# h-Said-Ball基与h-Said-Ball曲线

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摘 要: h-Bézier曲线是Bézier曲线基于h-微积分意义下的推广模型.为增强Said-Ball曲线的造型能力,提高h-Bézier曲线递归求值速度,该文提出任意次的h-Said-Ball基函数,构造了h-Said-Ball曲线.通过分析Said-Ball曲线递归求值算法与 Bézier曲线的转化关系,结合h-Bézier曲线的递归求值算法和h-Bernstein基函数的构造方式,得到任意次h-Said-Ball基函数的表达式.h-Said-Ball基具有非负,单位分解,端点插值等优良性质,和h-Bernstein基之间存在显式转换矩阵.进一步,定义h-Said-Ball曲线并分析其基本性质,推导递归求值算法和包络表示,h-Said-Ball曲线的求值计算量是h-Bézier曲线的一半.借助从h-Said-Ball曲线到h-Bézier曲线的割角算法,证明了h-Said-Ball基是全正基,从而h-Said-Ball曲线具有变差缩减性和保凸性.数值实例显示了h-Said-Ball曲线相比Said-Ball曲线的造型优势和灵活性.

关键词: h-Bézier曲线; Said-Ball曲线; h-Said-Ball基函数; h-Said-Ball曲线; 全正基; 递 归求值算法

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1974年, 英国数学家Ball首次构造了一种称为Ball曲线的有理三次参数曲线, 并将其成功应用到英国航空公司的CONSURF机身曲面造型系统<sup>[1-3]</sup>.为了探究Ball曲线的本质特征, 1987年, 王国瑾<sup>[4]</sup>将三次Ball基推广到了任意次. 1989年, Said等人<sup>[5]</sup>将Ball基推广到了任意奇数次. 胡事民等人<sup>[6]</sup>于1996年将其推广到了任意偶数次, 此后为方便区分, 这两类曲线被分别称为Wang-Ball曲线和Said-Ball曲线.

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Said-Ball曲线的主要优点是基函数的次数分布呈阶梯式,递归求值算法的运行速度 是Bézier曲线de Casteljau算法的两倍,且Said-Ball基函数具有一个全正的配置矩阵<sup>[7]</sup>. Wang-Ball基函数的特点是次数按序号呈脊形分布,两头低中间高,相邻两个基函数的次数一般相差二 次,且首尾两个基函数只有二次.与此同时,这两种曲线还保持着计算稳定性,对称性,凸包性, 端点插值性,几何不变性等与Bézier曲线类似的优良性质<sup>[8]</sup>.

为了调整曲线的形状或改变曲线的位置,学者们通过增加参数构造广义Ball曲线.2000年邬 弘毅等人<sup>[9]</sup>在基函数中加入一个位置参数,构造了介于Wang-Ball曲线和Said-Ball曲线之间和介 于Said-Ball曲线和Bézier曲线之间的两类曲线族.2010年张莉等人<sup>[10]</sup>通过对基函数加入两种位 置参数,给出了一种Bézier曲线,Said-Ball曲线和Wang-Ball曲线的统一表示.2012年汪志华等 人<sup>[11]</sup>给出了介于Bézier曲线,Said-Ball曲线和Wang-Ball曲线之间的曲线族.为了在不改变控制 顶点的情况下,使曲线能够形状可调.严兰兰等人<sup>[12]</sup>对低次Said-Ball基函数中加入了形状参数 进行推广.熊建等人<sup>[13]</sup>构造了任意次带形状参数的广义Said-Ball曲线,并给出两段曲线*G*<sup>1</sup>的拼 接条件.

随着量子微积分<sup>[14]</sup>的发展,出现了*h*-Bernstein多项式和*q*-Bernstein多项式<sup>[15]</sup>等Bernstein多项式的推广形式. 1968年, Stancu<sup>[16]</sup>首次引入*h*-Bernstein多项式及包含该多项式的正线性逼近算子. 1985年, Goldman<sup>[17]</sup>通过Pólya Urn模型实验生成了*h*-Bernstein多项式,进而定义了*h*-Bézier曲线,并分析其性质,发现*h*-Bézier曲线拥有许多与Bézier曲线类似的几何性质,例如变差缩减性,保凸性等性质. 由于带有参数*h*,可以通过改变*h*的取值调整曲线形状. 2011年,Simeonov等人<sup>[18]</sup>定义*h*-开花算法,利用其开花形式推导了曲线的细分算法和Marsden恒等式. 2019年,孙一皓等人<sup>[19]</sup>通过增加正实数权因子构造了有理*h*-Bézier曲线,利用二次有理*h*-Bézier曲线精确表示圆锥曲线. 2022年,李林等人<sup>[20]</sup>构造曲率单调的组合二次*h*-Bézier曲线.

本文将Bézier曲线推广到*h*-Bézier曲线的方法应用到Said-Ball曲线上,构造*h*-Said-Ball曲线. 根据Said-Ball曲线及Bézier曲线的递归求值算法的关系,在*h*-Said-Ball曲线和*h*-Bézier曲线的递 归求值算法上找到相同的关系.将曲线的递归求值算法反向递推,可得到基函数递推生成算法. 根据算法得到任意次*h*-Said-Ball基函数的显式表达式,并依照Said-Ball基函数和*h*-Bernstein基 函数的性质,分析了其类似的优良性质,例如单位分解性,端点性,对称性等.根据*h*-Said-Ball基 函数,构造了*h*-Said-Ball曲线并分析其性质,并利用割角算法证明了*h*-Said-Ball基是全正基,由 此得到了*h*-Said-Ball曲线的变差缩减性和保凸性.给出了*h*-Said-Ball曲线的递归求值算法,并在 求值速度上与*h*-Bézier曲线的de Castelijau算法作了比较.

## §2 预备知识

本节给出Bézier曲线, *h*-Bézier曲线及Said-Ball曲线的相关定义. 给定平面或空间中的n + 1个点 $Q_i$ ( $i = 0, 1, \dots, n$ ), *n*次Bézier曲线<sup>[21]</sup>为

$$\boldsymbol{P}\left(t\right) = \sum_{i=0}^{n} \boldsymbol{Q}_{i} b_{i}^{n}\left(t\right),$$

其中 $b_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i}$ 为[0,1]区间上的n次Bernstein基函数<sup>[22]</sup>.

给定平面或空间中的n + 1个点 $V_i$ ( $i = 0, 1, \dots, n$ ), n次h-Bézier曲线<sup>[15]</sup>为

$$\boldsymbol{H}\left(t\right) = \sum_{i=0}^{n} \boldsymbol{V}_{i} b_{i}^{n}\left(t;h\right),$$

其中 $b_i^n(t;h) = \binom{n}{i} \frac{\prod\limits_{k=0}^{i-1} (t+kh) \prod\limits_{k=0}^{n-i-1} (1-t+kh)}{\prod\limits_{k=0}^{n-1} (1+kh)}$ 为[0,1]区间上的n次h-Bernstein基函数. 当h = 0

0时, *h*-Bernstein基函数退化为经典的Bernstein基函数, *h*-Bézier曲线退化为经典的Bézier曲线.

