}\left\{\int_{\partial\Omega_1^*}|u|^{-\alpha}u\mathrm{d}\sigma_u(D_u^\alpha)^{i-1}f(u+x,v+y)\right\}(D_v^\beta)^{k-2}\mathrm{d}\sigma_vH_{i,k-1}(u,v)|v|^{-\beta}v+$$

$$\sum_{i=1}^k(-1)^i(-1)^{k-2}\int_{\Omega_2^*}\left\{\int_{\Omega_1^*}|u|^{-\alpha}u\mathrm{d}\sigma_u(D_u^\alpha)^{i-1}f(u+x,v+y)\right\}(D_v^\beta)^{k-2}H_{i,k-2}(u,v)\mathrm{d}v=$$

$$\cdots=$$

$$\sum_{i=1}^k\sum_{j=1}^k(-1)^i(-1)^j\int_{\partial\Omega_1^*\times\partial\Omega_2^*}|u|^{-\alpha}uH_{i,j}(u,v)\mathrm{d}\sigma_u\big((D_u^\alpha)^{i-1}f(u+x,v+y)(D_v^\beta)^{j-1}\big)\mathrm{d}\sigma_v|v|^{-\beta}v.$$

因为$f(u+x,v+y)$在$\Omega_1^*\times\Omega_2^*$中是一个加双权的双$k$-正则函数, 则$(D_u^\alpha)^k f(u+x,v+y)(D_v^\beta)^k=0$,

$$\sum_{i=1}^{k}\sum_{j=1}^{k}(-1)^{i+j}\int_{\partial\Omega_1^*\times\partial\Omega_2^*}H_{i,j}(u,v)|u|^{-\alpha}u\mathrm{d}\sigma_u\big((D_u^\alpha)^{i-1}f(u+x,v+y)(D_v^\beta)^{j-1}\big)\mathrm{d}\sigma_v|v|^{-\beta}v=0.$$

定理4.5 (平均值定理) 若$f(x,y)$为$\Omega_1^*\times\Omega_2^*$ 中的加双权的双k-正则函数. 对任意$(x,y)\in\Omega_1\times\Omega_2$, 假定$\overline{B_1}=\overline{B(x,r_1)}\subset\Omega_1$, $\overline{B_2}=\overline{B(x,r_2)}\subset\Omega_2$, $B_1^*\times B_2^*=\{(u,v)|u+x\in B_1,x\in\overline{B_1};v+y\in B_2,y\in\overline{B_2}\}$, $0<\alpha,\beta<\dfrac{1}{k}$, 则

$$f(x,y)=\sum_{i=1}^{k}\sum_{j=1}^{k}(-1)^{i+j}\frac{A_iB_j}{r_1^{n-(i-1)\alpha}r_2^{m-(j-1)\beta}}\Big\{\int_{B_1^*\times B_2^*}(-n)\big((D_u^\alpha)^{i-1}f(u+x,v+y)(D_v^\beta)^j\big)|v|^\beta+$$
$$(-m)|u|^\alpha\big((D_u^\alpha)^if(u+x,v+y)(D_v^\beta)^{j-1}\big)+mn\big((D_u^\alpha)^{i-1}f(u+x,v+y)(D_v^\beta)^{j-1}\big)+$$
$$|u|^\alpha\big((D_u^\alpha)^if(u+x,v+y)(D_v^\beta)^j\big)|v|^\beta\mathrm{d}u\mathrm{d}v\Big\}.$$

证 根据定理4.3可得

$$f(x,y)=\sum_{i=1}^{k}\sum_{j=1}^{k}(-1)^{i+j}\int_{\partial B_1^*\times\partial B_2^*}|u|^{-\alpha}uH_{i,j}(u,v)\mathrm{d}\sigma_u\big((D_u^\alpha)^{i-1}f(u+x,v+y)(D_v^\beta)^{j-1}\big)\mathrm{d}\sigma_v|v|^{-\beta}v=$$
$$\sum_{i=1}^{k}\sum_{j=1}^{k}(-1)^{i+j}\frac{A_iB_j}{r_1^{n-(i-1)\alpha}r_2^{m-(j-1)\beta}}\int_{\partial B_1^*\times\partial B_2^*}u\mathrm{d}\sigma_u\big((D_u^\alpha)^{i-1}f(u+x,v+y)(D_v^\beta)^{j-1}\big)\mathrm{d}\sigma_v v.$$

令$H(u+x,v+y)=\int_{\partial B_2^*}f(u+x,v+y)(D_v^\beta)^{j-1}\mathrm{d}\sigma_v v$, 则由引理2.2可得

$$f(x,y)=\sum_{i=1}^{k}\sum_{j=1}^{k}(-1)^{i+j}\frac{A_iB_j}{r_1^{n-(i-1)\alpha}r_2^{m-(j-1)\beta}}\int_{\partial B_1^*}u\mathrm{d}\sigma_u(D_u^\alpha)^{i-1}H(u+x,v+y)=$$
$$\sum_{i=1}^{k}\sum_{j=1}^{k}(-1)^{i+j}\frac{A_iB_j}{r_1^{n-(i-1)\alpha}r_2^{m-(j-1)\beta}}\times$$
$$\int_{B_1^*}(uD_u)(D_u^\alpha)^{i-1}H(u+x,v+y)+u\big\{D_u(D_u^\alpha)^{i-1}H(u+x,v+y)\big\}\mathrm{d}u=$$
$$\sum_{i=1}^{k}\sum_{j=1}^{k}(-1)^{i+j}\frac{A_iB_j}{r_1^{n-(i-1)\alpha}r_2^{m-(j-1)\beta}}\int_{B_1^*}(-n)(D_u^\alpha)^{i-1}H(u+x,v+y)+|u|^\alpha(D_u^\alpha)^iH(u+x,v+y)\mathrm{d}u.$$

利用类似的方法, 可得

$$H(u+x,v+y)=\int_{B_2^*}(-m)f(u+x,v+y)(D_v^\beta)^{j-1}+f(u+x,v+y)(D_v^\beta)^j|v|^\beta\mathrm{d}v.$$

把$H(u+x,v+y)$代入可得

$$f(x,y)=\sum_{i=1}^{k}\sum_{j=1}^{k}(-1)^{i+j}\frac{A_iB_j}{r_1^{n-(i-1)\alpha}r_2^{m-(j-1)\beta}}\Big\{\int_{B_1^*\times B_2^*}(-n)\big((D_u^\alpha)^{i-1}f(u+x,v+y)(D_v^\beta)^j\big)|v|^\beta+$$
$$(-m)|u|^\alpha\big((D_u^\alpha)^if(u+x,v+y)(D_v^\beta)^{j-1}\big)+mn\big((D_u^\alpha)^{i-1}f(u+x,v+y)(D_v^\beta)^{j-1}\big)+$$
$$|u|^\alpha\big((D_u^\alpha)^if(u+x,v+y)(D_v^\beta)^j\big)|v|^\beta\mathrm{d}u\mathrm{d}v\Big\}.$$

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The Cauchy integral formula for the bi-k-monogenic function with bi-weight

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Abstract: Firstly, the definition of the bi-k-monogenic function with bi-weight in Clifford analysis is given. Secondly, the kernel function of the bi-k-monogenic function with bi-weight is given and its properties are studied. Then the Cauchy-Pompeiu formula, the Cauchy integral formula, the Cauchy integral theorem and mean value theorem for the bi-k-monogenic function with bi-weight are proved.

Keywords: bi-k-monogenic function with bi-weight; Cauchy-Pompeiu formula; Cauchy integral formula; Cauchy integral theorem

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