高校应用数学学报
2024, 39(1): 89-104

无界域上全纯Cliffordian函数的若干性质及其Cauchy型积分的边值特性

张 坤[1], 高 龙[2], 乔玉英*[2]

(1. 沧州交通学院, 河北沧州 061199;
2. 河北师范大学 数学科学学院, 河北石家庄 050024)

摘 要: 首先介绍了定义于Euclidean空间, 取值于实Clifford代数的全纯Cliffordian函数. 然后利用正则函数的性质讨论了全纯Cliffordian函数的若干性质和空间性质. 主要借用第一类拟置换得出全纯Cliffordian函数的等价条件, 研究了它与正则函数之间的关系. 接下来以Cauchy积分公式和Plemelj公式为基础, 得出了开拓定理. 最后定义了无界域上的Cauchy型积分, 并证得它在Cauchy主值意义下收敛. 同时利用一些重要的积分估值得出了无界域上的Plemelj公式.

关键词: 全纯Cliffordian函数; Cauchy型积分; Cauchy主值; Plemelj公式
中图分类号: O174.5
文献标识码: A　　**文章编号**: 1000-4424(2024)01-0089-16

§1 引 言

Clifford代数是W. K. Clifford创立的对乘法可结合, 不可交换的几何代数[1]. Clifford分析是随Clifford代数发展起来的一个新兴数学分支. 它在弹性力学, 流体力学, 机器人学, 量子力学, 计算生物学等很多现代科技领域被广泛地应用. 单复变中全纯函数理论向高维空间推广分为两个方向: 一个是多复变中的全纯函数理论; 另一个是Clifford分析中的正则函数理论. 其中关于Cauchy型积分算子性质的研究一直是国内外活跃的课题之一, 它在解决Clifford分析的边值问题上具有重要意义. 并且在许多领域中扮演重要角色. 例如: 调和分析, 算子代数, 位势理论, 偏微分方程和数值逼近等. 上世纪80年代, Clifford分析理论初步完成, 以研究正则函数的性质为中心形成了一个理论体系, 得到了一系列的成果. 例如: 正则函数的Cauchy积分公式, Cauchy-Pompeiu公式, 唯一性定理以及Cauchy型积分的边界性质等[2].

近期, 许多学者对Clifford分析的函数理论, 边值问题及奇异积分理论做了大量工作和深入研究. 2005年, 黄沙等[3-4]系统的研究了Clifford代数中正则函数的理论体系, 在此基础上给出了

收稿日期: 2022-10-01　　修回日期: 2023-07-28
*通讯作者, E-mail: yuyingqiao@163.com
基金项目: 河北省高等学校科学研究计划(ZC2022092)

一类非齐次偏微分的边值问题; 杜金元等[5]探讨了在Clifford代数和泛Clifford代数中Cauchy型积分的边界行为; 任广斌等[6]研究了Clifford分析中一类与复分析联系很密切的Slice正则函数的性质; 乔玉英等[7]研究了Clifford分析中非欧度量下的一类正则函数的性质; 谢永红等[8]研究了hypergenic-Cauchy型积分的边界性质; 王丽萍等[9]研究了T算子的性质; 杨贺菊等[10]研究了加α权k正则函数的Cauchy积分公式; 李尊凤等[11]讨论了复Clifford代数上的Cauchy积分公式; 史海盼等[12-14]研究了Clifford分析中双边Fourier变换, Clifford-Fourier变换以及双边分数阶Fourier变换的性质; Dinh等[15-16]讨论了k-正则函数, 广义(ki)-正则函数的表示; Blaya等[17]给出了次多正则函数的Cauchy积分公式; 许娜[18]研究了复泛Clifford代数上的Cauchy积分公式.

H. Leutwiler等对Dirac算子的修正促进了Clifford空间中超正则, k正则, k超正则等新函数类的推广. 全纯Cliffordian函数是一类比正则函数更广泛的函数类. G. Laville和I. Ramadanoff[19-22]把幂函数, 指数函数等常用全纯函数推广到高维空间, 引入了全纯Cliffordian函数. 然后给出其核函数, 有界域上的积分表达, Taylor和Laurent展式等. 并对解析型, 椭圆型和Jaccobi椭圆型Cliffordian函数做了相关研究. 扈玮玮[23]研究了有界域上全纯Cliffordian函数的Cauchy型积分, Plemelj公式, Plivalov定理等. 库敏讨论了复Clifford代数上全纯Cliffordian函数的相关性质. 这一新函数进一步拓宽了人们在实复Clifford分析上的研究视野和发展方向.