给定平面或空间中的n + 1个点 $\mathbf{R}_i$ ( $i = 0, 1, \dots, n$ ),称

$$\boldsymbol{B}\left(t\right) = \sum_{i=0}^{n} \boldsymbol{R}_{i} \boldsymbol{s}_{i}^{n}\left(t\right)$$

为n次Said-Ball曲线<sup>[6]</sup>,其中

$$s_{i}^{n}(t) = \begin{cases} \begin{pmatrix} \lfloor n/2 \rfloor + i \\ i \end{pmatrix} t^{i}(1-t)^{\lfloor n/2 \rfloor + 1}, 0 \le i \le \lceil n/2 \rceil - 1; \\ \begin{pmatrix} n \\ \lfloor n/2 \rfloor \end{pmatrix} t^{\lfloor n/2 \rfloor}(1-t)^{\lfloor n/2 \rfloor}, i = \lfloor n/2 \rfloor; \\ \begin{pmatrix} \lfloor n/2 \rfloor + n - i \\ n - i \end{pmatrix} t^{\lfloor n/2 \rfloor + 1}(1-t)^{n-i}, \lfloor n/2 \rfloor + 1 \le i \le n. \end{cases}$$
(1)

为n次Said-Ball基函数<sup>[6]</sup>,其中[x]表示小于或等于x的最大整数, [x]表示大于或等于x的最小整数.

# §3 h-Said-Ball基函数

基于*h*-微积分理论,将Said-Ball基函数推广到带参数*h*的任意*n*次基函数,即*h*-Said-Ball基函数.

3.1 基函数的定义

**定义3.1** 设实数 $h \ge 0, n \in \mathbb{N}, \forall t \in [0, 1], n$ 次h-Said-Ball 基函数 $\{s_i^n(t; h)\}_{i=0}^n$ 定义为

$$s_{i}^{n}(t;h) = \begin{cases} \left( \begin{array}{c} \lfloor n/2 \rfloor + i \\ i \end{array} \right)^{\frac{1}{k=0}(t+kh)} \frac{\prod_{k=0}^{\lfloor n/2 \rfloor}(1-t+kh)}{\prod_{k=0}(1+kh)}, 0 \le i \le \lceil n/2 \rceil - 1; \\ \prod_{k=0}^{l}(t+kh) \prod_{k=0}^{l}(1-t+kh)} \left( \begin{array}{c} n \\ \lfloor n/2 \rfloor \end{array} \right)^{\frac{1}{k=0}(t+kh)} \frac{\prod_{k=0}^{l}(1-t+kh)}{\prod_{k=0}(1+kh)}, i = \lfloor n/2 \rfloor; \\ \left( \begin{array}{c} \lfloor n/2 \rfloor + n - i \\ n-i \end{array} \right)^{\frac{\lfloor n/2 \rfloor}{k=0}(t+kh)} \frac{\prod_{k=0}^{l}(1-t+kh)}{\prod_{k=0}(1+kh)}, \lfloor n/2 \rfloor + 1 \le i \le n. \end{cases}$$
(2)

其中,  $\lfloor x \rfloor$ 表示小于或等于x的最大整数,  $\lceil x \rceil$ 表示大于或等于x的最小整数.

下面给出低次h-Said-Ball基函数与Said-Ball基函数的实例进行对比.

**例1** 五次h-Said-Ball基函数和五次的Said-Ball基函数  

$$\begin{cases}
s_0^5(t;h) = \frac{(1-t)(1-t+h)(1-t+2h)}{(1+h)(1+2h)} \\
s_1^5(t;h) = \frac{3t(1-t)(1-t+h)(1-t+2h)}{(1+h)(1+2h)(1+3h)} \\
s_2^5(t;h) = \frac{6t(t+h)(1-t)(1-t+h)(1-t+2h)}{(1+h)(1+2h)(1+3h)(1+4h)} \\
s_3^5(t;h) = \frac{6t(t+h)(t+2h)(1-t)(1-t+h)}{(1+h)(1+2h)(1+3h)(1+4h)} \\
s_5^5(t;h) = \frac{3t(1-t)(t+h)(t+2h)}{(1+h)(1+2h)} \\
s_5^5(t;h) = \frac{t(t+h)(t+2h)}{(1+h)(1+2h)} \\
\end{cases}, \begin{cases}
s_0^5(t) = (1-t)^3 \\
s_1^5(t) = 3t(1-t)^3 \\
s_2^5(t) = 6t^2(1-t)^3 \\
s_3^5(t) = 6t^3(1-t)^2 \\
s_4^5(t) = 3t^3(1-t) \\
s_5^5(t) = 3t^3(1-t) \\
s_5^5(t) = t^3
\end{cases}$$
(3)

式(3)中左侧为五次的*h*-Said-Ball基函数,右侧为五次的Said-Ball基函数.通过显式表达式发现五次的*h*-Said-Ball基函数和五次的Said-Ball基函数的次数依次为3,4,5,5,4,3呈阶梯状对称分布.当h = 0时,五次的*h*-Said-Ball基函数退化为五次的Said-Ball基函数.



图 1 五次h-Said-Ball基函数(h = 1, 0.5, 0)

图1从左至右分别为h = 1, h = 0.5, h = 0时, 六次h-Said-Ball基函数的图像, 当h减小时, 除了首末基函数逐渐靠拢横轴, 其余基函数会逐渐远离横轴, 并且当h = 0时五次h-Said-Ball基函数退化为Said-Ball基函数.

