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分数阶Navier-Stokes方程解的爆破准则

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摘 要: 首先证明了分数阶三维不可压缩Navier-Stokes方程在齐次Sobolev空间$\dot{H}^s$中解的存在性, 其中$\alpha > \frac{1}{2}$, $\max\left\{\frac{5}{2}-2\alpha, 0\right\} < s < \frac{3}{2}$. 其次在最大时间$T_\nu{}^*$有限时, 利用Fourier变换的性质, 齐次Sobolev空间中的插值结果以及乘积定理, 研究了解在$\dot{H}^s$空间中的爆破性和L^2范数的衰减性, 以及解关于Fourier变换的L^1范数的下界估计. 这是对Benameur J等人(2010)对经典Navier-Stokes方程所得出结论的推广.

关键词: 分数阶Navier-Stokes方程; 存在性; 衰减性; 爆破准则

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§1 引 言

本文考虑分数阶不可压缩Navier-Stokes方程的初值问题

$$\begin{cases} \partial_t u + \nu(-\Delta)^\alpha u + (u\cdot\nabla)\, u = -\nabla p, \\ \mathrm{div} u = 0, \\ u\,(0) = u^0 \end{cases} \tag{1}$$

在最大存在时间附近解的性质以及在齐次Sobolev空间中的爆破性. 其中ν为流体的粘性系数, $\alpha > 0$表示“耗散强度”. 标量函数$p = p(t,x)$表示流体在$(t,x) \in \mathbf{R}^+ \times \mathbf{R}^3$所受的未知压力, 向量函数$u(t,x) = (u^1(t,x), u^2(t,x), u^3(t,x))$表示流体在$(t,x) \in \mathbf{R}^+ \times \mathbf{R}^3$的未知速度, $u^0 = (u_1^0(t,x), u_2^0(t,x), u_3^0(t,x))$是给定的初始速度, $\Delta = \sum_{j=1}^n \partial_{x_j}^2$是关于空间变量$x = (x_1, x_2, x_3)$的Laplacian微分算子.

对于经典不可压缩Navier-Stokes方程($\alpha = 1$), Kato T在文献[1]和Leray J在文献[2]中研究了解的局部存在性和唯一性. Fujita H, Kato T在文献[3]中研究了小初值下在临界Sobolev空间$\dot{H}^{\frac{1}{2}}$中强解的适定性. Chemin J Y在文献[4]中研究了$\mathbf{R}^3$中Navier-Stokes方程的适定性. Beale J T, Kato T, Majda A在文献[5]和Kato T, Ponce G在文献[6]中证明了极大解的爆破结果. Benameur J在文献[7]中研究了非齐次Sobolev空间中解的爆破准则, 即$\|u(t)\|_{\dot{H}^s} \geq (T_\nu{}^* - t)^{-\frac{s}{3}}$ $(s > \frac{5}{2})$, 其中$T_\nu{}^*$是最大存在时间. Robinson J C, Sadowski W, Silva R P在文献[8]中提高了爆破解的下界.

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对于分数阶Navier-Stokes方程(1), Ding Yong, Sun Xiaochun在文献[9]和[10]中分别研究了弱解的唯一性和分数阶Navier-Stokes-Coriolis系统的色散效应和局部适定性. Sun Xiaochun, Liu Jia在文献[11]中研究了其在Sobolev-Gevery空间中的长时间衰减性. Sun Xiaochun, Liu Huandi在文献[12]中研究了分数阶各向异性Navier-Stokes方程弱解的唯一性.

本文研究分数阶不可压缩Navier-Stokes方程(1)在齐次Sobolev空间$\dot{H}^s$中解的存在性, 爆破准则和L^2范数衰减以及解关于Fourier变换的L^1范数的下界估计.

§2 预备知识

首先回顾一些基本知识.