本文研究定义于$\mathbf{R}^{2m+2}$空间, 取值于Clifford代数$\mathbf{Cl}_{2m+1,0}(\mathbf{R})$上的全纯Cliffordian函数的若干性质. 主要安排如下: §2回顾与全纯Cliffordian函数有关的基本知识和常用引理, 并且给出了两个重要算子的不等式模估计. §3讨论了全纯Cliffordian函数的空间特性-构成一右$\mathbf{Cl}_{2m+1,0}(\mathbf{R})$模, 介绍了第一类拟置换, 给出全纯Cliffordian函数与正则函数之间的关系. 还以Cauchy积分表示式和Plemelj公式为基础, 得出了有界域上的开拓定理. §4受无界域上k正则函数高阶Cauchy核的启发, 构作了无界域上全纯Cliffordian函数的Cauchy核. 并介绍了$2m+1$次连续可微函数的Cauchy型积分及其Cauchy主值. 这里通过巧妙积分估值和增设条件, 将无界积分区域的边界分成有界和无界两部分进行讨论, 最后得出了Plemelj公式.

§2 预备知识

设$\mathbf{Cl}_{2m+1,0}(\mathbf{R})$是$2m+2$维实线性空间$\mathbf{R}^{2m+2}$的Clifford代数, 且$\{e_i, i=0,1,\cdots,2m+1\}$为$\mathbf{R}^{2m+2}$的一组标准正交基, 则$\mathbf{Cl}_{2m+1,0}(\mathbf{R})$是以

$$\mathrm{e}_0,\mathrm{e}_1,\cdots,\mathrm{e}_{2m+1};\mathrm{e}_1\mathrm{e}_2,\cdots,\mathrm{e}_{2m}\mathrm{e}_{2m+1};\cdots;\mathrm{e}_1\cdots\mathrm{e}_{2m+1}$$

为基的2^{2m+1}维空间, $\mathbf{Cl}_{2m+1,0}(\mathbf{R})$中的基元素可表示成$\mathrm{e}_A=\mathrm{e}_{\alpha_1}\cdots\mathrm{e}_{\alpha_h}$, 其中$A=\{\alpha_1,\cdots,\alpha_h\}\subseteq\{1,\cdots,2m+1\}$, 且$1\le\alpha_1<\alpha_2<\cdots<\alpha_h\le 2m+1$. 当$A=\varnothing$时, $\mathrm{e}_A=\mathrm{e}_0=1$. $\mathbf{Cl}_{2m+1,0}(\mathbf{R})$中任一元素$a$都可表示成$a=\sum\limits_A a_A\mathrm{e}_A$, $a_A\in\mathbf{R}$. 在$\mathbf{Cl}_{2m+1,0}(\mathbf{R})$中的乘法满足

$$\begin{cases}\mathrm{e}_0^2=1,\ \mathrm{e}_0\mathrm{e}_i=\mathrm{e}_i\mathrm{e}_0=\mathrm{e}_i,\ i=1,2,\ldots,2m+1,\\ \mathrm{e}_i^2=-1,\ i=1,2,\ldots,2m+1,\\ \mathrm{e}_i\mathrm{e}_j=-\mathrm{e}_j\mathrm{e}_i,\ 1\le i,j\le 2m+1, i\ne j,\\ \mathrm{e}_{\alpha_1}\mathrm{e}_{\alpha_2}\cdots\mathrm{e}_{\alpha_h}=\mathrm{e}_{\alpha_1\alpha_2\cdots\alpha_h},\ 1\le\alpha_1<\alpha_2<\cdots<\alpha_h\le 2m+1.\end{cases}\tag{2.1}$$

$$\begin{cases}\bar{\mathrm{e}_i}=-\mathrm{e}_i,\ i=1,2,\ldots,2m+1,\\ \overline{\lambda\mu}=\bar{\mu}\bar{\lambda}, \lambda,\mu\in\mathbf{Cl}_{2m+1,0}(\mathbf{R}).\end{cases}\tag{2.2}$$

$\mathbf{Cl}_{2m+1,0}(\mathbf{R})$中元素$a$的范数为$|a|=\sqrt{\sum\limits_A |a_A|^2}$. 且$\forall a,b\in\mathbf{Cl}_{2m+1,0}(\mathbf{R})$, 有

$$|a+b|\le|a|+|b|,\quad |ab|\le J|a||b|,$$

其中J为一正常数.

若$\mathbf{Cl}_{2m+1,0}(\mathbf{R})$中的元素$x=\sum\limits_{i=0}^{2m+1}\mathrm{e}_ix_i$, 则$x$的共轭元素为$\overline{x}=x_0-\sum\limits_{i=1}^{2m+1}\mathrm{e}_ix_i$, 且有

$$x\overline{x}=\overline{x}x=|x|^2,\ 其中|x|^2=\sum_{i=0}^{2m+1}x_i^2.$$

设

$$F_\Omega^{(r)}=\left\{f|f:\Omega\to\mathbf{Cl}_{2m+1,0}(\mathbf{R}),f(x)=\sum_A f_A(x)\mathrm{e}_A,f_A(x)\in C^r(\Omega,\mathbf{R})\right\},$$

其中$C^r(\Omega,\mathbf{R})$为r次连续可导函数集, Ω为$\mathbf{R}^{2m+2}$中一非空区域.

