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矩阵最小奇异值的两个估计

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摘　要: 利用行列式和Frobenius范数，给出了非奇异矩阵最小奇异值的两个估计. 在某些情况下，数值例子表明所得的估计优于已存在的结果.

关键词: 最小奇异值; Frobenius范数; 行列式

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§1　引　言

设M_n为$n \times n$复矩阵集合, I_n表示n阶单位矩阵, 矩阵$A \in M_n$的奇异值为$\sigma_i(i=1,\cdots,n)$, 其中$\sigma_1 \ge \sigma_2 \ge \cdots \ge \sigma_{n-1} \ge \sigma_n \ge 0$. 对于$A=[a_{ij}] \in M_n$的Frobenius范数定义为

$$\|A\|_F = (\sum_{i,j=1}^{n} |a_{ij}|^2)^{\frac{1}{2}} = (\mathrm{tr}A^H A)^{\frac{1}{2}},$$

其中A^H是A的共轭转置. 下文中的A均为非奇异矩阵.

Yu和Gu在[1]中得到奇异值σ_n的下界为

$$\sigma_n \ge |\mathrm{det}A|(\frac{n-1}{\|A\|_F^2})^{\frac{n-1}{2}} = l > 0. \tag{1.1}$$

Zou在[2]中改进了不等式(1.1), 得到了

$$\sigma_n \ge |\mathrm{det}A|(\frac{n-1}{\|A\|_F^2 - l^2})^{\frac{n-1}{2}} = l_0.$$

Lin和Xie在[3]中得到了矩阵最小奇异值的下界, 证明a是方程

$$x^2(\|A\|_F^2 - x^2)^{n-1} = |\mathrm{det}A|^2(n-1)^{n-1}$$

的最小正解且$\sigma_n \ge a > l_0$.

Shun在[4]中得到了如下的不等式

$$\sigma_n \ge (l_0^2 + |\mathrm{det}(l_0^2 I_n - A^H A)|(\frac{n-1}{\|A\|_F^2 - n l_0^2})^{n-1})^{\frac{1}{2}} = l_1,$$

证明b就是方程

$$x = (l_0^2 + |\mathrm{det}(l_0^2 I_n - A^H A)|(\frac{n-1}{\|A\|_F^2 - x^2 - (n-1)l_0^2})^{n-1})^{\frac{1}{2}}$$

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的最小正解且$\sigma_n \geq b > l_1$.

Liu等在[5]中得到了如下的不等式

$$\sigma_n \geq (l_1^2 - l_0^2 + |\det((l_1^2 - l_0^2)I_n - A^H A)|(\frac{n-1}{\|A\|_F^2 - n(l_1^2 - l_0^2)})^{n-1})^{\frac{1}{2}} = l_2,$$

同时还在[5]中证明了c是方程

$$x = (l_1^2 - l_0^2 + |\det((l_1^2 - l_0^2)I_n - A^H A)|(\frac{n-1}{\|A\|_F^2 - x^2 - (n-1)(l_1^2 - l_0^2)})^{n-1})^{\frac{1}{2}}$$

的最小正解且$\sigma_n \geq c > l_2$.

本文将给出非奇异矩阵最小奇异值的两个新的下界l_3和d. 在某些情况下, 数值例子表明所得下界优于已存在的结果.

§2 主要结果

定理2.1　设$A \in M_n$是非奇异矩阵,

$$(l_1^2 - (l_0^2 - l^2) + |\det((l_1^2 - (l_0^2 - l^2))I_n - A^H A)|(\frac{n-1}{\|A\|_F^2 - n(l_1^2 - (l_0^2 - l^2))})^{n-1})^{\frac{1}{2}} = l_3,$$

则$\sigma_n \geq l_3$, 其中$l = |\det A|(\frac{n-1}{\|A\|_F^2})^{\frac{n-1}{2}}, l_0 = |\det A|(\frac{n-1}{\|A\|_F^2 - l^2})^{\frac{n-1}{2}}$,

$$l_1 = (l_0^2 + |\det(l_0^2 I_n - A^H A)|(\frac{n-1}{\|A\|_F^2 - n l_0^2})^{n-1})^{\frac{1}{2}}.$$