- 函数f的Fourier变换可定义为
$$\mathcal{F}(f)(\xi)=\widehat{f}(\xi)=\int_{\mathbf{R}^3}\exp(-\mathrm{i}x\cdot\xi)f(x)\,\mathrm{d}x,\xi=(\xi_1,\xi_2,\xi_3)\in\mathbf{R}^3,f\in L^1(\mathbf{R}^3).$$
- 分数阶Laplacian微分算子通过Fourier变换定义为
$$\mathcal{F}\left((-\Delta)^{\alpha}u\right)(t,\xi)=|\xi|^{2\alpha}\mathcal{F}u(t,\xi)\,.$$
- $\mathbf{R}^3$上函数$f(x),g(x)$的卷积为
$$f*g(x)=\int_{\mathbf{R}^3}f(x-y)\,g(y)\,\mathrm{d}y=\int_{\mathbf{R}^3}f(y)\,g(x-y)\,\mathrm{d}y.$$
- 若$f=(f_1,f_2,f_3)$和$g=(g_1,g_2,g_3)$是两个向量场, 则
$$f\otimes g:=(g_1f,g_2f,g_3f)\,,$$
$$\operatorname{div}(f\otimes g):=(\operatorname{div}(g_1f),\operatorname{div}(g_2f),\operatorname{div}(g_3f))\,.$$
- 齐次Sobolev空间定义为$\dot{H}^s\left(\mathbf{R}^3\right)=\left\{f\in S'\left(\mathbf{R}^3\right),\widehat{f}\in L^1_{loc},|\xi|^s\widehat{f}\in L^2\left(\mathbf{R}^3\right)\right\}$, 其中$f$的范数为
$$\|f\|_{\dot{H}^s}=\left(\int_{\mathbf{R}^3}|\xi|^{2s}\left|\widehat{f}(\xi)\right|^2\mathrm{d}\xi\right)^{\frac{1}{2}}.$$

引理2.1[13] 设$a,b>0$, 那么对任意的$\lambda>0$有
$$\lambda^a\mathrm{e}^{-b\lambda}\le a^a(\mathrm{e}b)^{-a}.$$

引理2.2[14] 设X是一个具有范数$\|\cdot\|$的Banach空间, $B:X\times X\rightarrow X$是一个双线性算子, 使得对任意的$x_1,x_2\in X,\|B(x_1,x_2)\|\le\eta\|x_1\|\|x_2\|$, 对任意的$y\in X,4\eta\|y\|<1$, 则方程$x=y+B(x,x)$存在唯一的解$x\in X$, 并且此解满足$\|x\|\le2\|y\|$及$\|x\|<\frac{1}{2\eta}$.

引理2.3[15] 设$s,s'\in\mathbf{R}$, $s<\frac{3}{2},s+s'>0$且$f,g\in\dot{H}^s\cap\dot{H}^{s'}$, 则存在常数$C=C(s,s')$, 使得
$$\|fg\|_{\dot{H}^{s+s'-\frac{3}{2}}}\le C\left(\|f\|_{\dot{H}^s}\|g\|_{\dot{H}^{s'}}+\|f\|_{\dot{H}^{s'}}\|g\|_{\dot{H}^s}\right).$$
若$s,s'<\frac{3}{2},s+s'>0$, $f\in\dot{H}^s,g\in\dot{H}^{s'}$, 则存在常数$C=C(s,s')$, 使得
$$\|fg\|_{\dot{H}^{s+s'-\frac{3}{2}}}\le C\|f\|_{\dot{H}^s}\|g\|_{\dot{H}^{s'}}.$$

引理2.4[16] 设$s_1,s_2>0,s_1<s_2,\theta\in(0,1),f\in\dot{H}^{s_1}\cap\dot{H}^{s_2}$, 则成立
$$\|f\|_{\dot{H}^{\theta(s_1+(1-\theta)s_2)}}\le\|f\|^{\theta}_{\dot{H}^{s_1}}\|f\|^{1-\theta}_{\dot{H}^{s_2}}.$$

引理2.5[17] (Young不等式) 令$a>0,b>0,\varepsilon>0,1<p,q<\infty,\frac{1}{p}+\frac{1}{q}=1$, 有不等式
$$ab\le\varepsilon a^p+C_{\varepsilon}b^q,$$

其中$C_\varepsilon=(\varepsilon p)^{-\frac{q}{p}}q^{-1}$.

