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求解双层规划问题的松弛序列二次规划方法

杜梦琪[1]， 徐梦薇[1]， 段庆松[2]

(1. 河北工业大学 理学院， 天津 300401;

2. 中银金融科技有限公司 智能风控中心, 上海 200120)

摘　要: 考虑一类具有特殊结构的双层规划问题, 其下层问题为凸问题. 首先通过内点罚方法将下层的约束函数惩罚到目标函数, 使得下层问题近似为一系列无约束优化问题. 然后使用KKT条件替换无约束的下层问题的最优解集, 那么双层规划问题被一系列松弛的单层问题近似. 文中设计了一种光滑的序列二次规划算法求解该松弛问题, 并证明了当罚因子趋近于0时, 该算法生成的迭代点列收敛到双层规划问题的弱稳定点. 数值实验验证了算法的可行性.

关键词: 双层规划; Tikhonov-regularized interior-penalty; 序列二次规划方法

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§1　引　言

双层规划问题(Bilevel Programming)是一个约束区域由另一个数学规划问题隐式确定的优化问题(见[1]). 它是一种具有上下两层的系统优化问题, 上下层的问题都有各自的目标函数和约束条件, 上层问题的目标函数和约束条件不仅与上层决策变量有关, 而且还依赖于下层问题的最优解, 而下层问题的最优解又受上层决策变量的影响. 由于大量现实生活中的问题, 如交通运输, 管理与经济, 工程设计, 供应链规划, 委托代理问题, 健康保险等都可以转化为双层规划问题(见[2, 3]), 所以有效地解决双层规划问题是非常重要的.

在本文考虑双层规划问题

$$\begin{aligned}(\mathrm{BP})\ \min_{x,y}\ & F(x,y)\\ \text{s.t.}\ & G_i(x,y)\le 0,\quad i=1,2,\cdots,m,\\ & y\in S(x).\end{aligned}\tag{1}$$

其中$S(x)$表示下层问题

$$(\mathrm{P}_x)\ \min_{y\in Y(x)}\ f(x,y)\tag{2}$$

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的最优解集, $Y(x) := \{y \in \mathbf{R}^{n_2} | g_i(x,y) \leq 0,\ i = 1,2,\cdots,l\}$. $F,\ G_i, (i = 1,2,\cdots,m) : \mathbf{R}^{n_1} \times \mathbf{R}^{n_2} \mapsto \mathbf{R}$是连续可微的函数. $f,\ g_i, (i = 1,2,\cdots,l) : \mathbf{R}^{n_1} \times \mathbf{R}^{n_2} \mapsto \mathbf{R}$是二阶连续可微的函数, 并且关于变量$y$是凸函数. 用$\mathcal{F}$表示问题(BP)的可行集.

众所周知, 双层规划问题的双层结构使其难以求解. 事实上, 即使在线性情况下, 双层规划问题也被证明是强NP难的. 从连续优化的角度来看, 发展有效的数值算法通常需要将双层规划问题转化为单层优化问题. 沿着这个方向, 学者们提出了几种双层规划的转换方法. 这些转换方法一般可分为以下几类: 基于下层解函数的转换, 基于下层最优值函数的转换和基于下层最优性条件的转换.

如果下层规划(P_x)对任意给定的$x \in \mathbf{R}^{n_1}$有唯一最优解$y(x)$, 基于下层解函数的转换是利用$y(x)$将双层规划(BP)转化为单层优化问题

$$\begin{aligned} (\mathrm{SP}) \quad \min \quad & F(x,y(x)) \\ \text{s.t.} \quad & G(x,y(x)) \leq 0. \end{aligned} \tag{3}$$

这种转换需要严格的条件, 以确保下层最优解的存在性及唯一性, 在下层问题解函数$y(x)$已知的情况下, 即使下层问题和上层问题都是可微的和凸的, 上面问题(SP)的目标函数一般既不是可微的也不是凸的. 因此这种方法通常难以应用于求解一般的双层规划.

基于下层最优值函数的转换是利用下层问题的最优值函数$v(x) := f(x,y(x)),\ y(x) \in S(x)$将双层规划问题转换为一个等价的, 非光滑的且非凸的优化问题

$$\begin{aligned} (\mathrm{VP}) \quad \min \quad & F(x,y) \\ \text{s.t.} \quad & f(x,y) \leq v(x), \\ & G_i(x,y) \leq 0, i = 1,2,\cdots,m, \\ & g_j(x,y) \leq 0, j = 1,2,\cdots,l. \end{aligned} \tag{4}$$

该转换模型首次在文献[4]中提出, 在通常情况下, $v(x)$是非光滑函数. [5]研究了$v(x)$的局部Lipschitz连续性, 并建立了最优性条件. [6, 7]以及[8, 9, 10]利用下层最优值函数, 研究了双层规划问题的各种最优性条件和约束规范. 通过计算可行集切锥的近似集合, [11]探讨了值函数模型的二阶最优性条件. [12]通过求解广义纳什均衡模型, 得到了松弛的值函数模型的稳定点, 并阐述了在很多情况下, 该算法可以得到原问题的近似全局最优解. 值函数模型的最优性理论已经相对完善, 但由于其所有可行点都不满足非光滑的MFCQ(Mangasarian-Fromovitz constraint qualification), 因此求解相对困难, 算法大多集中在求解特殊问题, 如线性问题和凸问题.

因为上面两种转换方法涉及到下层问题的最优解或最优值函数, 所以难以应用于求解双层规划. 目前最流行的方法是利用下层问题的最优性条件将双层规划(BP)转换为均衡约束数学规划问题(mathematical program with equilibrium constraints, MPEC), 即

$$\begin{aligned} (\mathrm{KP}) \quad \min_{x,y,u} \ & F(x,y) \\ \text{s.t.} \ & \nabla_y f(x,y) + u^\top \nabla_y g(x,y) = 0, \\ & g(x,y) \leq 0,\ u \geq 0,\ \langle g(x,y), u \rangle = 0, \\ & G(x,y) \leq 0, \end{aligned} \tag{5}$$

其中下层问题的解集被它的KKT条件代替, 并且$u \in \mathbf{R}^l$. 用$\mathcal{F}_{\mathrm{KP}}$表示问题(KP)的可行域. 由于MPEC的可行域具有组合结构, 因此它不是一个普通的约束优化问题. 此外它在任何可行点都不满足MFCQ, 这意味着在非线性规划中已经建立的优化算法在求解MPEC时可能不稳定. 在过去的几十年里, 人们提出了许多近似算法来求解MPEC. [13]提出了求解一般约束优化问题的

对偶内点方法, 在不假设正则性条件的情况下, 证明了强收敛理论, 并结合不等式松弛技术, 用该方法求解MPEC模型. [14]开发了求解MPEC问题的序列二次规划方法, 该方法在每次迭代步都需求解一个线性规划和一个二次规划子问题, 并证明了该方法的强收敛性. 关于MPEC理论和数值算法的最新发展, 请读者参阅[15, 16, 17, 18, 19, 20]和其中的参考文献.

由于问题(KP)包含互补约束, 求解难度很大. 文献[21]利用Tikhonov-regularized interior-penalty(log-barrier)函数

$$\phi(x,y)=-\sum_{i=1}^{m}\ln\{-g_i(x,y)\}+\tfrac{\mu}{2}\|y\|^2, \tag{6}$$

将下层问题的凸约束函数惩罚到凸目标函数, 当正则化参数$r>0$时, 得到一个严格凸或强凸的目标函数. 此时下层问题的罚问题为

$$\min_{y}\ f_\varepsilon^r(x,y)=f(x,y)-\varepsilon\sum_{i=1}^{l}\ln\{-g_i(x,y)\}+\varepsilon\tfrac{r}{2}\|y\|^2. \tag{7}$$

用$S^\varepsilon(x)$表示罚问题(7)的最优解集.

本文利用问题(7)的KKT条件替换问题(7)的最优解集, 得到双层规划的松弛问题

$$\begin{aligned}(\mathrm{P}^\varepsilon)\quad &\min\ F(x,y)\\ &\text{s.t.}\ \ G_i(x,y)\le 0,\quad i=1,2,\cdots,m,\\ &\qquad\ \nabla_y f_\varepsilon^\mu(x,y)=0.\end{aligned} \tag{8}$$

对每个$\varepsilon>0$, 将问题(P^ε)的可行域表示为$\mathcal{F}^\varepsilon$.

本文通过求解一系列松弛问题(P^ε)进而得到原双层规划问题的稳定点. 本文证明了$\mathcal{F}^\varepsilon$中可行点的任意聚点包含在$\mathcal{F}$中. 并且证明了当问题(KP)满足MPEC-LICQ(MPEC linear independence constraint qualification), 且ε充分小时, 问题(P^ε)满足LICQ. 本文定义了一个关于问题(BP)的弱稳定点, 进一步, 设计了序列二次规划(sequential quadratic programming, SQP)算法来求解问题(P^ε), 从而得到原双层规划问题的解. $\varepsilon\searrow 0$, 且一定的约束规范满足的情况下, 证明了算法生成的迭代序列的任何聚点都是问题(BP)的弱稳定点.