$ \left\{ \begin{array}{l} s_{0}^{6}\left(t;h\right) = \frac{(1-t)(1-t+h)(1-t+2h)(1-t+3h)}{(1+h)(1+2h)(1+3h)} \\ s_{1}^{6}\left(t;h\right) = \frac{4t(1-t)(1-t+h)(1-t+2h)(1-t+3h)}{(1+h)(1+2h)(1+3h)(1+4h)} \\ s_{2}^{6}\left(t;h\right) = \frac{10t(t+h)(1-t)(1-t+h)(1-t+2h)(1-t+3h)}{(1+h)(1+2h)(1+3h)(1+4h)(1+5h)} \\ s_{3}^{6}\left(t;h\right) = \frac{20t(t+h)(t+2h)(1-t)(1-t+h)(1-t+2h)}{(1+h)(1+2h)(1+3h)(1+4h)(1+5h)} \\ s_{4}^{6}\left(t;h\right) = \frac{10t(t+h)(t+2h)(1+3h)(1-t+h)}{(1+h)(1+2h)(1+3h)(1+4h)(1+5h)} \\ s_{5}^{6}\left(t;h\right) = \frac{4t(1-t)(t+h)(t+2h)(t+3h)}{(1+h)(1+2h)(1+3h)(1+4h)} \\ s_{6}^{6}\left(t;h\right) = \frac{t(t+h)(t+2h)(t+3h)}{(1+h)(1+2h)(1+3h)} \end{array} \right\} $	$s_{0}^{6}(t) = (1-t)^{4}$ $s_{1}^{6}(t) = 4t(1-t)^{4}$ $s_{2}^{6}(t) = 10t^{2}(1-t)^{4}$ $s_{3}^{6}(t) = 20t^{3}(1-t)^{3}  . \qquad (4)$ $s_{4}^{6}(t) = 10t^{4}(1-t)^{2}$ $s_{5}^{6}(t) = 4t^{4}(1-t)$ $s_{6}^{6}(t) = t^{4}$
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式(4)中左侧为六次的*h*-Said-Ball基函数,右侧为六次的Said-Ball基函数. 六次*h*-Said-Ball基函数和六次的Said-Ball基函数的次数依次为4,5,6,6,5,4呈阶梯状对称分布. 当h = 0时,六次*h*-Said-Ball基函数退化为六次Said-Ball基函数.



图 2 六次h-Said-Ball基函数(h = 1,0.5,0)

图2从左至右分别为h = 1, h = 0.5, h = 0的六次h-Said-Ball基函数的图像, 当h = 0时六次h-Said-Ball基函数退化为Said-Ball基函数.

3.2 基函数的性质

根据基函数的定义, h-Said-Ball基函数在 $h \ge 0$ 时有如下基本性质.

**性质3.1** 非负性:  $s_i^n(t;h) \ge 0, i = 0, 1, \dots, n, t \in [0,1]$ .

**性质3.2** 单位分解性:  $\sum_{i=0}^{n} s_{i}^{n}(t;h) = 1, t \in [0,1]$ . (*h*-Said-Ball基函数的单位分解性可由数 学归纳法证明)

**性质3.3** 对称性:  $s_i^n(t;h) = s_{n-i}^n(1-t;h), i = 0, 1, \dots, n$ .

**性质3.4** 线性无关性: 当参数h固定时,  $\{s_0^n(t;h), s_1^n(t;h), \dots, s_n^n(t;h)\}$ 是线性无关的, 可作为n次多项式空间一组基函数.

性质3.5 递推性: 2m次与2m+1次h-Said-Ball基函数存在如下递推关系.

$$s_{i}^{2m}(t;h) = \begin{cases} s_{i}^{2m+1}(t;h), 0 \le i \le m-1\\ s_{i}^{2m+1}(t;h) + s_{i+1}^{2m+1}(t;h), i = m\\ s_{i+1}^{2m+1}(t;h), m+1 \le i \le 2m \end{cases}$$
(5)

$$s_{i}^{2m+1}(t;h) = \begin{cases} s_{i}^{2m}(t;h), 0 \leq i \leq m-1\\ s_{i}^{2m}(t;h) \frac{1-t+mh}{1+2mh}, i = m\\ s_{i-1}^{2m}(t;h) \frac{t+mh}{1+2mh}, i = m+1\\ s_{i-1}^{2m}(t;h), m+1 \leq i \leq 2m \end{cases}$$
(6)

注 式(5),式(6)推广了文献[8]中给出的2m次与2m + 1次Said-Ball基函数之间的递推关系, 当h = 0时,式(5),式(6)退化为文献[8]中的Said-Ball基函数之间的递推关系式.

**性质3.6** 端点性质:  $s_i^n(0;h) = \begin{cases} 1, i = 0 \\ 0, i \neq 0 \end{cases}$ ,  $s_i^n(1;h) = \begin{cases} 1, i = n \\ 0, i \neq n \end{cases}$ . **性质3.7** 退化性质: 当h = 0时, n次h-Said-Ball基函数退化为n次Said-Ball基函数, 即 $s_i^n(t;0) = s_i^n(t)$ .

### 3.3 h-Said-Ball基函数和h-Bernstein基函数之间的转换

为了实现*h*-Said-Ball曲线与*h*-Bézier曲线的相互转化,本节分析了*h*-Said-Ball基函数与*h*-Bernstein基函数之间的转换,并给出两组基函数之间相互转换的矩阵表示.

**命题3.1**  $\{s_i^n(t;h)\}_{i=0}^n$ 为一组n次h-Said-Ball基函数,  $\{b_i^n(t;h)\}_{i=0}^n$ 为一组n次h-Bernstein

基函数,则有

 $(s_0^n(t;h), s_1^n(t;h), \cdots, s_n^n(t;h))^{\mathrm{T}} = (\alpha_{ij})_{(n+1)\times(n+1)} (b_0^n(t;h), b_1^n(t;h), \cdots, b_n^n(t;h))^{\mathrm{T}},$ 其中

$$\alpha_{ij} = \begin{cases} \left( \begin{array}{c} \lfloor n/2 \rfloor + i \\ i \end{array} \right) \left( \begin{array}{c} \lceil n/2 \rceil - i - 1 \\ j - i \end{array} \right), i \leq \lceil n/2 \rceil - 1, i \leq j \leq \lceil n/2 \rceil - 1 \\ \left( \begin{array}{c} n \\ j \end{array} \right) \\ \left( \begin{array}{c} \lfloor 3n/2 \rfloor - i \\ n - i \end{array} \right) \left( \begin{array}{c} i - 1 - \lfloor n/2 \rfloor \\ i - j \end{array} \right), i \geq \lfloor n/2 \rfloor + 1, \lfloor n/2 \rfloor + 1 \leq j \leq i \\ \left( \begin{array}{c} n \\ j \end{array} \right) \\ 1, i = j = n/2 \\ 0, \overline{\Delta} \mathbb{N} \end{bmatrix} \end{cases}$$