引理2.6[16] (Gronwall不等式) 设$y(t)$是$[0,T]$上的非负绝对连续函数, $h(t),g(t)$是$[0,T]$上的非负可积函数, 且满足

$$y'(t)\le h(t)\,y(t)+g(t)\,,\text{a.e. } t\in[0,T],$$

那么

$$y(t)\le \mathrm{e}^{\int_0^t h(\tau)\mathrm{d}\tau}\left[y(0)+\int_0^t g(\tau)\,\mathrm{d}\tau\right].$$

§3 主要结果

定理3.1 设$\alpha>\frac{1}{2},\max\{\frac{5}{2}-2\alpha,0\}<s<\frac{3}{2},u_0\in\dot{H}^s$且$\mathrm{div}u_0=0$, 则存在${T_\nu}^*>0$, 方程(1)存在解$u\in C([0,{T_\nu}^*);\dot{H}^s)$.

证 考虑方程(1)的解

$$u(t)=\mathrm{e}^{-\mu t(-\Delta)^\alpha}u_0-\int_0^t \mathrm{e}^{-\mu(t-\tau)(-\Delta)^\alpha}\mathrm{P}\mathrm{div}\,(u\otimes u)\,(\tau)\,\mathrm{d}\tau=\mathrm{e}^{-\mu t(-\Delta)^\alpha}u_0+B(u,u),$$

这里投影算子$\mathrm{P}:=\mathrm{I_d}+\nabla(-\Delta)^{-1}\mathrm{div}$.

$$B(u,v)=-\int_0^t \mathrm{e}^{-\mu(t-\tau)(-\Delta)^\alpha}\mathrm{P}\mathrm{div}\,(u\otimes v)\,(\tau)\,\mathrm{d}\tau.$$

$$\begin{aligned}\|\mathrm{e}^{-\mu t(-\Delta)^\alpha}u_0\|_{\dot{H}^s}&=\left(\int_{\mathbf{R}^3}|\xi|^{2s}\mathrm{e}^{-2\mu t|\xi|^{2\alpha}}|\widehat{u_0}|^2\mathrm{d}\xi\right)^{\frac{1}{2}}\\&\le\left(\int_{\mathbf{R}^3}|\xi|^{2s}|\widehat{u_0}|^2\mathrm{d}\xi\right)^{\frac{1}{2}}=\|u_0\|_{\dot{H}^s}.\end{aligned}$$

由引理2.1和引理2.3有

$$\begin{aligned}\|B(u,v)\|_{\dot{H}^s}&\le\int_0^t\|\mathrm{e}^{-\mu(t-\tau)|\xi|^{2\alpha}}\mathrm{P}\mathrm{div}\,(u\otimes v)\,(\tau)\|_{\dot{H}^s}\mathrm{d}\tau\\&\le\int_0^t\left(\int_{\mathbf{R}^3}|\xi|^{2s+2}\mathrm{e}^{-2\mu(t-\tau)|\xi|^{2\alpha}}\left|\widehat{u\otimes v}\right|^2\mathrm{d}\xi\right)^{\frac{1}{2}}\mathrm{d}\tau\\&=\int_0^t\left(\int_{\mathbf{R}^3}\left(|\xi|^{2\alpha}\right)^{\frac{5-2s}{2\alpha}}\mathrm{e}^{-2\mu(t-\tau)|\xi|^{2\alpha}}|\xi|^{4s-3}\left|\widehat{u\otimes v}\right|^2\mathrm{d}\xi\right)^{\frac{1}{2}}\mathrm{d}\tau\\&\le C\int_0^t(t-\tau)^{-\frac{5-2s}{4\alpha}}\left(\int_{\mathbf{R}^3}|\xi|^{4s-3}\left|\widehat{u\otimes v}\right|^2\mathrm{d}\xi\right)^{\frac{1}{2}}\mathrm{d}\tau\\&\le C\int_0^t(t-\tau)^{-\frac{5-2s}{4\alpha}}\|u\otimes v\|_{\dot{H}^{2s-\frac{3}{2}}}\mathrm{d}\tau\\&\le C\int_0^t(t-\tau)^{-\frac{5-2s}{4\alpha}}\|u\|_{\dot{H}^s}\|v\|_{\dot{H}^s}\mathrm{d}\tau\\&\le C{T_\nu}^{*\frac{2s+4\alpha-5}{4\alpha}}\|u\|_{L^\infty_{T_\nu^*}(\dot{H}^s)}\|v\|_{L^\infty_{T_\nu^*}(\dot{H}^s)}.\end{aligned}$$

由引理2.2得方程(1)存在解$u\in C([0,{T_\nu}^*);\dot{H}^s)$.