§2介绍一些预备知识, 并讨论了双层规划问题的松弛问题(P^ε). §3设计了求解双层规划松弛问题的SQP算法, 并证明了其收敛性. §4给出了13个问题的实验结果. §5是结论.

§2 预备知识和双层规划的松弛问题

本文采用了下面的符号. 对任意两个向量$a\in\mathbf{R}^n, b\in\mathbf{R}^n$, 用$a^\top b$表示它们的内积. 给定一个函数$F:\ \mathbf{R}^n\to\mathbf{R}^m$, 其雅可比矩阵表示为$\nabla F(z)\in\mathbf{R}^{m\times n}$. 如果$m=1$, 则将梯度$\nabla F(z)\in\mathbf{R}^n$作为列向量. 对于矩阵$A\in\mathbf{R}^{n\times m}$, $A^\top$是它的转置. $\|\cdot\|, \|\cdot\|_1, \|\cdot\|_\infty$分别表示欧氏范数, 1-范数和$\infty$-范数. 此外表示函数$[\cdot]_+=\max\{\cdot,0\}$.

考虑约束优化问题

$$\begin{aligned}(\mathrm{P})\qquad &\min\qquad f(x)\\ &\ \text{s.t.}\qquad g_i(x)\le 0,\ i=1,2,\cdots,p,\\ &\qquad\qquad\ h_j(x)=0,\ j=p+1,\cdots,q.\end{aligned}$$

其中目标函数和约束函数f, $g_i(i=1,2,\cdots,p)$, $h_j(j=p+1,\cdots,q):\mathbf{R}^n\to\mathbf{R}$是连续可微的. 对于问题(P)的可行解$\bar{x}\in\mathbf{R}^n$, 用$I(\bar{x}):=\{i:\ g_i(\bar{x})=0, i=1,2,\cdots,p\}$表示在$\bar{x}$处的积极集.

定义 2.1 (MFCQ) 问题(P)在可行点$\bar{x}$处满足MFCQ, 如果存在向量$v \in \mathbf{R}^n$使得

$$\nabla g_i(\bar{x})^\top v < 0,\ \forall\, i \in I(\bar{x}),$$
$$\nabla h_j(\bar{x})^\top v = 0,\ \forall\, j \in \{p+1, \cdots, q\}$$

成立, 并且等式约束梯度集$\{\nabla h_j(\bar{x}), j \in \{p+1, \cdots, q\}\}$是线性无关的.

根据Fritz-John型必要最优性条件, 定义了下面的约束规范, 它比MFCQ弱.

定义 2.2 (NNAMCQ) 问题(P)在可行点$\bar{x}$处满足NNAMCQ(no nonzero abnormal multiplier constraint qualification), 如果

$$0 \in \sum_{i \in I(\bar{x})} \lambda_i \nabla g_i(\bar{x}) + \sum_{j=p+1}^{q} \lambda_j \nabla h_j(\bar{x}),\ \lambda_i \geq 0, i \in I(\bar{x}),$$
$$\Rightarrow \lambda_i = 0, \lambda_j = 0, i \in I(\bar{x}), j = p+1, \cdots, q.$$

在没有等式约束的情况下, NNAMCQ等价于MFCQ, [22]已经证明了这一点. 为了处理数值算法中不可行的聚点, 现在将NNAMCQ和MFCQ扩展到不可行的聚点.

定义 2.3 (ENNAMCQ) 问题(P)在$\bar{x}$处满足ENNAMCQ(extended no nonzero abnormal multiplier constraint qualification), 如果

$$\sum_{i=1}^{p} \lambda_i \nabla g_i(\bar{x}) + \sum_{j=p+1}^{q} \lambda_j \nabla h_j(\bar{x}) = 0,\ \lambda_i \geq 0, i = 1, \cdots, p,$$
$$\sum_{i=1}^{p} \lambda_i g_i(\bar{x}) + \sum_{j=p+1}^{q} \lambda_j h_j(\bar{x}) \geq 0,$$

意味着$\lambda_i = 0, \lambda_j = 0$对任意的$i \in \{1, \cdots, p\}, j \in \{p+1, \cdots, q\}$.

定义 2.4 (EMFCQ) 问题(P)在$\bar{x}$处满足EMFCQ(extended Mangasarian-Fromovitz constraint qualification), 如果下面两个条件成立.

(i) $\nabla h_j(\bar{x}), j = p+1, \cdots, q$是线性无关的.

(ii) 存在方向d使得

$$g_i(\bar{x}) + \nabla g_i(\bar{x})^\top d < 0,\ i = 1, \cdots, p,$$
$$h_j(\bar{x}) + \nabla h_j(\bar{x})^\top d = 0,\ j = p+1, \cdots, q$$

成立.

注意到当$\bar{x}$是问题(P)的可行点时, ENNAMCQ和EMFCQ分别退化为NNAMCQ和MFCQ. 下面的定理表明了ENNAMCQ和EMFCQ的等价性.

定理 2.1 关系

$$\text{ENNAMCQ} \Leftrightarrow \text{EMFCQ}$$

总是成立的.

该定理证明类似于文献[23]中定理2.2的证明, 所以这里省略了.

下面考虑了双层规划的松弛问题(P^ε). 为了证明问题(BP)与问题(P^ε)的可行域间的关系, 给出以下假设和引理.

假设 2.1 对于固定点$\bar{x}$, 集值映射$Y(x)$在$\bar{x}$附近一致有界, 即存在$\bar{x}$的邻域$\mathbb{U}$使得集合$\bigcup_{x\in\mathbb{U}}Y(x)$是有界的.

假设 2.2 对任意的$x\in\mathbf{R}^{n_1}$, 下层问题(P_x)满足Slater条件: 存在$\mathring{y}(x)$使得

$$g_i(x,\mathring{y}(x))<0, i=1,2,\cdots,l.$$

根据假设2.1易知

$$\limsup_{x\to\bar{x}}\{\min_{y\in S(x)}\|y\|^2\}<+\infty. \tag{9}$$

在本节中, 假设值函数$v(x)$是Lipschitz连续的. 有很多文献已经研究了值函数的Lipschitz连续性, 下面的命题2.1是Clarke[24, Theorem 6.5.2]的一个特例, 在[25]中可以找到值函数Lipschitz连续性的其他较弱的充分条件以及其次微分的估计.

命题 2.1 如果假设2.1-2.2成立. 则值函数$v(x)$在$\bar{x}$附近Lipschitz连续.

引理 2.1 (i) ([21, Proposition 2]) 对任意的$r\geq 0$和任意的$\varepsilon>0$, 如果问题(7)的解$y^\varepsilon(x)$存在, 则有条件

$$v(x)\leq f(x,y^\varepsilon(x))\leq v(x)+l\varepsilon+\varepsilon\frac{r}{2}\min_{y\in S(x)}\|y\|^2 \tag{10}$$

成立.

(ii) ([21, Lemma 2]) 如果假设2.2成立, 并且条件(9)在$\bar{x}\in\mathbf{R}^{n_1}$处成立, 则对任意的$i=1,2,\cdots,l$, 有

$$0\leq\limsup_{\varepsilon\searrow 0,x\to\bar{x}}\frac{-\varepsilon}{g_i(x,y^\varepsilon(x))}<+\infty. \tag{11}$$

定理 2.2 若假设2.1-2.2成立. 对任意的$\varepsilon>0$, 如果$(x^\varepsilon,y^\varepsilon)\in\mathcal{F}^\varepsilon$且$g(x^\varepsilon,y^\varepsilon)<0$, $(\bar{x},\bar{y})$是序列$\{(x^\varepsilon,y^\varepsilon)\}$的任意聚点, 则有$(\bar{x},\bar{y})\in\mathcal{F}$.

证 因为$(x^\varepsilon,y^\varepsilon)\in\mathcal{F}^\varepsilon$, 则有$G(x^\varepsilon,y^\varepsilon)\leq 0$. 由假设条件知对任意的$\varepsilon>0$, $g(x^\varepsilon,y^\varepsilon)<0$. 因此, 根据函数$G$和$g$的连续性, 有$G(\bar{x},\bar{y})\leq 0$以及$g(\bar{x},\bar{y})\leq 0$.