**证** 由于该证明在*n*为奇数次与*n*为偶数次的证法类似,下面只对*n*为偶数(*n* = 2*m*)情况进行证明.

$$\begin{split} \stackrel{\text{de}}{=} & 0 \leq i \leq m - 1 \text{H}^{j}, \\ s_{i}^{2m}(t;h) &= \binom{m+i}{i} \int_{k=0}^{\frac{i-1}{m+i}} \frac{(t+kh)\prod_{k=0}^{m}(1-t+kh)}{\prod_{k=0}^{m+i}(1+kh)} \frac{\left(1-t+(m+1)h\right)}{\left(1+(m+i+1)h\right)} + \frac{t+ih}{1+(m+i+1)h} \right) \cdots \left(\frac{1-t+(2m-i-1)h}{1+(2m-1)h} + \frac{t+ih}{1+(2m-1)h}\right) \\ &= \sum_{j=0}^{m-i-1} \frac{\binom{m+i}{i} \binom{m-i-1}{j}}{\binom{n}{i+j}} \frac{\prod_{k=0}^{i+j-1}(t+kh)\prod_{k=0}^{2m-i-j-1}(1-t+kh)}{\prod_{k=0}^{2m-1}(1+kh)} \\ &= \sum_{j=0}^{m-1} \frac{\binom{m+i}{i} \binom{m-i-1}{j-i}}{\binom{n}{j}} b_{j}^{2m}(t;h), \end{split}$$

$$\begin{split} s_{n-i}^{2m}(t;h) &= \binom{m+i}{i} \int \frac{\prod\limits_{k=0}^{m} (t+kh) \prod\limits_{k=0}^{i-1} (1-t+kh)}{\prod\limits_{k=0}^{m+i} (1+kh)} \left( \frac{t+(m+1)h}{1+(m+i+1)h} + \frac{1-t+ih}{1+(m+i+1)h} \right) \cdots \left( \frac{1-t+ih}{1+(2m-1)h} + \frac{t+(2m-i-1)h}{1+(2m-1)h} \right) \\ &= \sum_{j=0}^{m-i-1} \frac{\binom{m+i}{j} \binom{m-i-1}{j}}{\binom{2m}{2m-i-j}} \binom{2m}{2m-i-j} \frac{\prod\limits_{k=0}^{2m-i-j-1} (t+kh) \prod\limits_{k=0}^{i+j-1} (1-t+kh)}{\prod\limits_{k=0}^{2m-1} (1+kh)} \\ &= \sum_{j=0}^{m-1} \frac{\binom{3m-i}{2m-i} \binom{i-1-m}{i-j}}{\binom{2m}{j}} b_{2m-j}^{2m}(t;h) \,. \end{split}$$

 $\stackrel{\textrm{\tiny def}}{=} m \boxplus, \, s_m^{2m}\left(t;h\right) = b_m^{2m}\left(t;h\right), \, \boxplus \alpha_{mm} = 1.$ 

命题3.2  $\{s_i^n(t;h)\}_{i=0}^n$ 为一组n次h-Said-Ball基函数,  $\{b_i^n(t;h)\}_{i=0}^n$ 为一组n次h-Bernstein 基函数, 则有

$$(b_{0}^{n}(t;h), b_{1}^{n}(t;h), \cdots, b_{n}^{n}(t;h))^{\mathrm{T}} = (\beta_{ij})_{(n+1)\times(n+1)} (s_{0}^{n}(t;h), s_{1}^{n}(t;h), \cdots, s_{n}^{n}(t;h))^{\mathrm{T}},$$

其中

$$\beta_{ij} = \begin{cases} \frac{\binom{(-1)^{j-i}}{i}\binom{n}{j}\binom{\lceil n/2\rceil - i - 1}{j-i}}{\binom{\lfloor n/2\rceil + j}{j}}, i \leq \lceil n/2\rceil - 1, i \leq j \leq \lceil n/2\rceil - 1\\ \frac{\binom{\lfloor n/2\rfloor + j}{j}}{\binom{\lfloor -1 \rfloor^{i-j}\binom{n}{i}\binom{i-1 - \lfloor n/2\rfloor}{i-j}}{\binom{\lfloor 3n/2\rfloor - j}{n-j}}, i \geq \lfloor n/2\rfloor + 1, \lfloor n/2\rfloor + 1 \leq j \leq i\\ 1, i = j = n/2\\ 0, \overline{A} \bigcup \end{cases} \end{cases}$$

h-Said-Ball基与h-Bernstein基的转换矩阵可由h-Bernstein基到h-Said-Ball基的转换矩阵的 逆矩阵得到,即

$$(\beta_{ij})_{(n+1)\times(n+1)} = \left( (\alpha_{ij})_{(n+1)\times(n+1)} \right)^{-1}$$

根据命题3.1和命题3.2给出了 $n = 3\pi n = 4$ 时h-Said-Ball基函数与h-Bernstein基函数互相 转化的变换矩阵.

**例3** 四次*h*-Said-Ball基函数与*h*-Bernstein基函数互相转化的两个变换矩阵.

(	$s_0^4(t;h)$		$\begin{pmatrix} 1 \end{pmatrix}$	1/4	0	0	0 \	1	$\left(\begin{array}{c} b_0^4\left(t;h\right)\end{array}\right)$		$\left(\begin{array}{c} b_0^4\left(t;h\right)\end{array}\right)$		( 1	-1/3	0	0	0)	۱.	$\left( \begin{array}{c} s_0^4\left(t;h\right) \end{array} \right)$
	$s_{1}^{4}\left(t;h\right)$		0	3/4	0	0	0		$b_{1}^{4}\left( t;h\right)$		$b_{1}^{4}\left( t;h\right)$		0	4/3	0	0	0		$s_{1}^{4}\left( t;h\right)$
	$s_{2}^{4}\left( t;h\right)$	=	0	0	1	0	0	.	$b_{2}^{4}\left( t;h\right)$	,	$b_{2}^{4}\left( t;h\right)$	=	0	0	1	0	0	.	$s_{2}^{4}\left( t;h ight)$
	$s_{3}^{4}\left( t;h\right)$		0	0	0	3/4	0		$b_{3}^{4}\left( t;h\right)$		$b_{3}^{4}\left( t;h\right)$		0	0	0	4/3	0		$s_{3}^{4}\left( t;h\right)$
ĺ	$s_4^4(t;h)$		0	0	0	1/4	1 /		$b_4^4(t;h)$		$\left\langle b_{4}^{4}\left( t;h ight)  ight angle$		0	0	0	-1/3	1 /		$\left( s_{4}^{4}\left( t;h ight)  ight)$

例4 五次*h*-Said-Ball基函数与*h*-Bernstein基函数互相转化的两个变换矩阵.

( :	$s_0^5(t;h)$		( 1	2/5	1/10	0	0	0 )		$\left(\begin{array}{c} b_0^5(t;h) \end{array}\right)$	١	$(b_0^5(t;h))$	۱	$\begin{pmatrix} 1 \end{pmatrix}$	-2/3	1/6	0	0	0)		$\left( \begin{array}{c} s_{0}^{5}\left( t;h\right) \end{array} \right)$
	$s_{1}^{5}\left(t;h ight)$		0	3/5	3/10	0	0	0		$b_{1}^{5}\left( t;h\right)$		$b_{1}^{5}\left( t;h\right)$		0	7/6	5/6	0	0	0		$s_{1}^{5}\left( t;h\right)$
-	$s_{2}^{5}\left(t;h\right)$	_	0	0	3/5	0	0	0		$b_{2}^{5}\left( t;h\right)$		$b_{2}^{5}\left(t;h ight)$	_	0	0	7/6	0	0	0		$s_{2}^{5}\left(t;h ight)$
	$s_{3}^{5}(t;h)$	_	0	0	0	3/5	0	0	ŀ	$b_{3}^{5}\left(t;h ight)$	,	$b_{3}^{5}\left(t;h ight)$	-	0	0	0	7/6	0	0	·	$s_{3}^{5}\left(t;h ight)$
	$s_4^5(t;h)$		0	0	0	3/10	3/5	0		$b_{4}^{5}\left( t;h ight)$		$b_{4}^{5}\left(t;h ight)$		0	0	0	5/6	7/6	0		$s_{4}^{5}\left(t;h ight)$
( ;	$s_{5}^{5}(t;h)$		0	0	0	1/10	2/5	1 )		$\left( \begin{array}{c} b_{4}^{5}\left( t;h ight) \end{array} \right)$		$b_4^5(t;h)$	/	0	0	0	1/6	-2/3	1 )		$\left( s_{5}^{5}\left( t;h ight)  ight)$

例3和例4中的矩阵都为分块对角矩阵,其中左上的子矩阵为上三角矩阵,右下的子矩阵为 下三角矩阵.