定理3.2 设$1<\alpha<\frac{5}{2},\frac{5}{2}-\alpha<s<\frac{3}{2},\limsup\limits_{t\to T_\nu^*}\|u(t)\|^2_{\dot{H}^s}=\infty$且$u\in C([0,{T_\nu}^*);\dot{H}^s)$是方程(1)的极大解, 若${T_\nu}^*<\infty$, 则有

$$\frac{c\nu^{\frac{s}{5-2\alpha}}}{(T_\nu^*-t)^{\frac{s}{5-2\alpha}}}\le\|u(t)\|_{L^2}^{\frac{2s}{5-2\alpha}-1}\|u(t)\|_{\dot{H}^s}.$$

证 方程(1)在$\dot{H}^s$空间下与u取内积

$$\langle \partial_t u, u\rangle_{\dot{H}^s} + \nu\langle(-\Delta)^\alpha u, u\rangle_{\dot{H}^s} + \langle u\cdot\nabla u, u\rangle_{\dot{H}^s} = -\langle \nabla p, u\rangle_{\dot{H}^s}.$$

则有

$$\begin{aligned}\frac{1}{2}\partial_t \|u\|_{\dot{H}^s}^2 + \nu\|\Lambda^\alpha u\|_{\dot{H}^s}^2 &\le |\langle u\cdot\nabla u, u_{\dot{H}^s}\rangle| = \int_{\mathbf{R}^3}|\xi|^{2s}|\widehat{u\cdot\nabla u}|\cdot|\widehat{u}|\mathrm{d}\xi\\ &= \int_{\mathbf{R}^3}|\xi|^{2s+1}|\widehat{u\otimes u}|\cdot|\widehat{u}|\mathrm{d}\xi\\ &= \int_{\mathbf{R}^3}|\xi|^{s-\alpha+1}|\widehat{u\otimes u}||\xi|^s|\widehat{\Lambda^\alpha u}|\mathrm{d}\xi\\ &\le \left(\int_{\mathbf{R}^3}|\xi|^{2(s-\alpha+1)}|\widehat{u\otimes u}|^2\mathrm{d}\xi\right)^{\frac{1}{2}}\left(\int_{\mathbf{R}^3}|\xi|^{2s}|\widehat{\Lambda^\alpha u}|^2\mathrm{d}\xi\right)^{\frac{1}{2}}\\ &= \|u\otimes u\|_{\dot{H}^{s-\alpha+1}}\|\Lambda^\alpha u\|_{\dot{H}^s}.\end{aligned}$$

由引理2.3及不等式$ab\le\frac{a^2}{2}+\frac{b^2}{2}$有

$$\begin{aligned}\frac{1}{2}\partial_t \|u\|_{\dot{H}^s}^2 + \nu\|\Lambda^\alpha u\|_{\dot{H}^s}^2 &\le C\|u\|_{\dot{H}^s}\|u\|_{\dot{H}^{\frac{5}{2}-\alpha}}\|\Lambda^\alpha u\|_{\dot{H}^s}\\ &\le C\nu^{-1}\|u\|_{\dot{H}^s}^2\|u\|_{\dot{H}^{\frac{5}{2}-\alpha}}^2 + \frac{\nu}{2}\|\Lambda^\alpha u\|_{\dot{H}^s}^2,\end{aligned}$$

则

$$\partial_t \|u\|_{\dot{H}^s}^2 + \nu\|\Lambda^\alpha u\|_{\dot{H}^s}^2 \le C\nu^{-1}\|u\|_{\dot{H}^s}^2\|u\|_{\dot{H}^{\frac{5}{2}-\alpha}}^2.$$

由引理2.6, 对于$0\le a\le t< T_\nu^*$,

$$\|u(t)\|_{\dot{H}^s}^2 \le \|u(a)\|_{\dot{H}^s}^2\,\mathrm{e}^{C\nu^{-1}\int_a^t\|u(\tau)\|_{\dot{H}^{\frac{5}{2}-\alpha}}^2\mathrm{d}\tau}.$$