由命题2.1, 值函数$v(x)$是Lipschitz连续的, 则对任意的$\bar{x}\in\mathbf{R}^{n_1}$有$\lim\limits_{x\to\bar{x}}v(x)=v(\bar{x})$. 由(9)知

$$\lim_{\varepsilon\searrow 0,x\to\bar{x}}\varepsilon\frac{r}{2}\min_{y\in S(x)}\|y\|^2=0.$$

因为问题(7)的目标函数是严格凸或强凸的, 并且该问题没有约束, 则对于任意的$\varepsilon>0$, 若$\nabla_y f_\varepsilon^r(x^\varepsilon,y)=0$的解$y^\varepsilon$满足$g(x^\varepsilon,y^\varepsilon)<0$则有$y^\varepsilon\in S^\varepsilon(x^\varepsilon)$. 因此根据夹逼原则和(10)可得

$$f(\bar{x},\bar{y})=\lim_{\varepsilon\searrow 0}f(x^\varepsilon,y^\varepsilon)=v(\bar{x}),$$

则$\bar{y}\in S(\bar{x})$, 进一步$(\bar{x},\bar{y})\in\mathcal{F}$.

定义 2.5 问题(KP)在可行点$(\bar{x},\bar{y},\bar{u})$处满足MPEC-LICQ(MPEC linear independence constraint qualification), 若不存在非零的乘子$\lambda_i^G,i\in I^G$, $\lambda_i^g,i\in I^g$和$\lambda_i^h,i\in\{1,2,\cdots,n_2\}$使得

$$\sum_{i\in I^G}\lambda_i^G\nabla G_i(\bar{x},\bar{y})+\sum_{i=1}^{n_2}\lambda_i^h[\nabla(\nabla_{y_i}f(\bar{x},\bar{y}))+\sum_{j=1}^{l}\bar{u}_j\nabla(\nabla_{y_i}g_j(\bar{x},\bar{y}))]+\sum_{i\in I^g}\lambda_i^g\nabla g_i(\bar{x},\bar{y})=0,$$

$$\sum_{j=1}^{n_2}\lambda_j^h\nabla_{y_j}g_i(\bar{x},\bar{y})=0, i\notin I^u,$$

其中$I^G := \{i : G_i(\bar{x}, \bar{y}) = 0, i = 1, 2, \cdots, m\}$, $I^g := \{i : g_i(\bar{x}, \bar{y}) = 0, i = 1, 2, \cdots, l\}$, $I^u := \{i : u_i = 0, i = 1, 2, \cdots l\}$.

定理 2.3 若假设2.1-2.2成立. 对于$\varepsilon > 0$, 假设$(\bar{x}, \bar{y})$是序列$\{(x^\varepsilon, y^\varepsilon)\}$的聚点, 其中$(x^\varepsilon, y^\varepsilon)$是问题$(\mathrm{P}^\varepsilon)$的可行点并且$g(x^\varepsilon, y^\varepsilon) < 0$. 如果对任意Lagrange乘子$\bar{u}$, 问题(KP)在$(\bar{x}, \bar{y}, \bar{u}) \in \mathcal{F}_{\mathrm{KP}}$处满足MPEC-LICQ. 则当$\varepsilon$充分小的时候, 问题$(\mathrm{P}^\varepsilon)$在$(x^\varepsilon, y^\varepsilon)$处满足LICQ.

证 因为$(x^\varepsilon, y^\varepsilon)$是问题$(\mathrm{P}^\varepsilon)$的可行点, 则有

$$\nabla_y f(x^\varepsilon, y^\varepsilon) + \sum_{j=1}^{l} \frac{-\varepsilon}{g_j(x^\varepsilon, y^\varepsilon)} \nabla_y g_j(x^\varepsilon, y^\varepsilon) + \varepsilon r y^\varepsilon = 0.$$

根据引理2.1, 不妨假设存在$\bar{u} \in \mathbf{R}^l$使得$\lim\limits_{\varepsilon \searrow 0} \frac{-\varepsilon}{g(x^\varepsilon, y^\varepsilon)} = \bar{u} \geq 0$, 于是上式对$\varepsilon \searrow 0$取极限得到$\nabla_y f(\bar{x}, \bar{y}) + \sum\limits_{j=1}^{l} \bar{u}_j \nabla_y g_j(\bar{x}, \bar{y}) = 0$. 当$g_j(\bar{x}, \bar{y}) < 0$, 显然$\bar{u}_j = 0$, 则$\bar{u} = (\bar{u}_1, \bar{u}_2, \cdots, \bar{u}_l)^\top$是下层问题$P_{\bar{x}}$在$\bar{y}$处的Lagrange乘子, 根据定理2.2, $(\bar{x}, \bar{y}) \in \mathcal{F}$, 那么$(\bar{x}, \bar{y}, \bar{u})$是问题(KP)的可行点.

利用反证法, 假设存在子序列$\varepsilon \searrow 0$使得问题(P^ε)在$(x^\varepsilon, y^\varepsilon)$处不满足LICQ, 即存在不全为0的乘子$\beta_i^{\varepsilon,G}, i \in \{1, 2, \cdots, m\}$和$\beta_i^{\varepsilon,h}, i \in \{1, 2, \cdots, n_2\}$使下面的条件成立.

$$\begin{aligned}
0 &= \sum_{i=1}^{m} \beta_i^{\varepsilon,G} \nabla G_i(x^\varepsilon, y^\varepsilon) + \nabla(\nabla_y f_\varepsilon^r(x^\varepsilon, y^\varepsilon)^\top \beta^{\varepsilon,h}) \\
&= \sum_{i=1}^{m} \beta_i^{\varepsilon,G} \nabla G_i(x^\varepsilon, y^\varepsilon) + \nabla[\nabla_y f(x^\varepsilon, y^\varepsilon) - \sum_{j=1}^{l} \frac{\varepsilon}{g_j(x^\varepsilon, y^\varepsilon)} \nabla_y g_j(x^\varepsilon, y^\varepsilon) + \varepsilon r y^\varepsilon]^\top \beta^{\varepsilon,h} \\
&= \sum_{i=1}^{m} \beta_i^{\varepsilon,G} \nabla G_i(x^\varepsilon, y^\varepsilon) + \nabla(\nabla_y f(x^\varepsilon, y^\varepsilon)^\top \beta^{\varepsilon,h}) + \sum_{j=1}^{l} [\frac{\varepsilon \nabla g_j(x^\varepsilon, y^\varepsilon)}{(g_j(x^\varepsilon, y^\varepsilon))^2} \nabla_y g_j(x^\varepsilon, y^\varepsilon)^\top \beta^{\varepsilon,h}] \\
&\quad + \sum_{j=1}^{l} [\frac{-\varepsilon}{g_j(x^\varepsilon, y^\varepsilon)} \nabla(\nabla_y g_j(x^\varepsilon, y^\varepsilon)^\top \beta^{\varepsilon,h})] + \sum_{i=1}^{n_2} \varepsilon r \beta_i^{\varepsilon,h} e_{n_1+i},
\end{aligned} \tag{12}$$

其中$\beta_i^{\varepsilon,G} = 0, i \notin I^{\varepsilon,G} := \{i : G_i(x^\varepsilon, y^\varepsilon) = 0\}$, 并且$e_{n_1+i} \in \mathbf{R}^{n_1+n_2}$表示第$n_1 + i$个元素为1, 其余元素为0的单位向量.

不失一般性, 假设序列$\{\beta^{\varepsilon,G}, \beta^{\varepsilon,h}\}_{\varepsilon>0}$有界, 则不妨假设存在$(\beta^G, \beta^h)$使得$\lim\limits_{\varepsilon \searrow 0} (\beta^{\varepsilon,G}, \beta^{\varepsilon,h}) = (\beta^G, \beta^h)$. 令$p_j^\varepsilon := \frac{\varepsilon}{(g_j(x^\varepsilon, y^\varepsilon))^2} \nabla_y g_j(x^\varepsilon, y^\varepsilon) \top \beta^{\varepsilon,h}$, $j \in \{1, 2, \cdots, l\}$.

若$\lim\limits_{\varepsilon \searrow 0} \|p^\varepsilon\| = \infty$, 则存在$\bar{p} \in \mathbf{R}^l$使得$\lim\limits_{\varepsilon \searrow 0} \frac{p^\varepsilon}{\|p^\varepsilon\|} = \bar{p} \neq 0$. 在(12)的两边同时除以$\|p^\varepsilon\|$并且取极限$\varepsilon \searrow 0$, 可以得到

$$\sum_{j=1}^{l} \bar{p}_j \nabla g_j(\bar{x}, \bar{y}) = 0.$$

当$j \notin I^g$, 即$g_j(\bar{x}, \bar{y}) < 0$, 根据$p^\varepsilon$的定义有$\bar{p}_j = 0$. 这与MPEC-LICQ矛盾. 因此$\|p^\varepsilon\|$有界. 不失一般性, 对任意$j \in \{1, 2, \cdots, l\}$, 假设$\lim\limits_{\varepsilon \searrow 0} p_j^\varepsilon = p_j$. 在等式(12)的两边同时取极限$\varepsilon \searrow 0$得到

$$\sum_{i=1}^{m} \beta_i^G \nabla G_i(\bar{x}, \bar{y}) + \nabla(\nabla_y f(\bar{x}, \bar{y}) + \sum_{j=1}^{l} \bar{u}_j \nabla_y g_j(\bar{x}, \bar{y}))^T \beta^h + \sum_{i=1}^{l} p_i \nabla g_i(\bar{x}, \bar{y}) = 0,$$