### §4 h-Said-Ball曲线

基于上述h-Said-Ball基函数, 定义了h-Said-Ball曲线并分析其优良性质. 构造了h-Said-Ball曲线的递归求值算法并分析了其包络.

### 4.1 *h*-Said-Ball曲线定义及基本性质

**定义4.1** 给n + 1个空间向量 $P_i \in \mathbf{R}^3$  ( $i = 0, 1, \dots, n$ )和实数 $h \ge 0$ ,称参数曲线段 7)

$$\mathbf{S}(t) = \sum_{i=0}^{n} \mathbf{P}_{i} s_{i}^{n}(t;h), t \in [0,1]$$
(7)

为一条n次h-Said-Ball曲线. 其中 $P_i \in \mathbf{R}^3$  ( $i = 0, 1, \dots, n$ )称为控制顶点, 依次用直线段连接相 邻两个控制顶点所得的折线多边形称为曲线的控制多边形.

根据h-Said-Ball基函数的性质, 推得h-Said-Ball曲线的基本性质如下.

性质4.1 几何不变性和仿射不变性

因为*h*-Said-Ball基函数具有单位分解性质,对*h*-Said-Ball曲线进行仿射变换得到新的*h*-Said-Ball曲线.新的曲线是由原控制顶点经过相同的仿射变换后得到的新的控制顶点所构造的曲线.由此说明*h*-Said-Ball曲线不依赖于坐标系的选取,具有几何不变性和仿射不变性.

**性质4.2** 凸包性

由于*h*-Said-Ball基函数的非负性和单位分解性,可知对于固定的*t*,曲线为各控制顶点的加权平均,说明*h*-Said-Ball在*h* > 0时落在了控制顶点的凸包中.

性质4.3 端点插值性

根据*h*-Said-Ball基函数的端点性质,可得曲线的端点插值性,即 $S(0) = P_0, S(n) = P_n$ .

性质4.4 形状可调性

由于*h*-Said-Ball曲线带有参数*h*,故当控制顶点固定时,*h*-Said-Ball曲线的形状可通过参数*h*的取值进行调整.





如图3所示,当控制顶点不变时,随着参数h的取值增大,曲线逐渐远离控制多边形并逐渐靠 近连接首末端点的直线.

性质4.5 退化性质

当h = 0时, h-Said-Ball曲线退化为Said-Ball曲线.

4.2 全正基

在参数曲线曲面中,全正基函数与造型设计中的变差缩减性和保凸性有密切关联. 当基函数为全正基时,则由其构成的曲线具有变差缩减性和保凸性. 首先给出全正基的相关概念.

定义4.2<sup>[23]</sup> 若实矩阵A的所有子式都非负,则称A为全正矩阵.

定义4.3<sup>[23]</sup> 在区间I上,若基函数序列

$$\{U_0(t), U_1(t), \cdots, U_n(t)\}$$

关于I上任意点序列

 $0 \le t_0 < t_1 < \dots < t_n \le 1$ 

的配置矩阵 $(U_j(t_i))_{i,j=0}^n$ 是全正矩阵,则称这组基函数为全正基.若这组全正基满足单位分解性,则称其为标准全正基.

CAGD中许多用于曲线上一点的求值算法都是割角算法<sup>[24]</sup>(文献[25]将其称为割角变换), 即每一步都是由控制点的凸组合组成的算法.例如Bézier曲线的de Casteljau算法和样条曲线 的de Boor-Cox算法.割角算法也可用于全正基的证明<sup>[26]</sup>.

**引理4.1**<sup>[26]</sup> 假设{ $\phi_i^n(t)$ } $_{i=0}^n$ , { $\varphi_i^n(t)$ } $_{i=0}^n$ 是向量空间的两组基, 其中{ $\varphi_i^n(t)$ } $_{i=0}^n$ 为标准全 正基, 若存在从控制顶点 $C_0, \dots, C_n$ 到 $D_0, \dots, D_n$ 的割角算法, 使得

$$\sum_{i=0}^{n} C_{i} \phi_{i}^{n}(t) = \sum_{i=0}^{n} D_{i} \varphi_{i}^{n}(t),$$

则 $\{\phi_i^n(t)\}_{i=0}^n$ 为标准全正基.

命题4.1 假设

$$\left\{s_{i}^{2m+1}\left(t;h\right)\right\}_{i=0}^{2m+1}$$

为2m + 1次的h-Said-Ball基函数,

$$\left\{b_{i}^{2m+1}\left(t;h\right)\right\}_{i=0}^{2m+1}$$

为2m + 1次的h-Bernstein基函数,那么从

$$S(t) = \sum_{i=0}^{2m+1} P_i s_i^{2m+1}(t;h)$$

到

$$P(t) = \sum_{i=0}^{2m+1} V_i b_i^{2m+1}(t;h)$$

的割角算法分为两种,即

左割角变换

$$\begin{cases}
P_i^0 = P_i, i = 0, \cdots, m; \\
P_0^b = P_0, k = 1, \cdots, m; \\
P_i^j = \frac{m+j}{m+j+i} P_i^{j-1} + \frac{i}{m+j+i} P_{i-1}^j, j = 1, \cdots, m + 1 - i; i = 1, \cdots, m. \\
V_0 = P_0; \\
V_k = P_i^{m+1-i}, i = 1, \cdots, m.
\end{cases}$$
(8)