因为$\limsup\limits_{t\to T_\nu^*}\|u(t)\|_{\dot{H}^s}^2=\infty$, 所以$\int_a^{T_\nu^*}\|u(t)\|_{\dot{H}^{\frac{5}{2}-\alpha}}^2\mathrm{d}\tau=\infty$. 由引理2.4得到插值结果

$$\|u\|_{\dot{H}^{\frac{5}{2}-\alpha}} \le \|u\|_{L^2}^{1-\frac{5-2\alpha}{2s}}\|u\|_{\dot{H}^s}^{\frac{5-2\alpha}{2s}}.$$

从而有

$$\begin{aligned}\frac{\|u\|_{\dot{H}^{\frac{5}{2}-\alpha}}^2}{\|u\|_{L^2}^{2-\frac{5-2\alpha}{s}}} \le \|u\|_{\dot{H}^s}^{\frac{5-2\alpha}{s}} &\le \|u(a)\|_{\dot{H}^s}^{\frac{5-2\alpha}{s}}\,\mathrm{e}^{C\nu^{-1}\int_a^t\|u(\tau)\|_{\dot{H}^{\frac{5}{2}-\alpha}}^2\mathrm{d}\tau},\\ \|u\|_{\dot{H}^{\frac{5}{2}-\alpha}}^2 \le \|u\|_{L^2}^{2-\frac{5-2\alpha}{s}}\|u(a)\|_{\dot{H}^s}^{\frac{5-2\alpha}{s}}\,\mathrm{e}^{C\nu^{-1}\int_a^t\|u(\tau)\|_{\dot{H}^{\frac{5}{2}-\alpha}}^2\mathrm{d}\tau},\\ \|u\|_{\dot{H}^{\frac{5}{2}-\alpha}}^2\,\mathrm{e}^{-C\nu^{-1}\int_a^t\|u(\tau)\|_{\dot{H}^{\frac{5}{2}-\alpha}}^2\mathrm{d}\tau} \le \|u\|_{L^2}^{2-\frac{5-2\alpha}{s}}\|u(a)\|_{\dot{H}^s}^{\frac{5-2\alpha}{s}}.\end{aligned}$$

在$[a,T]$上对上式积分有

$$1-\mathrm{e}^{-C\nu^{-1}\int_a^T\|u(\tau)\|_{\dot{H}^{\frac{5}{2}-\alpha}}^2\mathrm{d}\tau} \le C\nu^{-1}\|u\|_{L^2}^{2-\frac{5-2\alpha}{s}}\|u(a)\|_{\dot{H}^s}^{\frac{5-2\alpha}{s}}(T-a).$$

当$T\to T_\nu^*$时有

$$1\le C\nu^{-1}\|u\|_{L^2}^{2-\frac{5-2\alpha}{s}}\|u(a)\|_{\dot{H}^s}^{\frac{5-2\alpha}{s}}(T_\nu^*-a).$$

对于$0\le t_1<T_\nu^*$, 考虑下面分数阶Navier-Stokes方程

$$\begin{cases}\partial_t v+\nu(-\Delta)^\alpha v+(v\cdot\nabla)v=-\nabla p,\\ \mathrm{div}v=0,\\ v(0)=v^0.\end{cases}\tag{2}$$

设该方程的极大解为v且$v\in C([0,t_\nu^*),\dot{H}^s)$, 其中$t_\nu^*=T_\nu^*-t_1$. 对于$0\le t<t_\nu^*$,

$$\begin{aligned}1\le C\nu^{-1}\|v(0)\|_{L^2}^{2-\frac{5-2\alpha}{s}}\|v(t)\|_{\dot{H}^s}^{\frac{5-2\alpha}{s}}(T_\nu^*-t_1-t),\\ 1\le C\nu^{-1}\|u(t_1)\|_{L^2}^{2-\frac{5-2\alpha}{s}}\|u(t+t_1)\|_{\dot{H}^s}^{\frac{5-2\alpha}{s}}(T_\nu^*-t_1-t).\end{aligned}$$

取$t=0$则有

$$1\leq C\nu^{-1}\left\|u\left(t\right)\right\|_{L^2}^{2-\frac{5-2\alpha}{s}}\left\|u\left(t\right)\right\|_{\dot{H}^s}^{\frac{5-2\alpha}{s}}\left(T_\nu^*-t\right),\quad \frac{C\nu^{\frac{s}{5-2\alpha}}}{\left(T_\nu^*-t\right)^{\frac{s}{5-2\alpha}}}\leq\left\|u\left(t\right)\right\|_{L^2}^{\frac{2s}{5-2\alpha}-1}\left\|u\left(t\right)\right\|_{\dot{H}^s}.$$

从而结论得证.