其中$\{\beta_i^G, i \in \{1, 2, \cdots, m\}; \beta_i^h, i \in \{1, 2, \cdots, n_2\}; p_i, i \in \{1, 2, \cdots, l\}\}$不全为0. 对于$i \notin I^G$, 即$G_i(\bar{x}, \bar{y}) < 0$, 根据函数$G(x, y)$的连续性, 对充分小的$\varepsilon$, $G_i(x^\varepsilon, y^\varepsilon) < 0$, 这意味着$i \notin I^{\varepsilon,G}$,

于是$\beta_i^G = 0, i \notin I^G$. 当$i \notin I^g$, $g_i(\bar{x},\bar{y}) < 0$, 根据p^ε的定义有$p_i = 0$. 若$i \notin I^u$, 即$\bar{u}_i = \lim\limits_{\varepsilon\searrow 0} \frac{-\varepsilon}{g_i(x^\varepsilon,y^\varepsilon)} > 0$, 则必有$g_i(x^\varepsilon,y^\varepsilon) \to 0$, 从而$\frac{-\varepsilon}{g_i(x^\varepsilon,y^\varepsilon)^2} \to +\infty$, 此时根据$\|p^\varepsilon\|$的有界性一定有$\nabla_y g_i(x^\varepsilon,y^\varepsilon)^\top \beta^{\varepsilon,h} \to 0$, 即$\nabla_y g_i(\bar{x},\bar{y})^\top \beta^h = 0, i \notin I^u$. 这与MPEC-LICQ矛盾.

综上所述, 当ε足够小时, 问题(P$^\varepsilon$)在$(x^\varepsilon,y^\varepsilon)$处满足LICQ.

§3 求解双层规划松弛问题的SQP方法

本节设计了一种新的求解双层规划问题的SQP算法, 并证明了该算法生成的迭代序列的任何聚点都是如下定义的双层规划的弱稳定点.

定义 3.1 $(\bar{x},\bar{y},u) \in \mathbf{R}^{n_1} \times \mathbf{R}^{n_2} \times \mathbf{R}^l$是问题(BP)的弱稳定点, 如果存在$\mu \in \mathbf{R}^{n_2}, \eta^g \in \mathbf{R}^l, \eta^G \in \mathbf{R}^m, \tilde{\eta} \in \mathbf{R}^l$使得下面条件成立.

$$\begin{aligned}
&0 = \nabla F(\bar{x},\bar{y}) + \sum_{i=1}^{l} \eta_i^g \nabla g_i(\bar{x},\bar{y}) + \sum_{i=1}^{m} \eta_i^G \nabla G_i(\bar{x},\bar{y}) + \nabla\Big[\nabla_y f + \sum_{i=1}^{l} u_i \nabla_y g_i\Big](\bar{x},\bar{y})^\top \mu,\\
&0 \le \eta_i^G \perp G_i(\bar{x},\bar{y}) \le 0, i = 1,2,\cdots,m,\\
&\nabla_y f(\bar{x},\bar{y}) + \sum_{j=1}^{l} u_j \nabla_y g_j(\bar{x},\bar{y}) = 0,\\
&\eta_i^g = 0, i \in \{i : g_i(\bar{x},\bar{y}) \ne 0, u_i = 0\},\ \nabla_y g_i(\bar{x},\bar{y})^\top \mu = 0, i \in \{i : g_i(\bar{x},\bar{y}) = 0, u_i \ne 0\},\\
&\eta_i^g \nabla_y g_i(\bar{x},\bar{y})^\top \mu \ge 0,\ \forall i \in \{i : g_i(\bar{x},\bar{y}) = 0, u_i = 0\}.
\end{aligned} \tag{13}$$

易知问题(P$^\varepsilon$)是一个具有m个不等式约束和n_2个等式约束的非线性约束优化问题, 设计SQP算法来求解问题(P$^\varepsilon$), 当$\varepsilon \searrow 0$, 且满足某些约束规范的情况下可以证明该算法生成的迭代序列的任意聚点都是问题(BP)的弱稳定点. 具体解决方法如下所述.

令(x_k,y_k)是当前的迭代点, B_k是问题(P$^\varepsilon$)的Lagrange函数关于(x,y)的Hesse矩阵的近似, 且B_k为正定矩阵. 通过求解如下二次规划子问题寻找下降方向$d_k \in \mathbf{R}^{n_1+n_2}$:

$$\begin{aligned}
(\mathrm{QP})_k \quad &\min_{d\in\mathbf{R}^{n_1+n_2}} \quad \nabla F(x_k,y_k)^\top d + \tfrac{1}{2} d^\top B_k d\\
&\quad\text{s.t.} \quad G_i(x_k,y_k) + \nabla G_i(x_k,y_k)^\top d \le 0,\ i = 1,2,\cdots,m,\\
&\qquad\qquad \nabla_{y_i} f^r_{\varepsilon^k}(x_k,y_k) + \nabla(\nabla_{y_i} f^r_{\varepsilon^k}(x_k,y_k))^\top d = 0, i = 1,2,\cdots,n_2.
\end{aligned} \tag{14}$$

假设子问题$(\mathrm{QP})_k$是可行的. 如果d_k是$(\mathrm{QP})_k$的解, 则该子问题的KKT条件可以写为

$$\begin{aligned}
&\nabla F(x_k,y_k) + B_k d_k + \sum_{i=1}^{m} \lambda_k^i \nabla G_i(x_k,y_k) + \sum_{i=1}^{n_2} \mu_k^i \nabla(\nabla_{y_i} f(x_k,y_k) -\\
&\varepsilon^k \sum_{j=1}^{l} \frac{\nabla_{y_i} g_j(x_k,y_k)}{g_j(x_k,y_k)} + \varepsilon^k r y_k^i) = 0,\\
&0 \le \lambda_k^i \perp (G_i(x_k,y_k) + \nabla G_i(x_k,y_k)^\top d_k) \le 0,\ i = 1,2,\cdots,m,\\
&\nabla_{y_i} f^r_{\varepsilon^k}(x_k,y_k) + \nabla(\nabla_{y_i} f^r_{\varepsilon^k}(x_k,y_k))^\top d_k = 0, i = 1,2,\cdots,n_2.
\end{aligned} \tag{15}$$

其中λ_k, μ_k是相应的Lagrange乘子.

令$\varepsilon > 0, \sigma > 0$, 定义$l_1$价值函数

$$\varphi_\sigma^{\varepsilon,r}(x,y) = F(x,y) + \sigma\Big[\ \|\ \nabla_y f^r_\varepsilon(x,y)\ \|_1 + \|\ [G(x,y)]_+\ \|_1\ \Big]. \tag{16}$$

为了证明算法的收敛性, 给出下面的假设.

算法 1 求解双层规划松弛问题的SQP算法

输入:

初始化: 令η是$(0,\frac{1}{2})$中的常数, $\{\rho,\rho_1,\delta,\hat{\eta}\}$是$(0,1)$中的常数. 选择初始点$(x_0,y_0,\lambda_0,\mu_0)\in\mathbf{R}^{n_1}\times\mathbf{R}^{n_2}\times\mathbf{R}^{m}\times\mathbf{R}^{n_2}$, $\varepsilon^0>0$, $\sigma_{-1}>0$, 正则参数r以及正定矩阵$B_0\in\mathbf{R}^{(n_1+n_2)\times(n_1+n_2)}$. 令$k:=0$.

1: 如果满足停止准则, 则终止算法. 否则, 转步2.

2: 解二次规划子问题$(\mathrm{QP})_k$从而得到d_k以及相应的Lagrange乘子(λ_k,μ_k).

3: 令$\tau=\max\{\|\lambda_k\|_\infty,\|\mu_k\|_\infty\}$. 如果$\sigma_{k-1}\geq\tau+\delta$, 则令$\sigma_k=\sigma_{k-1}$. 否则, 令$\sigma_k=\tau+2\delta$.

4: 选择步长α_k为$\{1,\rho,\rho^2,\cdots\}$中的最大值, 使得

$$\varphi_{\sigma_k}^{\varepsilon^k,r}(x_k+\alpha_k d_{xk},y_k+\alpha_k d_{yk})-\varphi_{\sigma_k}^{\varepsilon^k,r}(x_k,y_k)\leq\eta\alpha_k D(\varphi_{\sigma_k}^{\varepsilon^k,r}(x_k,y_k);d_k). \tag{17}$$

令$x_{k+1}:=x_k+\alpha_k d_{xk}$, $y_{k+1}:=y_k+\alpha_k d_{yk}$.

5: 令

$$\varepsilon^{k+1}=\begin{cases}\rho_1\varepsilon^k, & \text{若 } \|d_k\|_1\leq\hat{\eta}\varepsilon^k,\\ \varepsilon^k, & \text{否则}.\end{cases}$$

更新正定矩阵B_{k+1}. 令$k:=k+1$, 转步1.