右割角变换

$$\begin{cases} P_{i}^{0} = P_{i}, i = m + 1, \cdots, 2m + 1; \\ P_{2m+1}^{k} = P_{2m+1}, k = 1, \cdots, m; \\ P_{2m+1-i}^{j} = \frac{i}{m+j+i} P_{2m+2-i}^{j} + \frac{m+j}{m+j+i} P_{2m+1-i}^{j-1}, j = 1, \cdots, m + 1 - i; i = 1, \cdots, m. \\ \begin{cases} V_{2m+1} = P_{0}; \\ V_{2m+1-i} = P_{m+1-i}^{2m+1-i}, i = 1, \cdots, m. \end{cases} \end{cases}$$

$$(9)$$

先证左割角变换成立. 已知S(t)为n = 2m + 1次的h-Said-Ball曲线, 用数学归纳法证明(8)式

在 j = 1, …, m时成立. 当 j = 1时, 对S(t)的第一项乘以<sup>1-t+(m+1)h</sup>/<sub>1+(m+1)h</sub> + <sup>t</sup>/<sub>1+(m+1)h</sub>,

S(t) = P\_0  $\frac{\prod_{k=0}^{m} (1-t+kh)}{\prod_{k=0}^{m} (1+kh)} \times \left(\frac{1-t+(m+1)h}{1+(m+1)h} + \frac{t}{1+(m+1)h}\right) + \sum_{i=1}^{2m+1} P_i s_i^n(t;h)$ =  $P_0 \frac{\prod_{k=0}^{m+1} (1-t+kh)}{\prod_{k=0}^{m+1} (1+kh)} + P_1^1 \frac{t}{\prod_{k=0}^{m} (1-t+kh)} \prod_{k=0}^{2m+1} P_i s_i^n(t;h)$ =  $\sum_{i=0}^{1} P_1^1 \left(\frac{m+2}{1}\right) \frac{\prod_{k=0}^{i-1} (t+kh) \prod_{k=0}^{m+1-i} (1-t+kh)}{\prod_{k=0}^{m+1} (1+kh)} + \sum_{i=2}^{2m+1} P_i s_i^n(t;h).$ 

根据上式, 在j = 1时有

$$P_0^1 = P_0^0, P_1^1 = \frac{1}{m+2}P_0^1 + \frac{m+1}{m+2}P_1.$$

假设当j = k时有

$$S(t) = \sum_{i=0}^{k} P_i^{k+1-i} \left( \begin{array}{c} m+k+1\\ i \end{array} \right) \frac{\prod_{k=0}^{i-1} (1-t+kh) \prod_{k=0}^{m+k-i} (1-t+kh)}{\prod_{k=0}^{m+k} (1+kh)} + \sum_{i=k+1}^{2m+1} P_i s_i^{2m+1}(t;h).$$

当j = k + 1时, 对S(t)前k + 1项分别乘以{ $E_i(t)$ } = { $\frac{1-t+(m+k+1-i)h}{1+(m+k+1)h} + \frac{t+ih}{1+(m+k+1)h}$ },  $i = 0, \dots, k,$ 则有

$$\begin{split} S\left(t\right) &= \sum_{i=0}^{k} P_{i}^{k+1-i} \left( \begin{array}{c} m+k+1\\ i \end{array} \right) \frac{\prod_{k=0}^{i-1} \left(1-t+kh\right) \prod_{k=0}^{m+k-i} \left(1-t+kh\right)}{\prod_{k=0}^{m+k} \left(1+kh\right)} \left\{ Q_{i}\left(t\right) \right\} \\ &+ P_{k+1} s_{k+1}^{2m+1}\left(t;h\right) + \sum_{i=k+2}^{2m+1} P_{i} s_{i}^{2m+1}\left(t;h\right) \\ &= \sum_{i=0}^{k+1} P_{i}^{k+2-i} \left( \begin{array}{c} m+k+2\\ i \end{array} \right) \frac{\prod_{k=0}^{i-1} \left(1-t+kh\right) \prod_{k=0}^{m+k+1-i} \left(1-t+kh\right)}{\prod_{k=0}^{m+k+1} \left(1+kh\right)} + \sum_{i=k+2}^{2m+1} P_{i} s_{i}^{2m+1}\left(t;h\right), \end{split}$$

由此得到

$$S(t) = \sum_{i=0}^{m} V_i b_i^{2m+1}(t;h) + \sum_{i=j+1}^{2m+1} P_i s_i^{2m+1}(t;h).$$

下证右割角变换成立. 用数学归纳法证明(9)式在j = 1, · · · , m时成立.

当
$$j = 1$$
时, 对 $S(t)$ 的最后一项乘 $\frac{t+(m+1)h}{1+(m+1)h} + \frac{1-t}{1+(m+1)h}$ ,  
 $S(t) = P_{2m+1} \frac{\prod\limits_{k=0}^{m} (t+kh)}{\prod\limits_{k=0}^{m} (1+kh)} \times \left[ \frac{t+(m+1)h}{1+(m+1)h} + \frac{1-t}{1+(m+1)h} \right] + \sum_{i=0}^{2m} P_i s_i^n(t;h)$   
 $= P_{2m+1} \frac{\prod\limits_{k=0}^{m+1} (t+kh)}{\prod\limits_{k=0}^{m+1} (1+kh)} + P_{2m}^1 \frac{(1-t)\prod\limits_{k=0}^{m} (t+kh)}{\prod\limits_{k=0}^{m} (1+kh)} + \sum_{i=0}^{2m-1} P_i s_i^n(t;h)$ .  
假设当 $j = k$ 时,可得

$$S\left(t
ight) =$$

$$\sum_{i=2m+1-k}^{2m+1} P_{2m+1-i}^{k+1-i} \left( \begin{array}{c} m+k+1\\ i \end{array} \right) \frac{\prod\limits_{k=0}^{m+k-i} (1-t+kh) \prod\limits_{k=0}^{i-1} (1-t+kh)}{\prod\limits_{k=0}^{m+k} (1+kh)} + \sum_{i=0}^{2m+1-(k+1)} P_i s_i^{2m+1} \left( t; h \right).$$

则有

$$S(t) = \sum_{i=0}^{m} V_{2m+1-i} b_{2m+1-i}^{2m+1}(t;h) + \sum_{i=0}^{m} P_i s_i^{2m+1}(t;h).$$

命题4.1得到了奇数次h-Said-Ball曲线到h-Bézier曲线的割角算法,又由任意次h-Bernstein 基函数在其多项式空间中为标准全正基<sup>[27]</sup>,根据引理4.1可知奇数次h-Said-Ball基函数为标准全 正基.

对于偶数次的曲线,即当n = 2m时,根据递推公式(5)式得

$$s^{2m}(t;h) = s^{2m+1}(t;h) M, M = (m_{ij})_{(2m+2)\times(2m+1)},$$
$$m_{ij} = \begin{cases} 1, 0 \le i = j \le m, m+1 \le i = j+1 \le 2m+1, \\ 0, \overleftarrow{\alpha} \not \blacksquare. \end{cases}$$

易证M为全正矩阵, 则n = 2m时h-Said-Ball基函数为标准全正基.