定理3.3 设$\alpha>\frac{1}{2}, \limsup\limits_{t\to T_\nu^*}\|\widehat{u}(t)\|_{L^1}=\infty$且$u\in C\left([0,T_\nu^*);\dot{H}^s\right)$是(1)的极大解, 若$T_\nu^*<\infty$, 则有

$$\frac{C\nu^{\frac{1}{2\alpha}}}{\left(T_\nu^*-t\right)^{\frac{2\alpha-1}{2\alpha}}}\leq\|\widehat{u}(t)\|_{L^1}.$$

证 对方程(1)取Fourier变换

$$\frac{1}{2}\partial_t|\widehat{u}\left(t,\xi\right)|^2+\nu|\xi|^{2\alpha}|\widehat{u}\left(t,\xi\right)|^2+(\widehat{u\cdot\nabla u})(t,\xi)\cdot\widehat{u}(t,-\xi)=0.$$

对任意的$\varepsilon>0$,

$$\frac{1}{2}\partial_t|\widehat{u}\left(t,\xi\right)|^2=\frac{1}{2}\partial_t\left(|\widehat{u}\left(t,\xi\right)|^2+\varepsilon\right)=\sqrt{|\widehat{u}(t,\xi)|^2+\varepsilon}\,\partial_t\left(\sqrt{|\widehat{u}(t,\xi)|^2+\varepsilon}\right).$$

那么

$$\partial_t\left(\sqrt{|\widehat{u}(t,\xi)|^2+\varepsilon}\right)+\nu|\xi|^{2\alpha}\frac{|\widehat{u}(t,\xi)|^2}{\sqrt{|\widehat{u}(t,\xi)|^2+\varepsilon}}+\frac{(\widehat{u\cdot\nabla u})(t,\xi)\cdot\widehat{u}(t,-\xi)}{\sqrt{|\widehat{u}(t,\xi)|^2+\varepsilon}}=0,$$

$$\partial_t\left(\sqrt{|\widehat{u}(t,\xi)|^2+\varepsilon}\right)+\nu|\xi|^{2\alpha}\frac{|\widehat{u}(t,\xi)|^2}{\sqrt{|\widehat{u}(t,\xi)|^2+\varepsilon}}\leq\left|(\widehat{u\cdot\nabla u})(t,\xi)\right|.$$

在$[a,t]\subset[0,T_\nu^*)$上对上式积分得

$$\sqrt{|\widehat{u}(t,\xi)|^2+\varepsilon}+\nu|\xi|^{2\alpha}\int_a^t\frac{|\widehat{u}(\tau,\xi)|^2}{\sqrt{|\widehat{u}(\tau,\xi)|^2+\varepsilon}}\mathrm{d}\tau\leq\sqrt{|\widehat{u}(a,\xi)|^2+\varepsilon}+\int_a^t\left|(\widehat{u\cdot\nabla u})(\tau,\xi)\right|\mathrm{d}\tau.$$

当$\varepsilon\to0$时,

$$\begin{aligned}|\widehat{u}(t,\xi)|+\nu|\xi|^{2\alpha}\int_a^t|\widehat{u}(\tau,\xi)|\,\mathrm{d}\tau&\leq|\widehat{u}(a,\xi)|+\int_a^t\left|(\widehat{u\cdot\nabla u})(\tau,\xi)\right|\mathrm{d}\tau\\&\leq|\widehat{u}(a,\xi)|+\int_a^t|\widehat{u}|*\left|\widehat{\nabla u}\right|(\tau,\xi)\,\mathrm{d}\tau,\end{aligned}$$

$$\|\widehat{u}(t,\xi)\|_{L^1}+\nu\int_a^t\|\widehat{(-\Delta)^\alpha}u\|_{L^1}\leq\|\widehat{u}(a,\xi)\|_{L^1}+\int_a^t\|\widehat{u}(\tau,\xi)\|_{L^1}\|\widehat{\nabla}u(\tau,\xi)\|_{L^1}\mathrm{d}\tau.$$