证　当$0 < \eta < \zeta < \lambda < \sigma_n^2$有

$$\begin{aligned}&|(\lambda - (\zeta - \eta) - \sigma_1^2)(\lambda - (\zeta - \eta) - \sigma_2^2)\cdots(\lambda - (\zeta - \eta) - \sigma_{n-1}^2)|\\ &\leq (\frac{\sigma_1^2 + \sigma_2^2 + \cdots + \sigma_{n-1}^2 - (n-1)(\lambda - (\zeta - \eta))}{n-1})^{n-1}.\end{aligned}$$

于是

$$\begin{aligned}&|(\lambda - (\zeta - \eta) - \sigma_1^2)(\lambda - (\zeta - \eta) - \sigma_2^2)\cdots(\lambda - (\zeta - \eta) - \sigma_{n-1}^2)|\\ &= \frac{|(\lambda - (\zeta - \eta) - \sigma_1^2)(\lambda - (\zeta - \eta) - \sigma_2^2)\cdots(\lambda - (\zeta - \eta) - \sigma_n^2)|}{\sigma_n^2 - (\lambda - (\zeta - \eta))}\\ &= \frac{|\det((\lambda - (\zeta - \eta))I_n - A^H A)|}{\sigma_n^2 - (\lambda - (\zeta - \eta))}.\end{aligned}$$

则

$$(\frac{\sigma_1^2 + \sigma_2^2 + \cdots + \sigma_{n-1}^2 - (n-1)(\lambda - (\zeta - \eta))}{n-1})^{n-1} \geq \frac{|\det((\lambda - (\zeta - \eta))I_n - A^H A)|}{\sigma_n^2 - (\lambda - (\zeta - \eta))},$$

$$\sigma_n^2 \geq \lambda - (\zeta - \eta) + |\det((\lambda - (\zeta - \eta))I_n - A^H A)|(\frac{n-1}{\sigma_1^2 + \cdots + \sigma_{n-1}^2 - (n-1)(\lambda - (\zeta - \eta))})^{n-1},$$

即

$$\sigma_n \geq (\lambda - (\zeta - \eta) + |\det((\lambda - (\zeta - \eta))I_n - A^H A)|(\frac{n-1}{\sigma_1^2 + \cdots + \sigma_{n-1}^2 - (n-1)(\lambda - (\zeta - \eta))})^{n-1})^{\frac{1}{2}}.$$

由$l < \sigma_n, l^2 < \sigma_n^2, l_0 < \sigma_n, l_0^2 < \sigma_n^2, l_1 < \sigma_n, l_1^2 < \sigma_n^2$, 令$\eta = l^2, \zeta = l_0^2, \lambda = l_1^2$, 可得

$$\begin{aligned}&\sigma_n \geq\\ &(l_1^2 - (l_0^2 - l^2) + |\det((l_1^2 - (l_0^2 - l^2))I_n - A^H A)|(\frac{n-1}{\|A\|_F^2 - \sigma_n^2 - (n-1)(l_1^2 - (l_0^2 - l^2))})^{n-1})^{\frac{1}{2}}. \quad (2.1)\end{aligned}$$

故

$$\sigma_n \geq (l_1^2-(l_0^2-l^2)+|\det((l_1^2-(l_0^2-l^2))I_n-A^HA)|(\frac{n-1}{\|A\|_F^2-n(l_1^2-(l_0^2-l^2))})^{n-1})^{\frac{1}{2}}.$$

定理2.2 设$A\in M_n$是非奇异矩阵,

$$d_{k+1}=$$

$$(l_1^2-(l_0^2-l^2)+|\det((l_1^2-(l_0^2-l^2))I_n-A^HA)|(\frac{n-1}{\|A\|_F^2-(n-1)(l_1^2-(l_0^2-l^2))-d_k^2})^{n-1})^{\frac{1}{2}},$$

其中

$$l=|\det A|(\tfrac{n-1}{\|A\|_F^2})^{\frac{n-1}{2}},\quad l_0=|\det A|(\tfrac{n-1}{\|A\|_F^2-l^2})^{\frac{n-1}{2}},$$

$$l_1=(l_0^2+|\det(l_0^2I_n-A^HA)|(\tfrac{n-1}{\|A\|_F^2-nl_0^2})^{n-1})^{\frac{1}{2}},$$

$$d_1=(l_1^2-(l_0^2-l^2)+|\det((l_1^2-(l_0^2-l^2))I_n-A^HA)|(\tfrac{n-1}{\|A\|_F^2-(n-1)(l_1^2-(l_0^2-l^2))})^{n-1})^{\frac{1}{2}},$$

则$0<d_k<d_{k+1}\leq\sigma_n, k=1,2,\cdots,\lim_{k\longrightarrow\infty}d_k$存在.