设$f\in F_\Omega^{(r)}$, Dirac算子定义为

$$D:F_\Omega^{(r)}\to F_\Omega^{(r-1)}$$

$$Df=\sum_{i=0}^{2m+1}\mathrm{e}_i\frac{\partial f}{\partial x_i}=\sum_{i,A}\mathrm{e}_i\mathrm{e}_A\frac{\partial f_A}{\partial x_i},\ fD=\sum_{i=0}^{2m+1}\frac{\partial f}{\partial x_i}\mathrm{e}_i=\sum_{i,A}\mathrm{e}_A\mathrm{e}_i\frac{\partial f_A}{\partial x_i},$$

$$\overline{D}f=\frac{\partial f}{\partial x_0}-\sum_{i=1}^{2m+1}\mathrm{e}_i\frac{\partial f}{\partial x_i},\ f\overline{D}=\frac{\partial f}{\partial x_0}-\sum_{i=1}^{2m+1}\frac{\partial f}{\partial x_i}\mathrm{e}_i.$$

并且有性质

$$D\overline{D}=\overline{D}D=\Delta=\sum_{i=0}^{2m+1}\frac{\partial^2}{\partial x_i^2}.$$

定义2.1 设$f\in F_\Omega^{(1)}$, 其中$\Omega\subset\mathbf{R}^{2m+2}$为非空区域, 若对任意的$x\in\Omega$, 有$Df(x)=0$ $(f(x)D=0)$, 则称f为Ω 上的左(右)正则函数, 通常简称左正则函数为正则函数.

定义2.2 设$f\in F_\Omega^{(2m+1)}$, 其中$\Omega\subset\mathbf{R}^{2m+2}$为非空区域, 若对任意的$x\in\Omega$, 有$D\Delta^m f(x)=0$ $(f(x)D\Delta^m=0)$, 则称f为Ω上的左(右)全纯Cliffordian函数, 通常简称左全纯Cliffordian函数为全纯Cliffordian 函数.

注2.1 从定义容易看出, 全纯Cliffordian函数是正则函数的推广, 正则函数均为全纯Cliffordian函数, 但反之不真. 例如: $f(x)=x$为全纯Cliffordian函数, 而不是正则函数.

设Ω如上所述, $\partial\Omega$为可微, 定向, 紧致的Liapunov曲面. 若$f(x):\partial\Omega\to\mathbf{Cl}_{2m+1,0}(\mathbf{R})$ 满足

$$|f(x_1)-f(x_2)|\le C|x_1-x_2|^\beta,\ (0<\beta<1)$$

对任意的$x_1,x_2\in\partial\Omega$成立, 其中C为正常数, 则称函数$f(x)$在$\partial\Omega$上是Hölder连续的, 记$\partial\Omega$上指标为β的Hölder连续函数全体为$H_{\partial\Omega}^\beta$.

以下为$\mathbf{Cl}_{2m+1,0}(\mathbf{R})$值的$2m+1$次微分形式:

$$\mathrm{d}\widehat{y_i}=\mathrm{d}y_0\wedge\cdots\wedge\mathrm{d}y_{i-1}\wedge\mathrm{d}y_{i+1}\cdots\wedge\mathrm{d}y_{2m+1},$$

$$\mathrm{d}\sigma=\sum_{i=0}^{2m+1}(-1)^{i-1}\mathrm{e}_i\mathrm{d}\widehat{y_i},\ \overrightarrow{n}=\sum_{i=0}^{2m+1}\mathrm{e}_in_i, i=0,1,\cdots,2m+1,$$

其中n_i为单位外法向量$\overrightarrow{n}$的第i个分量. 则有$\mathrm{d}\sigma=\overrightarrow{n}\mathrm{d}S$, $\mathrm{d}S$为面积微元, $\mathrm{d}y$为体积微元, 且$\mathrm{d}y=\mathrm{d}y_0\wedge\cdots\wedge\mathrm{d}y_{2m+1}$.

称

$$N(x)=\varepsilon_m x^{-1} \tag{2.3}$$

为全纯Cliffordian函数的Cauchy核函数, 其中

$$x\in \mathbf{R}^{2m+2}/\{0\},\ \varepsilon_m=(-1)^m\frac{m+1}{2^{2m+1}m!\pi^{m+1}}.$$

并且有

$$\Delta^m N(x)=\frac{1}{\omega_{2m+2}}\frac{\overline{x}}{|x|^{2m+2}}=E(x), \tag{2.4}$$

其中$\omega_{2m+2}=\dfrac{2\pi^{m+1}}{\Gamma(m+1)}$是单位球表面积, $E(x)$为正则函数的Cauchy核.