由Hölder不等式

$$\begin{aligned}\|\widehat{\nabla}u(\tau,\xi)\|_{L^1}&=\int_{\mathbf{R}^3}|\xi|\,|\widehat{u}|\,\mathrm{d}\xi\\&=\int_{\mathbf{R}^3}|\xi|\,|\widehat{u}|^{\frac{1}{2\alpha}}|\widehat{u}|^{1-\frac{1}{2\alpha}}\mathrm{d}\xi\\&\leq\left(\int_{\mathbf{R}^3}|\xi|^{2\alpha}\,[\widehat{u}]\,\mathrm{d}\xi\right)^{\frac{1}{2\alpha}}\left(\int_{\mathbf{R}^3}|\widehat{u}|\,\mathrm{d}\xi\right)^{1-\frac{1}{2\alpha}}\\&=\|\widehat{(-\Delta)^\alpha}u\|_{L^1}^{\frac{1}{2\alpha}}\|\widehat{u}\|_{L^1}^{1-\frac{1}{2\alpha}}.\end{aligned}$$

则

$$\|\widehat{u}(t,\xi)\|_{L^1}+\nu\int_a^t\|\widehat{(-\Delta)^\alpha}u\|_{L^1}\leq\|\widehat{u}(a,\xi)\|_{L^1}+\int_a^t\|\widehat{u}(\tau,\xi)\|_{L^1}^{2-\frac{1}{2\alpha}}\|\widehat{(-\Delta)^\alpha}u(\tau,\xi)\|_{L^1}^{\frac{1}{2\alpha}}\mathrm{d}\tau.$$

由引理2.5得

$$\|\widehat{u}(t,\xi)\|_{L^1}+\nu\int_a^t\|\widehat{(-\Delta)^{\alpha}u}\|_{L^1}\mathrm{d}\tau\le$$

$$\|\widehat{u}(a,\xi)\|_{L^1}+\mathrm{C}\nu^{-\frac{1}{2\alpha-1}}\int_a^t\|\widehat{u}(\tau,\xi)\|_{L^1}^{\frac{4\alpha-1}{2\alpha-1}}\mathrm{d}\tau+\frac{\nu}{2}\int_a^t\|\widehat{(-\Delta)^{\alpha}u}(\tau,\xi)\|_{L^1}\mathrm{d}\tau,$$

$$\|\widehat{u}(t,\xi)\|_{L^1}+\frac{\nu}{2}\int_a^t\|\widehat{(-\Delta)^{\alpha}u}\|_{L^1}\mathrm{d}\tau\le\|\widehat{u}(a,\xi)\|_{L^1}+\mathrm{C}\nu^{-\frac{1}{2\alpha-1}}\int_a^t\|\widehat{u}(\tau,\xi)\|_{L^1}^{\frac{2\alpha}{2\alpha-1}}\|\widehat{u}(\tau,\xi)\|_{L^1}\mathrm{d}\tau.$$

由引理2.6得

$$\|\widehat{u}(t,\xi)\|_{L^1}\le\|\widehat{u}(a,\xi)\|_{L^1}\mathrm{e}^{C\nu^{-\frac{1}{2\alpha-1}}\int_a^t\|\widehat{u}(\tau,\xi)\|_{L^1}^{\frac{2\alpha}{2\alpha-1}}\mathrm{d}\tau}.$$

因为$\limsup\limits_{t\to T_\nu^*}\|\widehat{u}(t)\|_{L^1}=\infty$, 所以$\int_a^{T_\nu^*}\|\widehat{u}(\tau,\xi)\|_{L^1}^{\frac{2\alpha}{2\alpha-1}}\mathrm{d}\tau=\infty$.