证 对k用数学归纳法证明

$$\sigma_n\geq d_{k+1}>d_k>0.$$

由(2.1)有

$$\begin{aligned}\sigma_n &\geq (l_1^2-(l_0^2-l^2)+|\det((l_1^2-(l_0^2-l^2))I_n-A^HA)|(\tfrac{n-1}{\|A\|_F^2-\sigma_n^2-(n-1)(l_1^2-(l_0^2-l^2))})^{n-1})^{\frac{1}{2}}\\ &\geq (l_1^2-(l_0^2-l^2)+|\det((l_1^2-(l_0^2-l^2))I_n-A^HA)|(\tfrac{n-1}{\|A\|_F^2-(n-1)(l_1^2-(l_0^2-l^2))})^{n-1})^{\frac{1}{2}}=d_1.\end{aligned}$$

因此$\sigma_n\geq d_1$, 则有

$$\begin{aligned}&\sigma_n\geq (l_1^2-(l_0^2-l^2)+|\det((l_1^2-(l_0^2-l^2))I_n-A^HA)|(\tfrac{n-1}{\|A\|_F^2-\sigma_n^2-(n-1)(l_1^2-(l_0^2-l^2))})^{n-1})^{\frac{1}{2}}\\ &\geq (l_1^2-(l_0^2-l^2)+|\det((l_1^2-(l_0^2-l^2))I_n-A^HA)|(\tfrac{n-1}{\|A\|_F^2-(n-1)(l_1^2-(l_0^2-l^2))-d_1^2})^{n-1})^{\frac{1}{2}}=d_2\\ &> (l_1^2-(l_0^2-l^2)+|\det((l_1^2-(l_0^2-l^2))I_n-A^HA)|(\tfrac{n-1}{\|A\|_F^2-(n-1)(l_1^2-(l_0^2-l^2))})^{n-1})^{\frac{1}{2}}=d_1>0.\end{aligned}$$

当$k=1$时有

$$\sigma_n\geq d_2>d_1>0.$$

假定对$k=m$仍然成立, 即$\sigma_n\geq d_{m+1}>d_m>0$成立. 现在考虑$k=m+1$的情况, 由(2.1)有

$$\begin{aligned}&\sigma_n\geq (l_1^2-(l_0^2-l^2)+|\det((l_1^2-(l_0^2-l^2))I_n-A^HA)|(\tfrac{n-1}{\|A\|_F^2-\sigma_n^2-(n-1)(l_1^2-(l_0^2-l^2))})^{n-1})^{\frac{1}{2}}\\ &\geq (l_1^2-(l_0^2-l^2)+|\det((l_1^2-(l_0^2-l^2))I_n-A^HA)|(\tfrac{n-1}{\|A\|_F^2-d_{m+1}^2-(n-1)(l_1^2-(l_0^2-l^2))})^{n-1})^{\frac{1}{2}}=\\ &d_{m+2}\\ &> (l_1^2-(l_0^2-l^2)+|\det((l_1^2-(l_0^2-l^2))I_n-A^HA)|(\tfrac{n-1}{\|A\|_F^2-d_m^2-(n-1)(l_1^2-(l_0^2-l^2))})^{n-1})^{\frac{1}{2}}=\\ &d_{m+1}>0.\end{aligned}$$

因此$\sigma_n\geq d_{m+2}>d_{m+1}>0$. 这证明了$\sigma_n\geq d_{k+1}>d_k>0, k=1,2,\cdots$. 由单调收敛定理可知$\lim_{k\longrightarrow\infty}d_k$存在.

定理2.3 设$d=\lim_{k\longrightarrow\infty}d_k$, $A\in M_n$是非奇异矩阵,

$$f(x)=$$

$$(l_1^2-(l_0^2-l^2))+|\det((l_1^2-(l_0^2-l^2))I_n-A^HA)|(\frac{n-1}{\|A\|_F^2-x^2-(n-1)(l_1^2-(l_0^2-l^2))})^{n-1})^{\frac{1}{2}},$$

则d是方程$x=f(x)$的最小正解, 并且有$\sigma_n\geq d$.