以下为主要引理.

引理2.1[19] (Hile引理) 设$x,t\in\mathbf{R}^n$, $n(\geq 2), m(\geq 0)$为整数，则有

$$\left|\frac{x}{|x|^{m+2}}-\frac{t}{|t|^{m+2}}\right|\leq\frac{|x-t|P_m(x,t)}{|x|^{m+1}|t|^{m+1}},$$

其中

$$P_m(x,t)=\begin{cases}\sum\limits_{k=0}^{m}|x|^{m-k}|t|^k, & m>0,\\ 1, & m=0.\end{cases} \tag{2.5}$$

引理2.2[19] 设Ω为$\mathbf{R}^{2m+2}$的非空有界区域, $\partial\Omega$为可微, 定向, 紧致的Liapunov曲面. $f\in F_{\Omega}^{(2m+1)}$, 若f是Ω上的全纯Cliffordian函数, 且$x\in\Omega$, 则

$$\begin{aligned}f(x) \;=\; &\int_{\partial\Omega}(\Delta^m N(y-x))\,\mathrm{d}\sigma_y f(y)-\sum_{k=1}^{m}\int_{\partial\Omega}\left(\frac{\partial}{\partial n}\Delta^{m-k}N(y-x)\right)D\Delta^{k-1}f(y)\mathrm{d}S_y\\ &+\sum_{k=1}^{m}\int_{\partial\Omega}\left(\Delta^{m-k}N(y-x)\right)\frac{\partial}{\partial n}D\Delta^{k-1}f(y)\mathrm{d}S_y.\end{aligned} \tag{2.6}$$

引理2.3[23] 设Ω如上所述, $f\in F_{\Omega}^{(2m+1)}$, 且对任意的$x\in\Omega$, 若f满足(2.6)式, 则$f(x)$为Ω上的全纯Cliffordian函数.

引理2.4[24] 设$\Omega,\partial\Omega$如上所述, $f\in F_{\Omega}^{(2m+1)}, x\in\partial\Omega$, 则有

$$\int_{\partial\Omega}(\Delta^m N(y-x))\,\mathrm{d}\sigma_y f(y)=\frac{1}{2}f(x)+\int_{\partial\Omega}(\Delta^m N(y-x))\,\mathrm{d}\sigma_y[f(y)-f(x)].$$

上式左端的积分为Cauchy主值意义下的积分, 右端的积分为普通意义下的广义积分.

注2.2 当$f=1$时, 有$\displaystyle\int_{\partial\Omega}(\Delta^m N(y-x))\,\mathrm{d}\sigma_y=\frac{1}{2}$.

引理2.5[25] 设$\Omega,\partial\Omega$如上所述, $\Omega^+=\Omega, \Omega^-=\mathbf{R}^{2m+2}/\overline{\Omega}$, $f\in H_{\partial\Omega}^{\alpha}$, $0<\alpha<1$, 且$x_0\in\partial\Omega$, 其中$\Psi(x)=\displaystyle\int_{\partial\Omega}(\Delta^m N(y-x))\,\mathrm{d}\sigma_y f(y)$. $\Psi^+(x_0)$, $\Psi^-(x_0)$分别为$\Psi(x)$当$x\to x_0$时, $x\in\Omega^+, x\in\Omega^-$的极限, 则有

$$\begin{cases}\Psi^+(x_0)=\dfrac{1}{2}f(x_0)+\Psi(x_0),\\ \Psi^-(x_0)=-\dfrac{1}{2}f(x_0)+\Psi(x_0).\end{cases} \tag{2.7}$$

或

$$\begin{cases} \Psi^+(x_0)+\Psi^-(x_0)=2\Psi(x_0), \\ \Psi^+(x_0)-\Psi^-(x_0)=f(x_0). \end{cases} \tag{2.7}$$

引理2.6[23] (有界域上全纯Cliffordian函数的Plemelj公式)设$\Omega,\partial\Omega$如上所述, $f\in F_{\partial\Omega}^{(2m+1)}$, $x_0\in\partial\Omega$, $\Phi^+(x_0)$, $\Phi^-(x_0)$分别表示当$x\to x_0, x\in\Omega^+, x\in\Omega^-$时$\Phi(x)$的边界值, 其中

$$\begin{aligned}\Phi(x) &= \int_{\partial\Omega}(\Delta^m N(y-x))\,\mathrm{d}\sigma_y f(y)-\sum_{k=1}^{m}\int_{\partial\Omega}\left(\frac{\partial}{\partial n}\Delta^{m-k}N(y-x)\right)D\Delta^{k-1}f(y)\mathrm{d}S_y \\ &\quad +\sum_{k=1}^{m}\int_{\partial\Omega}\left(\Delta^{m-k}N(y-x)\right)\frac{\partial}{\partial n}D\Delta^{k-1}f(y)\mathrm{d}S_y.\end{aligned}$$