输出: (x_k,y_k), ε^k.

假设 3.1 存在两个正常数m和M, $m\leq M$, 使对每个k和每个$d\in\mathbf{R}^{n_1+n_2}$有

$$m\|d\|^2\leq d^\top B_k d\leq M\|d\|^2.$$

定理 3.1 令$d_k\in\mathbf{R}^{n_1+n_2}$是二次规划子问题$(\mathrm{QP})_k$的解, 则价值函数$\varphi_\sigma^{\varepsilon^k,r}$在$(x_k,y_k)$处沿方向$d_k$的方向导数$D(\varphi_\sigma^{\varepsilon^k,r}(x_k,y_k);d_k)$满足

$$\begin{aligned}&D(\varphi_\sigma^{\varepsilon^k,r}(x_k,y_k);d_k)\leq\\&-d_k^\top B_k d_k-(\sigma-\|\mu_k\|_\infty)\sum_{i=1}^{n_2}|\nabla_{y_i}f_{\varepsilon^k}^r(x_k,y_k)|-(\sigma-\|\lambda_k\|_\infty)\sum_{i=1}^{m}[G_i(x_k,y_k)]_+.\end{aligned}$$

证 根据Taylor展开定理和条件(15), 对于$\alpha\leq1$得到

$$\begin{aligned}&\varphi_\sigma^{\varepsilon^k,r}(x_k+\alpha d_{xk},y_k+\alpha d_{yk})-\varphi_\sigma^{\varepsilon^k,r}(x_k,y_k)\\
=&F(x_k+\alpha d_{xk},y_k+\alpha d_{yk})-F(x_k,y_k)+\\
&\sigma\Big[\|\nabla_y f_{\varepsilon^k}^r(x_k+\alpha d_{xk},y_k+\alpha d_{yk})\|_1-\|\nabla_y f_{\varepsilon^k}^r(x_k,y_k)\|_1\\
&+\|[G(x_k+\alpha d_{xk},y_k+\alpha d_{yk})]_+\|_1-\|[G(x_k,y_k)]_+\|_1\Big]\\
=&\alpha\nabla F(x_k,y_k)^\top d_k+\sigma\Big[\|\nabla_y f_{\varepsilon^k}^r(x_k,y_k)+\alpha\nabla(\nabla_y f_{\varepsilon^k}^r(x_k,y_k))^\top d_k\|_1-\|\nabla_y f_\varepsilon^r(x_k,y_k)\|_1\\
&+\|[G(x_k,y_k)+\alpha\nabla G(x_k,y_k)^\top d_k]_+\|_1-\|[G(x_k,y_k)]_+\|_1\Big]+o(\alpha)\\
\leq&\alpha\nabla F(x_k,y_k)^\top d_k+\sigma\Big[\|\nabla_y f_{\varepsilon^k}^r(x_k,y_k)-\alpha\nabla_y f_{\varepsilon^k}^r(x_k,y_k)\|_1-\|\nabla_y f_\varepsilon^r(x_k,y_k)\|_1\\
&+\|[G(x_k,y_k)-\alpha G(x_k,y_k)]_+\|_1-\|[G(x_k,y_k)]_+\|_1\Big]+o(\alpha)\\
=&\alpha\Big[\nabla F(x_k,y_k)^\top d_k-\sigma(\|\nabla_y f_{\varepsilon^k}^r(x_k,y_k)\|_1+\|[G(x_k,y_k)]_+\|_1)\Big]+o(\alpha).\end{aligned}$$

然后对上式除以$\alpha > 0$, 并且两边同时取$\alpha \to 0^+$的极限, 得到

$$D(\varphi_\sigma^{\varepsilon^k,r}(x_k,y_k);d_k) \le \nabla F(x_k,y_k)^\top d_k - \sigma\Big[\|\nabla_y f_\varepsilon^r(x_k,y_k)\|_1 + \|[G(x_k,y_k)]_+\|_1\Big]. \tag{18}$$

由d_k是二次规划子问题$(\mathrm{QP})_k$的最优解, λ_k和μ_k是二次规划子问题$(\mathrm{QP})_k$相应的Lagrange乘子, 根据条件(15)可知

$$\begin{aligned}\nabla F(x_k,y_k)^\top d_k &= -d_k^\top B_k d_k - \lambda_k^\top \nabla G(x_k,y_k)d_k - \mu_k^\top \nabla(\nabla_y f_{\varepsilon^k}^r(x_k,y_k))d_k \\ &= -d_k^\top B_k d_k + G(x_k,y_k)^\top \lambda_k + (\nabla_y f_{\varepsilon^k}^r(x_k,y_k))^\top \mu_k.\end{aligned} \tag{19}$$

因此由(18)-(19)可得

$$\begin{aligned}&D(\varphi_\sigma^{\varepsilon^k,r}(x_k,y_k);d_k)\\ \le& -d_k^\top B_k d_k + G(x_k,y_k)^\top \lambda_k + (\nabla_y f_{\varepsilon^k}^r(x_k,y_k))^\top \mu_k - \sigma[\|\nabla_y f_\varepsilon^r(x_k,y_k)\|_1 + \|[G(x_k,y_k)]_+\|_1]\\ \le& -d_k^\top B_k d_k + \sum_{i=1}^m \lambda_k^i |G_i(x_k,y_k)_+| - \sigma[\sum_{i=1}^{n_2} |\nabla_{y_i} f_\varepsilon^r(x_k,y_k)| +\\ &\sum_{i=1}^m [G_i(x_k,y_k)]_+] + \sum_{i=1}^{n_2} |\mu_k^i||\nabla_{y_i} f_{\varepsilon^k}^r(x_k,y_k)|\\ \le& -d_k^\top B_k d_k - (\sigma - \|\mu_k\|_\infty)\sum_{i=1}^{n_2} |\nabla_{y_i} f_{\varepsilon^k}^r(x_k,y_k)| - (\sigma - \|\lambda_k\|_\infty)\sum_{i=1}^m [G_i(x_k,y_k)]_+.\end{aligned}$$

注 为了确保问题$(\mathrm{QP})_k$的解是价值函数$\varphi_\sigma^{\varepsilon^k,r}$在$(x_k,y_k)$处的下降方向, 罚参数$\sigma$按方式

$$\sigma_k = \begin{cases} \sigma_{k-1}, & 若\ \sigma_{k-1} \ge \tau + \delta, \\ \tau + 2\delta, & 若\ \sigma_{k-1} < \tau + \delta \end{cases}$$

更新, 其中$\tau = \max\{\|\mu_k\|_\infty, \|\lambda_k\|_\infty\}$, 并且$\delta > 0$, 则

$$D(\varphi_{\sigma_k}^{\varepsilon^k,r}(x_k,y_k);d_k) \le -d_k^\top B_k d_k. \tag{20}$$

因为B_k是正定的, 得到d_k是价值函数$\varphi_{\sigma_k}^{\varepsilon^k,r}$的下降方向.

定理 3.2 如果假设3.1成立, 并且假设算法1在有限次迭代中没有终止, 序列$\{(x_k,y_k)\}$, $\{\lambda_k\}$, $\{\mu_k\}$是有界的. 则$\bar{K} = \{k : \|d_k\| \le \hat{\eta}\varepsilon^k\}$是一个无限集, 且

$$\lim_{k\to\infty, k\in\bar{K}} \varepsilon^k = 0, \quad \lim_{k\to\infty, k\in\bar{K}} d_k = 0.$$

证 首先证明总是存在某个k使得$\{\|d_k\| \le \hat{\eta}\varepsilon^k\}$成立, 即$\bar{K}$是一个无限集.

利用反证法, 假设对每个k有$\|d_k\| \ge c_0 > 0$. 则由假设3.1, 条件(17)和(20), 可知存在一个正常数c使得$\varphi_{\sigma_k}^{\varepsilon^k,r}(x_{k+1},y_{k+1}) \le \varphi_{\sigma_k}^{\varepsilon^k,r}(x_k,y_k) - c$. 根据序列$\{\lambda_k\}, \{\mu_k\}$的有界性, 存在$\bar{k}_1$和$\bar{\sigma}$使得当$k \ge \bar{k}_1$时, 有$\sigma_k = \bar{\sigma}$; 并且根据$\varepsilon^k$的更新规则, 存在$\bar{k}_2$和$\bar{\varepsilon}$使得当$k \ge \bar{k}_2$时, 有$\varepsilon^k = \bar{\varepsilon}$. 取$\bar{k} = \max\{\bar{k}_1, \bar{k}_2\}$, 当$k \ge \bar{k}$时, 有$\sigma_k = \bar{\sigma}, \varepsilon^k = \bar{\varepsilon}$.