证 设x_0是方程$x=f(x)$的最小正解, 对k用数学归纳法得$x_0>d_k, k=1,2,\cdots$. 当$k=$

1时,

$$\begin{aligned} x_0 &= (l_1^2-(l_0^2-l^2)+|\det((l_1^2-(l_0^2-l^2))I_n-A^HA)|(\tfrac{n-1}{\|A\|_F^2-x_0^2-(n-1)(l_1^2-(l_0^2-l^2))})^{n-1})^{\frac{1}{2}} \\ &> (l_1^2-(l_0^2-l^2)+|\det((l_1^2-(l_0^2-l^2))I_n-A^HA)|(\tfrac{n-1}{\|A\|_F^2-(n-1)(l_1^2-(l_0^2-l^2))})^{n-1})^{\frac{1}{2}}=d_1. \end{aligned}$$

即$x_0>d_1$成立. 假设当$k=m$时也成立, 即$x_0>d_m$. 现在考虑当$k=m+1$时,

$$\begin{aligned} x_0 &= (l_1^2-(l_0^2-l^2)+|\det((l_1^2-(l_0^2-l^2))I_n-A^HA)|(\tfrac{n-1}{\|A\|_F^2-x_0^2-(n-1)(l_1^2-(l_0^2-l^2))})^{n-1})^{\frac{1}{2}} \\ &> (l_1^2-(l_0^2-l^2)+|\det((l_1^2-(l_0^2-l^2))I_n-A^HA)|(\tfrac{n-1}{\|A\|_F^2-d_m^2-(n-1)(l_1^2-(l_0^2-l^2))})^{n-1})^{\frac{1}{2}} \\ &= d_{m+1}, \end{aligned}$$

即$x_0>d_{m+1}$. 这证明了$x_0>d_k, k=1,2,\cdots,$ 成立. 因此d是方程$x=f(x)$的正解且$x_0>d_k, k=1,2,\cdots$, 那么$d=x_0$. 故d是方程$x=f(x)$的最小正解并且$\sigma_n\geq d$.

这样就得到了非奇异矩阵最小奇异值两个新的下界l_3和d.

§3 数值例子

这一节首先通过例3.1和3.2来比较l,l_0,l_1,l_2,l_3的值.

例3.1　设

$$A=\begin{bmatrix}4&2&0\\5&8&7\\7&5&8\end{bmatrix}.$$

计算可得最小奇异值$\sigma_n=2.5249$, 并且有

$$l=0.9054054, l_0=0.9079198, l_1=1.220167, l_2=1.399112.$$

本文的结果

$$l_3=1.626562.$$

例3.2　设

$$A=\begin{bmatrix}3&2&0\\1&9&5\\0&5&7\end{bmatrix}.$$

计算可得最小奇异值$\sigma_n=1.9619$, 并且有

$$l=1.030928, l_0=1.036607, l_1=1.343431, l_2=1.61175.$$

本文的结果

$$l_3=1.840419.$$

下面通过例3.3来比较c,d,l_3的值.

例3.3　设

$$A=\begin{bmatrix}3&2&0\\1&9&5\\0&5&7\end{bmatrix}.$$

计算可得$c=1.6123$. 本文的结果

$$d=1.840529,\quad l_3=1.840419.$$

这些例子表明本文得到的估计要比已有的结果更好.

参考文献:

[1] Yu Yisheng, Gu Dunhe. A note on a lower bound for the smallest singular value[J]. Linear Algebra Appl, 1997, 253: 25-38.

[2] Zou Limin. A lower bound for the smallest singular value[J]. J Math Inequal, 2012, 6: 625-629.

[3] Lin Minghua, Xie Mengyan. On some lower bounds for smallest singular value of matrices[J]. Appl Math Lett, 2021, 121: 107411.

[4] Shun Xu. Two new lower bounds for the smallest singular value[J]. J Math Inequal, 2022, 16: 63-68.

[5] Liu Wushuang, Hu Xingkai, Wang Yaoqun, et al. Two lower bounds for the smallest singular value of matrices[J]. J Math Inequal, 2023, 17: 885-891.

Two estimates for the smallest singular value of matrices

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Abstract: In this note, two estimates for the smallest singular value of nonsingular matrices are presented by the determinant and Frobenius norm of matrices. Moreover, numerical examples show that the estimates are better than the existing results under certain circumstances.

Keywords: smallest singular value; Frobenius norm; determinant

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