**定理4.1** *n*次多项式空间的*h*-Said-Ball基函数 $\{s_0^n(t;h), s_1^n(t;h), \dots, s_n^n(t;h)\}$ 为标准全正基.

*h*-Said-Ball基函数是标准全正基,根据文献[27],可知在I上,若序列 $\{U_0(t), \dots, U_n(t)\}$ 是全正函数序列,那么对于任意数 $a_0, \dots, a_n$ 有

$$S^{-}(a_{0}U_{0}(t) + \dots + a_{n}U_{n}(t)) \leq S^{-}(a_{0}, \dots, a_{n}).$$

所以对于h-Said-Ball基函数有

$$S^{-}(P_0s_0(t;h) + \dots + P_ns_n(t;h)) \leq S^{-}(P_0,\dots,P_n),$$

因此*h*-Said-Ball曲线具有变差缩减性.即*h*-Said-Ball曲线与所在平面内任一直线的交点个数不 会超过它的控制顶点与该直线的交点个数.*h*-Said-Ball曲线的保凸性是指如果*h*-Said-Ball曲线 的控制多边形是凸的,那么*h*-Said-Ball曲线也是凸的.保凸性是变差缩减性的特殊情况<sup>[26]</sup>.

推论4.1 任意n次h-Said-Ball曲线具有变差缩减性.

推论4.2 任意n次h-Said-Ball曲线具有保凸性.

#### 4.3 递归求值算法和包络定理

结合Said-Ball曲线的递归求值算法和*h*-Bézier曲线的*h*-de Castelijau算法,给出*h*-Said-Ball曲线的递归求值算法.

算法4.1 h-Said-Ball曲线的递归求值算法

步骤1 当h-Said-Ball曲线为奇数次时,先做一次递推降到偶数次曲线. n = 2m + 1时,

$$P_i^0 = \begin{cases} P_i, 0 \le i \le m-1; \\ \frac{1-t+(n-i-1)h}{1+(n-1)h} P_i + \frac{t+ih}{1+(n-1)h} P_{i+1}, i = m; \\ P_{i+1}, m+1 \le i \le 2m, \end{cases}$$

n = 2m时,  $P_i^0 = P_i, i = 0, \cdots, 2m$ .

步骤2 将2m次h-Said-Ball曲线转化为m+1次h-Bézier形式的曲线( $1 \le r \le m-1$ ),

$$P_i^r = \begin{cases} P_i^{r-1}, 0 \le i \le m - r - 1; \\ \frac{1 - t + (2m - r)h}{1 + (2m - r)h} P_i^{r-1} + \frac{t + ih}{1 + (2m - r)h} P_{i+1}^{r-1}, m - r \le i \le m; \\ P_{i+1}^{r-1}, m + 1 \le i \le 2m - r. \end{cases}$$

步骤3 对m + 1次h-Bézier形式的曲线执行h-de Casteljau算法( $m \le r \le 2m$ ),

$$P_{i}^{r} = \frac{1 - t + (2m - i - r)h}{1 + (2m - r)h} P_{i}^{r-1} + \frac{t + ih}{1 + (2m - r)h} P_{i+1}^{r-1}, i = 0, \cdots, 2m - r.$$

如果n为奇数,

$$S(t) = \sum_{i=0}^{2m} P_i^0(t) \, s_i^{2m}(t;h) = \dots = P_0^{2m}(t) \, ;$$

如果n为偶数,

$$S(t) = \sum_{i=0}^{2m-1} P_i^1(t) \, s_i^{2m-1}(t;h) = \dots = P_0^{2m}(t) \, .$$

任意*n*次*h*-Said-Ball曲线的递归求值算法用矩阵表示如下: 当*n*为偶数(n = 2m)时,  $P^r(t) = H_r(t) \cdots H_1(t)$ , 其中前m - 1个矩阵, 即 $r = 1, \cdots, m - 1$ 时,



后m个矩阵, 即 $r = m, \cdots, 2m$ 时

$$\boldsymbol{H}_{r}\left(t\right) = \left(\begin{array}{cccc} \frac{\frac{1-t+(2m-r)h}{1+(2m-r)h}}{\frac{1-t+(2m-r)h}{1+(2m-r)h}} & \frac{t}{1+(2m-r)h} \\ & \frac{1-t+(2m-r-1)h}{1+(2m-r)h} & \frac{t+h}{1+(2m-r)h} \\ & \ddots & \ddots \\ & & \frac{1-t}{1+(2m-r)h} & \frac{t+(2m-r)h}{1+(2m-r)h} \end{array}\right).$$

当n为奇数(n = 2m+1)时,  $P^{r}(t) = H_{r}(t) \cdots H_{1}(t)$ , 其中前m-1个矩阵, 即 $r = 1, \cdots, m-1$ 时

0



**注** 由算法4.1可以统计*h*-Said-Ball曲线递归求值算法的计算量. 当求*n*次*h*-Said-Ball曲线 上的一点时,执行算法4.1 *n*为奇数次和偶数次的乘法次数*M*<sup>*n*</sup><sub>*h*</sub>(*hS*)分别为

$$M_h^{2m}(hS) = 2\left(1+2+\dots+\frac{n}{2}+\left(\frac{n}{2}+1\right)+\frac{n}{2}+\dots+3+2\right) = \frac{(n+2)^2}{2}-2,$$
$$M_h^{2m+1}(hS) = 2\left(1+2+\dots+\frac{n-1}{2}+\frac{n+1}{2}+\frac{n-1}{2}+\dots+2+1\right) = \frac{(n+1)^2}{2}.$$

n为奇数次和偶数次的加法次数Anh (hS)分别为

$$A_{h}^{2m}(hS) = 1 + 2 + \dots + \frac{n}{2} + \left(\frac{n}{2} + 1\right) + \frac{n}{2} + \dots + 3 + 2 = \frac{(n+2)^{2}}{4} - 1,$$
  
$$A_{h}^{2m+1}(hS) = 1 + 2 + \dots + \frac{n-1}{2} + \frac{n+1}{2} + \frac{n-1}{2} + \dots + 2 + 1 = \frac{(n+1)^{2}}{4}.$$

求*n*次*h*-Bézier曲线上的一点,执行*h*-de Casteljau算法<sup>[18]</sup>的乘法次数( $M_h^n(hB)$ )和加法次数( $A_h^n(hB)$ )分别为

$$M_h^n(hB) = 2(1+2+\dots+n) = n(n+1), \quad A_h^n(hB) = 1+2+\dots+n = \frac{n(n+1)}{2}.$$

上式得到了Said-Ball曲线和Bézier曲线执行各自算法比较的相同结果<sup>[6]</sup>,即求*h*-Said-Ball曲线一点的值所需的运算近似是*h*-Bézier曲线的一半.计算量减少的同时,当取相同的*h*时,*h*-Said-Ball曲线相较于*h*-Bézier曲线更远离控制多边形.