根据$\varphi_{\sigma_k}^{\varepsilon^k,r}(x_{k+1},y_{k+1}) \le \varphi_{\sigma_k}^{\varepsilon^k,r}(x_k,y_k) - c, c \ge 0$, 可知序列$\{\varphi_{\bar{\sigma}}^{\bar{\varepsilon},r}(x_k,y_k)\}$是单调下降的. 根据序列$\{(x_k,y_k)\}$的有界性, 可以得到

$$\begin{aligned}\sum_{k\ge\bar{k}} c &\le \sum_{k\ge\bar{k}} (\varphi_{\bar{\sigma}}^{\bar{\varepsilon},r}(x_k,y_k) - \varphi_{\bar{\sigma}}^{\bar{\varepsilon},r}(x_{k+1},y_{k+1}))\\ &= \varphi_{\bar{\sigma}}^{\bar{\varepsilon},r}(x_{\bar{k}},y_{\bar{k}}) - \lim_{k\to\infty} \varphi_{\bar{\sigma}}^{\bar{\varepsilon},r}(x_k,y_k) < \infty,\end{aligned}$$

这出现了矛盾. 因此$\bar{K}$是一个无限集, 这也意味着当$k \to \infty, k \in \bar{K}$时, $\varepsilon^k \searrow 0$, $d_k \to 0$.

假设 3.2 对充分大的k, $j \in \{1,2,\cdots,l\}$, $\frac{-\varepsilon^k}{g_j(x_k,y_k)}$有界.

根据引理2.1, 可知假设3.2是合理的.

定理 3.3 令假设3.1-3.2成立, 假设算法1在有限次迭代中没有终止, 序列
$$\{(x_k, y_k)\},\ \{\lambda_k, \mu_k, p_k\}_{\bar{K}}$$
有界, 其中$p_j^k = \nabla_y g_j(x_k, y_k)^\top \mu_k \frac{\varepsilon^k}{(g_j(x_k,y_k))^2}$, $j \in \{1, 2, \cdots, l\}$, 则序列$\{(x_k, y_k)\}_{\bar{K}}$的任意聚点是问题(BP)的弱稳定点.

证 假设存在$(\bar{x}, \bar{y}) \in \mathbf{R}^{n_1} \times \mathbf{R}^{n_2}$使得$\lim\limits_{k\to\infty, k\in\bar{K}} (x_k, y_k) = (\bar{x}, \bar{y})$. 因为序列$\{\lambda_k, \mu_k, p_k\}_{\bar{K}}$是有界的, 不失一般性, 假设存在$(\bar{\lambda}, \bar{\mu}, \bar{p}) \in \mathbf{R}^m \times \mathbf{R}^{n_2} \times \mathbf{R}^l$使得$\lim\limits_{k\to\infty, k\in\bar{K}} (\lambda_k, \mu_k, p_k) = (\bar{\lambda}, \bar{\mu}, \bar{p})$, 其中$\bar{\lambda} \geq 0$. 因为对任意$j \in \{1, 2, \cdots, l\}$, 序列$\{\frac{-\varepsilon^k}{g_j(x_k,y_k)}\}_{\bar{K}}$是有界的, 则存在有界向量$\bar{u} \in \mathbf{R}^l$使得$\lim\limits_{k\to\infty, k\in\bar{K}} \frac{-\varepsilon^k}{g(x_k,y_k)} = \bar{u}$.

条件(15)的第一个式子可以具体表示为
$$\nabla F(x_k, y_k) + \sum_{i=1}^{m} \lambda_k^i \nabla G_i(x_k, y_k) + \sum_{i=1}^{n_2} \mu_k^i [\nabla(\nabla_{y_i} f(x_k, y_k)) + \sum_{j=1}^{l} \frac{-\varepsilon^k}{g_j(x_k, y_k)} \nabla(\nabla_{y_i} g_j(x_k, y_k))]$$
$$+ B_k d_k + \sum_{j=1}^{l} [(\nabla_y g_j(x_k, y_k)^\top \mu_k) \frac{\varepsilon^k}{(g_j(x_k, y_k))^2}] \nabla g_j(x_k, y_k) + \sum_{i=1}^{n_2} \mu_k^i (\varepsilon^k r \nabla(y_k^i)) = 0. \tag{21}$$
根据函数F和G的连续可微性, 以及函数f和g的二阶连续可微性, 存在子序列$\widetilde{K} \subseteq \bar{K}$, 在条件(15)中令$k \to \infty, k \in \widetilde{K}$, 得到
$$\begin{aligned} &\nabla F(\bar{x}, \bar{y}) + \sum_{i=1}^{m} \bar{\lambda}_i \nabla G_i(\bar{x}, \bar{y}) + \sum_{i=1}^{n_2} \bar{\mu}_i \Big[\nabla(\nabla_{y_i} f(\bar{x}, \bar{y})) + \sum_{j=1}^{l} \bar{u}_j \nabla(\nabla_{y_i} g_j(\bar{x}, \bar{y}))\Big] \\ &+ \sum_{j=1}^{l} \bar{p}_j \nabla g_j(\bar{x}, \bar{y}) = 0, \\ &0 \leq \bar{\lambda}_i \perp (G_i(\bar{x}, \bar{y})) \leq 0, i = 1, 2, \cdots, m, \\ &\nabla_{y_i} f(\bar{x}, \bar{y}) + \sum_{j=1}^{l} [\bar{u}_j \nabla_{y_i} g_j(\bar{x}, \bar{y})] = 0, i = 1, 2, \cdots, n_2. \end{aligned} \tag{22}$$
考虑以下三种情形.

(1) 若$g_j(\bar{x}, \bar{y}) \neq 0, \bar{u}_j = 0$, 由定义可知$\bar{p}_j = 0$.

(2) 若$g_j(\bar{x}, \bar{y}) = 0, \bar{u}_j \neq 0$, 根据假设3.2可知, $\lim\limits_{k\to\infty, k\in\bar{K}} \frac{\varepsilon^k}{(g_j(x_k,y_k))^2} = \infty$, 从而根据$\|p_k\|$的有界性, 一定有$\nabla_y g_j(x_k, y_k)^\top \mu_k \to 0$, 即$\nabla_y g_j(\bar{x}, \bar{y})^\top \bar{\mu} = 0$.

(3) 若$g_j(\bar{x}, \bar{y}) = 0, \bar{u}_j = 0$, 由于$(\nabla_y g_j(x_k, y_k)^\top \mu_k) \frac{\varepsilon^k}{g_j(x_k,y_k)^2} (\nabla_y g_j(x_k, y_k)^\top \mu_k) \geq 0$, 因此$\bar{p}_j(\nabla_y g_j(\bar{x}, \bar{y})^\top \bar{\mu}) \geq 0$.

综上$(\bar{x}, \bar{y})$是问题(BP)的弱稳定点.

下面两个定理给出了序列$\{\lambda_k, \mu_k, p_k\}$有界的充分条件, 其中$p_k$为定理3.3中所定义.

定理 3.4 若假设3.1-3.2成立, 假设算法1在有限次迭代中没有终止, 并且序列$\{(x_k, y_k)\}$有界. 令(x^*, y^*)是序列$\{(x_k, y_k)\}$的任意聚点, 若存在$j \in \{1, 2, \cdots, l\}$使得$g_j(x^*, y^*) = 0$, 且集

合$\{\nabla G_i(x^*,y^*), i\in I^G; \nabla(\nabla_{y_i} f(x^*,y^*))+\sum_{j=1}^{l}\bar{u}_j\nabla(\nabla_{y_i} g_j(x^*,y^*)), i=1,2,\cdots,n_2; \nabla g_i(x^*,y^*),$ $i\in I^g\}$是线性无关的, 则序列$\{\lambda_k,\mu_k,p_k\}_{k\in\bar{K}}$有界.

证 当存在$j\in\{1,2,\cdots,l\}$, 使得$g_j(x^*,y^*)=0$时, 根据假设3.2可知$\lim\limits_{k\to\infty}\varepsilon^k=\varepsilon^*=0$. 那么$\bar{K}=\{k:\|d_k\|\le\hat{\eta}\varepsilon^k\}$是一个无限集, 于是$\lim\limits_{k\to\infty,k\in\bar{K}} d_k=0$.

令$w_k=(\lambda_k,\mu_k,p_k)$. 利用反证法, 假设存在子序列$K_1\subseteq\bar{K}$使得$\lim\limits_{k\to\infty,k\in K_1}\|w_k\|=\infty$. 假设存在$\bar{w}=(\bar{\lambda},\bar{\mu},\bar{p})$使得

$$\lim_{k\to\infty,k\in K_1}\frac{w_k}{\|w_k\|}=\bar{w}\neq 0,$$

其中$\bar{\lambda}\ge 0$.