则有

$$\begin{cases} \Phi^+(x_0)=\dfrac{1}{2}f(x_0)+\Phi(x_0), \\ \Phi^-(x_0)=-\dfrac{1}{2}f(x_0)+\Phi(x_0). \end{cases} \tag{2.8}$$

或

$$\begin{cases} \Phi^+(x_0)+\Phi^-(x_0)=2\Phi(x_0), \\ \Phi^+(x_0)-\Phi^-(x_0)=f(x_0). \end{cases} \tag{2.8}$$

以下设$\mathbf{R}^{2m+2}$中的无界区域U具有Lipschitz连续的边界∂U, 且U的余集中包含非空开集, $0\notin\partial U$, 且$\forall t\in\partial U$, 0不在t处的切平面上. 设$E_*(y,x)=E(y-x)-E(y-x_0), x\in\mathbf{R}^{2m+2}/\partial U, x_0\in\mathbf{R}^{2m+2}/\overline{U}$, x_0为固定点.

引理2.7[26] 设$U,\partial U$如上所述, 设$\psi(x):\partial U\to\mathbf{Cl}_{2m+1,0}(\mathbf{R})$ 有界, 且设$\psi(x)\in H_{\partial U}^{\beta}$, $0<\beta<1$, 则对任意的$x\in\partial U$, 积分$\Psi(x)=\displaystyle\int_{\partial U}E_*(y,x)\,\mathrm{d}\sigma_y\psi(y)$在Cauchy主值下是有意义的.

引理2.8[26] 设$U,\partial U$如上所述, 设$\psi(x):\partial U\to\mathbf{Cl}_{2m+1,0}(\mathbf{R})$, 且$\psi(x)\in H_{\partial U}^{\beta}, 0<\beta<1$, $\Psi(x)=\displaystyle\int_{\partial U}E_*(y,x)\,\mathrm{d}\sigma_y\psi(y)$, 则对任意的$t\in\partial U$, 有

$$\Psi^{\pm}(t)=\pm\frac{1}{2}\psi(t)+\int_{\partial U}E_*(y,t)\mathrm{d}\sigma_y\psi(y).$$

引理2.9 若对任意的$y,x,x_0\in\mathbf{R}^{2m+2}$, 存在$M_1>0$, 使得

$$\frac{|y-x|}{|y-x_0|}\le M_1,\ \frac{|y-x_0|}{|y-x|}\le M_1,$$

则对$k=0,1,\cdots,m$, 存在$M>0$, 使得

$$\left|\frac{\partial}{\partial n}\Delta^{m-k}N(y-x)-\frac{\partial}{\partial n}\Delta^{m-k}N(y-x_0)\right|\le\frac{M|x-x_0|}{|y-x_0|^{2(m-k)+3}}.$$

证 因为$\Delta^m N(y-x)=\dfrac{1}{\omega_{2m+2}}\dfrac{\overline{y-x}}{|y-x|^{2m+2}}$, 且

$$\begin{aligned}&\frac{\partial}{\partial n}\Delta^{m-k}N(y-x)\\ &=\frac{1}{\omega_{2(m-k)+2}}\frac{\partial}{\partial n}\frac{\overline{y-x}}{|y-x|^{2(m-k)+2}}=\frac{1}{\omega_{2(m-k)+2}}\frac{\partial}{\partial y_0}\frac{\overline{y-x}}{|y-x|^{2(m-k)+2}}\cos(y_0,\overrightarrow{n})+\cdots\\ &\quad+\frac{1}{\omega_{2(m-k)+2}}\frac{\partial}{\partial y_{2m+1}}\frac{\overline{y-x}}{|y-x|^{2(m-k)+2}}\cos(y_{2m+1},\overrightarrow{n}),\end{aligned}$$