图 4 六次h-Bézier曲线和h-Said-Ball曲线

图4为n = 6时的两种曲线,其中实线表示h-Said-Ball曲线(从里向外h=0.5, 0.1, 0),点线 表示h-Bézier曲线(从里向外h=0.5, 0.1, 0).根据曲线的退化性可知当h取0时,h-Said-Ball曲线 和h-Bézier曲线分别退化为Said-Ball曲线和Bézier曲线.当h取相同的值时,h-Said-Ball曲线离控 制多边形较远,h-Bézier曲线离控制多边形较近.从参数h的影响来看,h的值越大,会使曲线越远 离控制多边形.

包络是一种曲线的几何生成法<sup>[28]</sup>,包络定理有助于研究参数曲线的几何性质.通过*h*-Said-Ball曲线的递归求值算法得到*h*-Said-Ball曲线的包络定理如下.

**定理4.2** n次h-Said-Ball曲线是n - 1次h-Said-Ball曲线族的包络,即

$$S_{n}(t) = env\left\{\widehat{S}_{n-1}(t,\lambda) = \sum_{i=0}^{n-1} s_{i}^{n-1}(t) P_{i}^{1}(\lambda), 0 \le t \le 1 | 0 \le \lambda \le 1\right\}.$$

证 h-Said-Ball曲线的递归求值算法可用以下形式表示.

当 n = 2m + 1时,

$$\widehat{P}_{i}(\lambda) = \begin{cases} P_{i}(\lambda), 0 \le i \le m - 1; \\ \frac{1 - \lambda + ((n-1)/2)h}{1 + (n-1)h} P_{m}(\lambda) + \frac{\lambda + ((n-1)/2)h}{1 + (n-1)h} P_{m+1}(\lambda), i = m; \\ P_{i+1}(\lambda), m + 1 \le i \le n - 1. \end{cases}$$

当 n = 2m 时,

$$\begin{cases} \tilde{P}_i\left(\lambda\right) = \frac{1-\lambda+mh}{1+(m+i)h}P_i + \frac{\lambda+ih}{1+(m+i)h}\tilde{P}_{i+1}, i = m-1, m-2, \cdots, 1, 0;\\ \tilde{P}_{n-i}\left(\lambda\right) = \frac{1-\lambda+ih}{1+(m+i)h}\tilde{P}_{n-i+1} + \frac{\lambda+mh}{1+(m+i)h}P_{n-i}, i = m-1, m-2, \cdots, 1, 0;\\ \tilde{P}_i\left(\lambda\right) = P_i\left(\lambda\right), i = m.\\ \tilde{P}_i\left(\lambda\right) = \begin{cases} \tilde{P}_i\left(\lambda\right), 0 \le i \le m-1,\\ \tilde{P}_{i+1}\left(\lambda\right), m \le i \le 2m-1. \end{cases}$$

由此可得到一族n - 1次的h-Said-Ball曲线 $\widehat{S}_{n-1}(t, \lambda) = \sum_{i=0}^{n-1} s_i^{n-1}(t) \tilde{P}_i(\lambda).$ 根据h-Said-Ball曲线的性质推得

$$\widehat{S}_{n-1}(t,t_0)|_{\lambda=t_0} = S_n(t_0), \frac{\partial S_n(t)}{\partial t} \| \frac{\partial S_{n-1}(t,\lambda)}{\partial t}|_{\lambda=t}.$$

推论4.3 n次h-Said-Ball曲线是n - r次h-Said-Ball曲线族的包络.

n次 $(n \ge 2)$ 的h-Said-Ball曲线可以看作是以 $P_1^i$ , $i = 0, \cdots, n-1$ 为控制顶点的n-1次曲线 族的包络. 依次类推可知n次h-Said-Ball曲线是n - r次h-Said-Ball曲线族的包络. 另一方面,用 直线段定义一次h-Said-Ball曲线族以后,则二次曲线可用其包络生成,如此递归进行就可以用包 络的方式几何地生成n次h-Said-Ball曲线.

### §5 结语

本文利用Bézier曲线和Said-Ball曲线的递归求值算法的特点,将Said-Ball曲线推广到了h-Said-Ball曲线. h-Said-Ball曲线具有与Said-Ball曲线类似的几何不变性,凸包性,变差缩减性等优良的几何性质.特别地,h-Said-Ball基函数是标准全正基,所以h-Said-Ball曲线具有变差缩减性和保凸性.h-Said-Ball曲线相比于Said-Ball曲线增加了参数h,可以在不改变控制多边形顶点的条件下,仅改变形状参数h的取值,就可以得到不同程度的逼近控制多边形的曲线,实现对曲线灵活而有效的整体或局部调控.在递归求值算法方面,本文给出h-Said-Ball曲线的递归算法,与h-Bézier曲线的h-de castelijau算法相比,运算量减少了一半.本文在§2给出了h-Said-Ball基函数和h-Bernstein基函数之间的转换公式,由此可以实现h-Said-Ball曲线与h-Bézier 曲线的相互转换.当h-Bézier曲线的计算较为复杂时,可将h-Bézier曲线转化为h-Said-Ball曲线从而减少计算量.

*h*-Said-Ball曲线作为Said-Ball曲线在*h*-微积分理论下的推广,保留了Said-Ball曲线求值高效等优良性质,参数*h*还可实现对曲线形状的调控.类似地,作者也研究了Wang-Ball曲线在*h*-微积分理论下的推广,Wang-Ball曲线和Said-Ball曲线在*q*-微积分理论下的推广,并将进一步推广 至张量积曲面和三角曲面情形.

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#### h-Said-Ball bases and h-Said-Ball curves

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Abstract: The *h*-Bézier curve is a generalized model of Bézier curve based on the sense of *h*-calculus. In order to enhance the modeling ability of Said-Ball curve and improve the speed of recursive valuation of *h*-Bézier curve, this paper proposes the *h*-Said-Ball basis function of arbitrary order and constructs the *h*-Said-Ball curve. By analyzing the transformation relationship between the recursive valuation algorithm of Said-Ball curve and Bézier curve, combining the recursive valuation algorithm of *h*-Bézier curve and the construction method of *h*-Bernstein basis function, the expressions of arbitrary times of *h*-Said-Ball basis function are obtained. The *h*-Said-Ball basis has excellent properties such as non-negativity, unit decomposition, and endpoint interpolation, and there is an explicit transformation matrix between it and the *h*-Bernstein basis. Further, the *h*-Said-Ball curve is defined and its basic properties are analyzed, and the recursive valuation algorithm and envelope representation are derived. *h*-Said-Ball curve is half the computational effort of the *h*-Bézier curve, it is shown that the *h*-Said-Ball basis is a fully positive basis, and thus the *h*-Said-Ball curve has variational reduction and convexity preservation. Numerical examples show the modeling advantages and flexibility of the *h*-Said-Ball curve.

**Keywords**: *h*-Bézier curve; Said-Ball curve; *h*-Said-Ball bases; *h*-Said-Ball curve; totally positive basis; recursive evaluation algorithm

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