$$\|\widehat{u}(t,\xi)\|_{L^1}\mathrm{e}^{-C\nu^{-\frac{1}{2\alpha-1}}\int_a^t\|\widehat{u}(\tau,\xi)\|_{L^1}^{\frac{2\alpha}{2\alpha-1}}\mathrm{d}\tau}\le\|\widehat{u}(a,\xi)\|_{L^1},$$

$$\|\widehat{u}(t,\xi)\|_{L^1}^{\frac{2\alpha}{2\alpha-1}}\mathrm{e}^{-C\nu^{-\frac{1}{2\alpha-1}}\int_a^t\left\|\widehat{u(\tau,\xi)}\right\|_{L^1}^{\frac{2\alpha}{2\alpha-1}}\mathrm{d}\tau}\le\|\widehat{u}(a,\xi)\|_{L^1}^{\frac{2\alpha}{2\alpha-1}}.$$

在$[a,T]\subset[0,T_\nu{}^*)$对上式积分得

$$\int_a^T\|\widehat{u}(t,\xi)\|_{L^1}^{\frac{2\alpha}{2\alpha-1}}\mathrm{e}^{-C\nu^{-\frac{1}{2\alpha-1}}\int_a^t\|\widehat{u}(\tau,\xi)\|_{L^1}^{\frac{2\alpha}{2\alpha-1}}\mathrm{d}\tau}\mathrm{d}t\le\|\widehat{u}(a,\xi)\|_{L^1}^{\frac{2\alpha}{2\alpha-1}}(T-a),$$

$$1-\mathrm{e}^{-C\nu^{-\frac{1}{2\alpha-1}}\int_a^t\|\widehat{u}(\tau,\xi)\|_{L^1}^{\frac{2\alpha}{2\alpha-1}}\mathrm{d}\tau}\le C\nu^{-\frac{1}{2\alpha-1}}\|\widehat{u}(a,\xi)\|_{L^1}^{\frac{2\alpha}{2\alpha-1}}(T-a).$$

当$T\to T_\nu{}^*$时有

$$1\le C\nu^{-\frac{1}{2\alpha-1}}\|\widehat{u}(a,\xi)\|_{L^1}^{\frac{2\alpha}{2\alpha-1}}(T_\nu^*-a).$$

对于$0\le t_1<T_\nu^*$, 考虑下面分数阶Navier-Stokes方程

$$\begin{cases}\partial_t v+\nu(-\Delta)^{\alpha}v+(v\cdot\nabla)v=-\nabla p,\\ \mathrm{div}v=0,\\ v(0)=v^0,\end{cases}\tag{3}$$

设该方程的极大解为v且$v\in C([0,t_\nu^*),\dot{H}^s)$, 其中$t_\nu^*=T_\nu^*-t_1$. 对于$0\le t<t_\nu^*$,

$$1\le C\nu^{-\frac{1}{2\alpha-1}}\|\widehat{v}(t,\xi)\|_{L^1}^{\frac{2\alpha}{2\alpha-1}}(T_\nu^*-t_1-t).$$

取$t=0$则有

$$1\le C\nu^{-\frac{1}{2\alpha-1}}\|\widehat{u}(t_1,\xi)\|_{L^1}^{\frac{2\alpha}{2\alpha-1}}(T_\nu^*-t_1),$$

$$\frac{C\nu^{\frac{1}{2\alpha}}}{(T_\nu^*-t)^{\frac{2\alpha-1}{2\alpha}}}\le\|\widehat{u}(t,\xi)\|_{L^1},$$

从而结论得证.

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On the blow-up criterion for solutions of 3D fractional Navier-Stokes equations

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Abstract: The existence of solutions to the fractional 3D incompressible Navier-Stokes equations in homogeneous Sobolev spaces $\dot{H}^s$ is firstly proved in this paper, where $\alpha > \frac{1}{2}, \max\{\frac{5}{2}-2\alpha, 0\} < s < \frac{3}{2}$. Secondly, when the maximum time $T_\nu{}^*$ is finite, the blow-up in $\dot{H}^s$ spaces and the decay in L^2 norm of the solution and the lower bounds estimate of the solution with respect to L^1 norm of Fourier transform are studied, via using the property of Fourier transform, interpolation results and product law in the homogeneous Sobolev spaces. Finally, it's a generalization of the results obtained by Benameur J, et al(2010) on the classical Navier-Stokes equations.

Keywords: fractional Navier-Stokes equation; existence; decay; blow-up criterion

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