又由
$$\frac{\partial}{\partial y_i}\frac{\overline{y-x}}{|y-x|^{2(m-k)+2}}=\frac{\overline{\mathrm{e}_i}}{|y-x|^{2(m-k)+2}}-\frac{\overline{y-x}(y_i-x_i)}{|y-x|^{2(m-k)+4}}[2(m-k)+2],$$
所以
$$\frac{\partial}{\partial n}\Delta^{m-k}N(y-x)$$
$$=\frac{1}{\omega_{2(m-k)+2}}\left[\sum_{i=0}^{2m+1}\left(\frac{\overline{\mathrm{e}_i}}{|y-x|^{2(m-k)+2}}-\frac{\overline{y-x}(y_i-x_i)}{|y-x|^{2(m-k)+4}}[2(m-k)+2]\right)\cos(y_i,\overrightarrow{n})\right].$$
故原不等式
$$\left|\frac{\partial}{\partial n}\Delta^{m-k}N(y-x)-\frac{\partial}{\partial n}\Delta^{m-k}N(y-x_0)\right|$$
$$=\frac{1}{\omega_{2(m-k)+2}}\left|\sum_{i=0}^{2m+1}\left[\left(\frac{\overline{\mathrm{e}_i}}{|y-x|^{2(m-k)+2}}-\frac{\overline{\mathrm{e}_i}}{|y-x_0|^{2(m-k)+2}}\right)\cos(y_i,\overrightarrow{n})\right.\right.$$
$$\left.\left.+\ [2(m-k)+2]\left(\frac{\overline{y-x_0}(y_i-x_{0i})}{|y-x_0|^{2(m-k)+4}}-\frac{\overline{y-x}(y_i-x_i)}{|y-x|^{2(m-k)+4}}\right)\cos(y_i,\overrightarrow{n})\right]\right|$$
$$\leq\frac{1}{\omega_{2(m-k)+2}}\sum_{i=0}^{2m+1}\left|\frac{1}{|y-x|^{2(m-k)+2}}-\frac{1}{|y-x_0|^{2(m-k)+2}}\right|$$
$$+\frac{2(m-k)+2}{\omega_{2(m-k)+2}}\sum_{i=0}^{2m+1}\left|\frac{\overline{y-x}(y_i-x_i)}{|y-x|^{2(m-k)+4}}-\frac{\overline{y-x_0}(y_i-x_{0i})}{|y-x_0|^{2(m-k)+4}}\right|,$$
通分变形得
$$\left|\frac{\partial}{\partial n}\Delta^{m-k}N(y-x)-\frac{\partial}{\partial n}\Delta^{m-k}N(y-x_0)\right|$$
$$\leq\frac{2m+2}{\omega_{2(m-k)+2}}\frac{|x-x_0|\left||y-x_0|^{2(m-k)+1}+|y-x_0|^{2(m-k)}|y-x|+\cdots+|y-x|^{2(m-k)+1}\right|}{|y-x|^{2(m-k)+2}|y-x_0|^{2(m-k)+2}}$$
$$+\frac{[2(m-k)+2](2m+2)}{\omega_{2(m-k)+2}}\left|\frac{\overline{y-x}}{|y-x|^{2(m-k)+4}}-\frac{\overline{y-x_0}}{|y-x_0|^{2(m-k)+4}}\right||y-x|$$
$$+\frac{[2(m-k)+2](2m+2)}{\omega_{2(m-k)+2}}\left|\frac{\overline{y-x_0}}{|y-x_0|^{2(m-k)+4}}\right||x-x_0|$$
$$=\frac{2m+2}{\omega_{2(m-k)+2}}\frac{|x-x_0|\left|1+\frac{|y-x|}{|y-x_0|}+\cdots+\frac{|y-x|^{2(m-k)+1}}{|y-x_0|^{2(m-k)+1}}\right|}{|y-x_0||y-x|^{2(m-k)+2}}$$
$$+\frac{[2(m-k)+2](2m+2)}{\omega_{2(m-k)+2}}\left|\frac{\overline{y-x}}{|y-x|^{2(m-k)+4}}-\frac{\overline{y-x_0}}{|y-x_0|^{2(m-k)+4}}\right||y-x|$$

$$+\frac{[2(m-k)+2](2m+2)}{\omega_{2(m-k)+2}}\frac{1}{|y-x_0|^{2(m-k)+3}}|x-x_0|,$$

由题设条件$\frac{|y-x|}{|y-x_0|}\le M_1$, 以及Hile引理2.1, 故上式

$$\begin{aligned}&\left|\frac{\partial}{\partial n}\Delta^{m-k}N(y-x)-\frac{\partial}{\partial n}\Delta^{m-k}N(y-x_0)\right|\\ \le\ &\frac{J_1|x-x_0|}{|y-x_0||y-x|^{2(m-k)+2}}\\ &+\frac{J_2|x-x_0|\sum\limits_{j=0}^{2(m-k)+2}|y-x_0|^{2(m-k)+2-j}|y-x|^j}{|y-x|^{2(m-k)+3}|y-x_0|^{2(m-k)+3}}|y-x|+\frac{J_3|x-x_0|}{|y-x_0|^{2(m-k)+3}}\\ =\ &\frac{J_1|x-x_0|}{|y-x_0||y-x|^{2(m-k)+2}}+\frac{J_2|x-x_0|\sum\limits_{j=0}^{2(m-k)+2}\frac{|y-x|^j}{|y-x_0|^j}}{|y-x|^{2(m-k)+2}|y-x_0|}+\frac{J_3|x-x_0|}{|y-x_0|^{2(m-k)+3}}\\ \le\ &\frac{J_1|x-x_0|}{|y-x_0||y-x|^{2(m-k)+2}}+\frac{J_4|x-x_0|}{|y-x|^{2(m-k)+2}|y-x_0|}+\frac{J_3|x-x_0|}{|y-x_0|^{2(m-k)+3}},\end{aligned}$$