对任意$j\in\{1,2,\cdots,l\}$, 令$\bar{u}_j=\lim\limits_{k\to\infty,k\in K_1}\frac{-\varepsilon^k}{g_j(x_k,y_k)}$. 等式(21)除以$\|w_k\|$并且两边同时取极限$k\to\infty,k\in K_1$, 有

$$\begin{aligned}&\sum_{i=1}^{m}\bar{\lambda}_i\nabla G_i(x^*,y^*)+\\&\sum_{i=1}^{n_2}\bar{\mu}_i\Big[\nabla(\nabla_{y_i} f(x^*,y^*))+\sum_{j=1}^{l}\bar{u}_j\nabla(\nabla_{y_i} g_j(x^*,y^*))\Big]+\sum_{j=1}^{l}\bar{p}_j\nabla g_j(x^*,y^*)=0.\end{aligned}\tag{23}$$

对条件(15)的第二个式子取极限$k\to\infty,k\in K_1$, 得到$\bar{\lambda}_i=0, i\notin I^G$. 此外对于$j\notin I^g$, 即$g_j(x^*,y^*)\neq 0$, 从而$\frac{\varepsilon^k}{g_j(x_k,y_k)^2}\to 0$, 则有$\bar{p}_j=0$. 结合条件(23), 与假设矛盾. 于是序列$\{\lambda_k,\mu_k,p_k\}_{\bar{K}}$是有界的.

定理 3.5 如果假设3.1成立, 假设算法1在有限次迭代中没有终止, 并且序列$\{(x_k,y_k)\}$有界. 令(x^*,y^*)和$\varepsilon^*>0$分别是序列$\{(x_k,y_k)\}$和ε^k的聚点. 如果$\forall j\in\{1,2,\cdots,l\}$, $g_j(x^*,y^*)\neq 0$, 且问题$(\mathrm{P}^{\varepsilon^*})$在$(x^*,y^*)$处满足EMFCQ, 则$\{d_k\}$和$\{\lambda_k,\mu_k,p_k\}$都有界.

证 由于问题$(\mathrm{P}^{\varepsilon^*})$在$(x^*,y^*)$处满足EMFCQ, 则存在向量$\hat{d}\in\mathbf{R}^{n_1+n_2}$使得

$$\nabla(\nabla_y f^r_{\varepsilon^*}(x^*,y^*))^\top\hat{d}+\nabla_y f^r_{\varepsilon^*}(x^*,y^*)=0,\quad \nabla G(x^*,y^*)^\top\hat{d}+\nabla G(x^*,y^*)<0,$$

并且$\nabla(\nabla_y f^r_{\varepsilon^*}(x^*,y^*))$行满秩. 由于$\forall j\in\{1,2,\cdots,l\}, g_j(x^*,y^*)\neq 0$, 那么对充分大的$k$, $\nabla(\nabla_y f^r_{\varepsilon^k}(x_k,y_k))$也是行满秩的, 因此存在子集$K$和序列$\hat{d}_k\to\hat{d}$, $k\to\infty,k\in K$, 使得

$$\nabla(\nabla_y f^r_{\varepsilon^k}(x_k,y_k))^\top\hat{d}_k+\nabla_y f^r_{\varepsilon^k}(x_k,y_k)=0.$$

此外对充分大的$k\in K$, $\nabla G(x_k,y_k)^\top\hat{d}_k+G(x_k,y_k)<0$. 注意到$\{\hat{d}_k\}_{k\in K}$有界, 并且$\hat{d}_k$是问题$(\mathrm{QP})_k$的可行点. 由于$d_k$是问题$(\mathrm{QP})_k$的最优解, 因此

$$\nabla F(x_k,y_k)^\top d_k+\frac{1}{2}d_k^\top B_k d_k\le\nabla F(x_k,y_k)^\top\hat{d}_k+\frac{1}{2}\hat{d}_k B_k\hat{d}_k.\tag{24}$$

根据假设3.1以及$\hat{d}_k$和$\{(x_k,y_k)\}$的有界性, 可知$\{d_k\}_{k\in K}$有界.

接下来证明序列$\{\lambda_k,\mu_k\}$有界. 首先令$w_k=(\lambda_k,\mu_k)$. 假设存在一个子序列$K_2\subseteq K$使得$\lim\limits_{k\to\infty,k\in K_2}\|w_k\|=\infty$. 则存在$\bar{w}=(\bar{\lambda},\bar{\mu})$使得

$$\lim_{k\to\infty,k\in K_2}\frac{w_k}{\|w_k\|}=\bar{w}\neq 0,$$

其中$\bar{\lambda}\ge 0$.

对条件(21)两边同时除以$\|w_k\|$并且令$k \in K_2, k \to \infty$, 得到

$$
\begin{aligned}
0 =& \sum_{i=1}^{n_2} \bar{\mu}_i \Big[\nabla(\nabla_{y_i} f(x^*, y^*)) + \sum_{j=1}^{l} \frac{-\varepsilon^*}{g_j(x^*, y^*)} \nabla(\nabla_{y_i} g_j(x^*, y^*)) \Big] + \sum_{i=1}^{n_2} \bar{\mu}_i \varepsilon^* r \nabla(y_i^*) \\
&+ \sum_{j=1}^{l} [\nabla_y g_j(x^*, y^*)^\top \bar{\mu} \frac{\varepsilon^*}{g_j(x^*, y^*)^2}] \nabla g_j(x^*, y^*) + \sum_{i=1}^{m} \bar{\lambda}_i \nabla G_i(x^*, y^*) \\
=& \sum_{i=1}^{n_2} \bar{\mu}_i \nabla(\nabla_{y_i} f_{\varepsilon^*}^r(x^*, y^*)) + \sum_{i=1}^{m} \bar{\lambda}_i \nabla G_i(x^*, y^*).
\end{aligned} \tag{25}
$$

对条件(15)的第2和第3个式子取极限$k \to \infty, k \in K_2$, 并且假设d^*是$\{d_k\}_{K_2}$的极限点, 则有

$$
\begin{aligned}
&0 \le \bar{\lambda}_i \perp (G_i(x^*, y^*) + \nabla G_i(x^*, y^*)^\top d^*) \le 0,\ i = 1, 2, \cdots, m, \\
&\nabla_{y_i} f_{\varepsilon^*}^r(x^*, y^*) + \nabla(\nabla_{y_i} f_{\varepsilon^*}^r(x^*, y^*))^\top d^* = 0, i = 1, 2, \cdots, n_2.
\end{aligned} \tag{26}
$$

在等式(25)的两边同乘d^*, 根据条件(26)可得

$$
\begin{aligned}
0 &= \sum_{i=1}^{m} \bar{\lambda}_i \nabla G_i(x^*, y^*)^\top d^* + \sum_{i=1}^{n_2} \bar{\mu}_i \nabla(\nabla_{y_i} f_{\varepsilon^*}^r(x^*, y^*))^\top d^* \\
&= -\sum_{i=1}^{m} \bar{\lambda}_i G_i(x^*, y^*) - \sum_{i=1}^{n_2} \bar{\mu}_i (\nabla_{y_i} f_{\varepsilon^*}^r(x^*, y^*)).
\end{aligned} \tag{27}
$$

因为问题($\mathrm{P}^{\varepsilon^*}$)在$(x^*, y^*)$处满足EMFCQ(等价于ENNAMCQ), 结合条件(25)和条件(27), 产生矛盾. 于是$\{\lambda_k\}$和$\{\mu_k\}$有界, 由定义可知, p_k有界.

从上面定理中可以直接得出以下推论.

推论 3.1 若定理3.4的条件成立, 则序列$\{(x_k, y_k)\}_{\bar{K}}$的任意聚点是问题(BP)的弱稳定点.

推论 3.2 若定理3.5的条件成立, 则序列$\{(x_k, y_k)\}_{\bar{K}}$的任意聚点是问题(BP)的弱稳定点.

§4 数值例子

本节应用算法1来求解一些双层规划问题, 以揭示算法的可行性, 并将数值结果与相应文献中算得的数值结果进行比较. 所有数值实验均在Matlab2016b中执行. 表1“Type”给出了问题的分类, 这些字符按规定的顺序分别表示上层目标函数, 上层约束, 下层目标函数和下层约束的函数类型, 分别为: **L**为线性, **Q**代表二次, **N**代表非线性非二次, **B**代表“箱子”约束. 如果没有约束, 则使用空白位置“_”.