又由题设$\frac{1}{|y-x|}\le\frac{M_1}{|y-x_0|}$, 因此得

$$\left|\frac{\partial}{\partial n}\Delta^{m-k}N(y-x)-\frac{\partial}{\partial n}\Delta^{m-k}N(y-x_0)\right|\le\frac{M|x-x_0|}{|y-x_0|^{2(m-k)+3}}.$$

注2.3　上述引理由点x,x_0的对称性可知有:

$$\left|\frac{\partial}{\partial n}\Delta^{m-k}N(y-x)-\frac{\partial}{\partial n}\Delta^{m-k}N(y-x_0)\right|\le\frac{M|x-x_0|}{|y-x|^{2(m-k)+3}}.$$

以下设$U,\partial U$如上所述, $N_0\in\partial\Omega$为任一固定点, 以N_0为原点, N_0处的外法向量为正的ξ_{2m+1}轴的正方向建立极坐标系, 则$\partial\Omega$能写成$\xi_{2m+1}=\xi_{2m+1}(\xi_0,\cdots,\xi_{2m})$的形式, 其中$\xi_{2m+1}$关于$\xi_i(i=0,\cdots,2m)$具有一阶偏微分.

令$d>0$充分小, 使其满足$bd^\alpha\le1$, 而且对任意的点$N\in\partial\Omega$, 令$\theta_0=\theta(N_0,N)$表示$\partial\Omega$上过N_0与N处的外法向量的夹角, 且以$\rho=|N_0N|$表示点N_0与点N的距离($\rho<d$), 这样可得到

$$\cos\theta_0\ge1-\frac{1}{2}\theta_0^2\ge1-\frac{1}{2}b^2\rho^{2\alpha}\ge0,$$

故有$\frac{1}{\cos\theta_0}\le\frac{1}{1-\frac{1}{2}b^2\rho^{2\alpha}}\le1+b^2\rho^{2\alpha}\le2$, 因而$\cos\theta_0\ge\frac{1}{2}$.

下面引入N_0点局部广义球坐标变换

$$\begin{cases}\xi_{2m}=\rho_0\cos\varphi_0\cos\varphi_1\cdots\cos\varphi_{2m-2}\cos\varphi_{2m-1},\\ \xi_{2m-1}=\rho_0\cos\varphi_0\cos\varphi_1\cdots\cos\varphi_{2m-2}\sin\varphi_{2m-1},\\ \cdots,\\ \xi_1=\rho_0\cos\varphi_0\sin\varphi_1,\\ \xi_0=\rho_0\sin\varphi_0.\end{cases}$$

其中ρ_0是ρ在N_0处切平面的投影, φ_i满足条件

$$|\varphi_i|\le\frac{\pi}{2},i=0,1,\cdots,2m-2,0\le\varphi_{2m-1}<2\pi,$$

由文献[20]有

$$\cos(\overrightarrow{n},\xi_{2m+1})\geq\frac{1}{2},\ \left|\frac{D(\xi_0,\xi_1,\cdots,\xi_{2m})}{D(\rho_0,\varphi_0,\cdots,\varphi_{2m-1})}\right|\leq\rho_0^{2m},$$

其中$\overrightarrow{n}$是N点处的法向量. 并且可以得到

$$\begin{aligned}|\mathrm{d}\sigma_x| &= |\mathrm{d}S_x|=\left|\frac{1}{\cos(\overrightarrow{n},\xi_{2m+1})}\mathrm{d}\xi_0\mathrm{d}\xi_1\cdots\mathrm{d}\xi_{2m}\right|\\ &\leq 2\left|\frac{D(\xi_0,\xi_1,\cdots,\xi_{2m})}{D(\rho_0,\varphi_0,\cdots,\varphi_{2m-1})}\right||\mathrm{d}\rho_0\mathrm{d}\varphi_0\mathrm{d}\varphi_1\cdots\mathrm{d}\varphi_{2m-1}|\leq C_0\rho_0^{2m}\mathrm{d}\rho_0\leq C_0\rho^{2m}\mathrm{d}\rho,\end{aligned}$$

其中$C_0>0$为常数.

上式为本文进行一些积分估值提供了关键性的依据.