在整个数值实验中, 算法中的参数为$\eta = 0.1$, $\rho = 0.5$, $\delta = 0.05$, $\hat{\eta} = 0.1$, $\sigma_{-1} = 1.25$, 选取固定的正则化参数$r = 1$, 对于例2-4, 例6-8, 例10-13选取参数$\rho_1 = 0.1$, 对于例1, 例5, 例9选取参数$\rho_1 = 0.01$, 并使用[26]提出的方法更新B_k. 令

$$
B_{k+1} = B_k - \tfrac{B_k s_k s_k^\top B_k}{s_k^\top B_k s_k} + \tfrac{z_k z_k^\top}{s_k^\top z_k}, \tag{28}
$$

其中$z_k = \theta_k y_k + (1 - \theta_k) B_k s_k$,

$$
\theta_k = \begin{cases} 1, & 若\ s_k^\top y_k \ge 0.2 s_k^\top B_k s_k, \\ \frac{0.8 s_k^\top B_k s_k}{s_k^\top B_k s_k - s_k^\top y_k}, & 若\ s_k^\top y_k < 0.2 s_k^\top B_k s_k, \end{cases}
$$

并且$s_k = \alpha_k d_k$, $\quad y_k = \nabla_{(x,y)} L_{\varepsilon^{k+1}}^r(x_{k+1}, y_{k+1}, \mu_{k+1}, \lambda_{k+1}) - \nabla_{(x,y)} L_{\varepsilon^{k+1}}^r(x_k, y_k, \mu_{k+1}, \lambda_{k+1})$, $L_\varepsilon^r(x, y, \mu, \lambda) = F(x, y) + \sum\limits_{i=1}^{m} \lambda_i G_i(x, y) + \sum\limits_{i=1}^{n_2} \mu_i \nabla_{y_i} f_\varepsilon^r(x, y)$. 数值结果如表2所示. 其中$(x_0, y_0)$表

表 1 测试问题

No.	Type	Author
Ex.1	NLNL	Zhou and Zemkoho[27]
Ex.2	Q_QB	Falk and Liu[28]
Ex.3	QLQB	Gümüs and Floudas [29]
Ex.4	Q_QL	Yezza[30]
Ex.5	QLNL	Zhou and Zemkoho[27]
Ex.6	QBQL	Zhou and Zemkoho[27]
Ex.7	QBNB	Zhou and Zemkoho[27]
Ex.8	LBQB	Ye and Zhu[6]
Ex.9	QBQQ	Friesz et al.[31]
Ex.10	QBQB	Vicente et al.[32]
Ex.11	QBQB	Vicente et al.[32]
Ex.12	NLNL	Ye et al.[33]
Ex.13	QLQL	Aiyoshi and Shimizu[34]

示迭代初始点; ε^0表示下层问题的初始罚参数; (x^*,y^*)表示计算得到的结果; "Iter"表示迭代数量; "Time"表示计算时间; "Fval"表示$F(x^*,y^*)$的值; ε^*表示下层问题罚参数的计算结果.

表 2 数值结果

Exam.	x_0	y_0	ε^0	x^*	y^*	Iter	Time	Fval	ε^*
1	7	4	0.1	5.0000	5.0000	53	1.7837	250.0015	1.0000e-15
2	(1, 1)	(1, 1)	0.1	(0.7500,0.7500)	(0.7500,0.7500)	29	0.0391	-2.2500	1.0000e-12
3	1	1	0.1	3.0000	5.0000	24	0.0966	9.0000	1.0000e-12
4	1	2	0.1	3.0000	1.0000	18	0.0386	0.5000	1.0000e-08
5	5	14	0.1	7.2000	12.8000	65	0.5734	2.3040e+03	1.0000e-15
6	20	10	0.1	93.3333	26.6667	25	0.0372	-3.2667e+03	1.0000e-07
7	1	2	0.1	-0.5671	5.6270e-09	11	0.0303	3.1663e-17	1.0000e-09
8	-0.7	0.6	0.1	-1.0000	-1.0000	46	0.6870	-2.0000	1.0000e-10
9	(0,0.1)	(1,1.2)	0.1	(0.5000,0.5000)	(0.5000,0.5000)	62	0.0313	-1.0000	1.0000e-13
10	(0.1,0.9)	0.8	0.1	(0.8000,0.2000)	1.0000	37	0.0273	3.1250e-12	1.0000e-07
11	(0.5,0.2)	0.1	0.1	(1.0000,0.4000)	0.8000	22	0.0650	3.1489e-16	1.0000e-09
12	0.5	0.2	0.1	1.0000	1.0000	16	0.0365	-1.0000	1.0000e-07
13	1	1	0.1	10.0000	10.0000	50	0.4315	100.0000	1.0000e-11

在文献中这些例子经常被用来测试算法的性能. 在这些例子中, 本文列出了相应文献算出的参考最优解, 并把算法1算得的计算结果与给出的参考最优解进行了比较. 算法1计算得到的最优解记为(x^*,y^*), 以及在最优解处的上层目标函数值和下层目标函数值分别记为$F(x^*,y^*)$和$f(x^*,y^*)$. 参考最优解表示为$(\bar{x},\bar{y})$, 在参考最优解处的上层目标函数值表示为$F(\bar{x},\bar{y})$并且在参考最优解处的下层目标函数值表示为$f(\bar{x},\bar{y})$.

本文算出的结果与相应文献算出的参考结果的比较见表3和4. 在表3和表4中, "Ref."列依次列出了文献算出的参考结果. 由表3可以看出, 对于例1, 例3-13, 算法1求出的解与相应文献中算

表 3 算法1求得的最优解与相关文献的参考解比较

Exam.	**Algorithm 1** (x^*, y^*)	**Ref.** $(\bar{x}, \bar{y})$
Ex.1	$(5, 5)$	$(5, 5)$
Ex.2	**(0.75,0.75,0.75,0.75)**	$(\sqrt{3}/2, \sqrt{3}/2, \sqrt{3}/2, \sqrt{3}/2)$
Ex.3	$(3, 5)$	$(3, 5)$
Ex.4	$(3, 1)$	$(3, 1)$
Ex.5	$(7.2, 12.8)$	$(7.2, 12.8)$
Ex.6	$(93.33, 26.67)$	$(93.33, 26.67)$
Ex.7	$(-0.5671, 0)$	$(-0.5671, 0)$
Ex.8	$(-1, -1)$	$(-1, -1)$
Ex.9	$(0.5, 0.5, 0.5, 0.5)$	$(0.5, 0.5, 0.5, 0.5)$
Ex.10	(0.8,0.2,1)	(0.8,0.2,1)
Ex.11	(1,0.4,0.8)	(1,0.4,0.8)
Ex.12	$(1, 1)$	$(1, 1)$
Ex.13	$(10, 10)$	$(10, 10)$

表 4 上层与下层最优目标函数值的比较

No.	**Algorithm 1** $F(x^*, y^*)$	$f(x^*, y^*)$	**Ref.** $F(\bar{x}, \bar{y})$	$f(\bar{x}, \bar{y})$
Ex.1	250.00	2.46e-18	250	0
Ex.2	**-2.25**	**1.84e-22**	-2.20	0
Ex.3	9.00	6.25e-22	9	0
Ex.4	0.50	2.50	0.5	2.5
Ex.5	2.30e+03	2.19e-17	2.30e+03	0
Ex.6	-3.27e+03	-711.11	-3.27e+03	-711.11
Ex.7	3.17e-17	-6.25e-25	0	0
Ex.8	-2.00	-1.00	-2	-1
Ex.9	-1.00	9.28e-18	-1	0
Ex.10	3.13e-12	-0.90	0	-0.9
Ex.11	3.15e-16	-0.32	0	-0.32
Ex.12	-1.00	-1.00	-1	-1
Ex.13	100	4.02e-11	100	0

出的解相等. 对于例2, 从表3和表4知算法1得到的解(x^*, y^*)与文献[28]中算出的参考解$(\bar{x}, \bar{y})$不一致, 经验证, 例2的最优解为$(0.75, 0.75, 0.75, 0.75)$, 即文献[28]没有算得最优解, 而本文的算法算到了最优解, 另外从表4中也可以看出本文的算法求得的解是要优于文献[28]算得的解.

因此算法1成功得到这些问题的最优解或近似最优解, 表明算法1是可行的.

§5 结论

本文研究了一类下层问题为凸问题的双层规划问题. 利用惩罚函数将下层问题转化为无约束凸优化问题. 提出用下层惩罚问题的KKT条件来代替原来的下层问题, 从而得到双层规划问题的松弛问题. 提出了一种求解双层规划松弛问题的SQP算法, 并证明了该算法的收敛性. 理论

和数值结果表明该算法具有良好的性能.

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A relaxed sequence quadratic programming method for solving bilevel programming problems

DU Meng-qi[1], XU Meng-wei[1], DUAN Qing-song[2]

(1. College of Science, Hebei University of Technology, Tianjin 300401, China;

2. Intelligent Risk Control Center, BOC Financial Technology Company Limited, Shanghai 200120, China)

Abstract: This article considers a class of bilevel programming problems with a special structure, where the lower-level problems are convex problems. First, the constraint function of the lower-level problem is penalized to the objective function by the interior point penalty method, so that the lower-level problem is approximated as a series of unconstrained optimization problems. Then the optimal set of solutions of the unconstrained lower-level problems is replaced by the KKT condition, and then the bilevel programming problem is approximated by a series of relaxed single-level problems. This article designs a smooth sequential quadratic programming algorithm to solve the relaxation problem and show that the iteration sequence generated by the algorithm converges to the weakly stationary point of the bilevel programming problem when the penalty factor converges to zero. The numerical experiments verify the feasibility of the algorithm.

Keywords: bilevel program; Tikhonov-regularized interior-penalty; sequential quadratic programming algorithm

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