§3 全纯Cliffordian函数空间的性质

由文献[3], 设$A=\{h_1,h_2,\cdots,h_k\}$, $0\leq h_1<h_2<\ldots<h_k\leq 2m+1$, 其中$h_i\in\mathbf{N},(i=1,2,\cdots,k)$ 且设$M\in\mathbf{N}$, 称

$$\overline{MA}=\begin{cases}A/\{M\}, & M\in A,\\ \\ \{g_1,g_2,\cdots,g_{k+1}\}, & M\overline{\in}A\end{cases}\tag{3.1}$$

为排列MA的第一类拟置换, 其中$g_i\in A\bigcup\{M\},0\leq g_1<g_2<\ldots<g_{k+1}$. 且规定$\overline{MM}=0,\overline{M0}=M$, 又称$\delta_{\overline{MA}}=(-1)^p$为第一类拟置换的符号, 其中$p$表示$A$中所有不为0的元素个数.

引理3.1[3] 设M,A如上所述, 则第一类拟置换$\overline{MA}$具有如下性质:

(1) 若$\overline{MA}=B$, 则$\overline{MB}=A$; (2)$\overline{MA}=\overline{AM}$;

(3) 设$M=0$, 那么$\delta_{\overline{MA}}=1$; (4) 若$\overline{MA}=B$ 且$M\neq 0$, 则$\delta_{\overline{MA}}=-\delta_{\overline{MB}}$;

(5)$e_Me_A=\delta_{\overline{MA}}e_{\overline{MA}}$.

定理3.1 $D\varDelta^m f(x)=0$的解集合$H(B)$, 即全纯Cliffordian函数空间构成一右$\mathbf{Cl}_{2m+1,0}(\mathbf{R})$模.

证 对任意的$f_i(x)\in(H(B))(i=1,2)$, 有$D\varDelta^m f_i(x)=0$, 及任意的$a,b\in\mathbf{Cl}_{2m+1,0}(\mathbf{R})$, $a=\sum\limits_A a_Ae_A$,$b=\sum\limits_B b_Be_B$, 其中$a_A,b_B\in\mathbf{R}$, 因为$D\varDelta^m(f_i\cdot e_A)=D\varDelta^m(f_i)\cdot e_A=0$, 所以

$$D\varDelta^m(f_i\cdot a)=D\varDelta^m(f_i\cdot\sum_A a_Ae_A)=\sum_A a_AD\varDelta^m(f_i\cdot e_A)=0,$$

故$f_i\cdot a\in(H(B))(i=1,2)$, 又由

$$D\varDelta^m\left(f_i\cdot(a+b)\right)=D\varDelta^m(f_i\cdot a)+D\varDelta^m(f_i\cdot b);$$

$$(f_1+f_2)\cdot a=f_1\cdot a+f_2\cdot a;f_i\cdot(ab)=(f_i\cdot a)\cdot b;f_i\cdot e_0=f_i,\ (i=1,2).$$

因而由右模定义定理得证.

注3.1 上述定理中$D\varDelta^m f(x)=0$的解集$H(B)$不构成一左$\mathbf{Cl}_{2m+1,0}(\mathbf{R})$模, 这是由于Clifford代数中元素的不可交换性导致的.

定理3.2 (全纯Cliffordian函数的充要条件) $D\varDelta^m f(x)=0$的充要条件是

$$\frac{\partial(\varDelta^m f_A)}{\partial x_0}=\sum_{i=1}^{2m+1}\delta_{\overline{iA}}\frac{\partial(\varDelta^m f_{\overline{iA}})}{\partial x_i}.$$

证 因为$D\Delta^m f=\sum\limits_{i=0}^{2m+1}\mathrm{e}_i\dfrac{\partial(\Delta^m f)}{\partial x_i}=\sum\limits_{i,A}\mathrm{e}_i\mathrm{e}_A\dfrac{\partial(\Delta^m f_A)}{\partial x_i}$, 由引理3.1知$\mathrm{e}_i\mathrm{e}_A=\delta_{\overline{iA}}\mathrm{e}_{\overline{iA}}$, $(1\leq i\leq 2m+1)$, 且当$i=0$时, $\delta_{\overline{iA}}=1$, 当$1\leq i\leq 2m+1$时, 若记$\overline{iA}$为B, 则$A=\overline{iB}$, $\delta_{\overline{iA}}=-\delta_{\overline{iB}}$, 故上式

$$\begin{aligned}D\Delta^m f &= \sum_{i,A}\delta_{\overline{iA}}\mathrm{e}_{\overline{iA}}\frac{\partial(\Delta^m f_A)}{\partial x_i}=\sum_A \mathrm{e}_A\frac{\partial(\Delta^m f_A)}{\partial x_0}+\sum_A\sum_{i=1}^{2m+1}\delta_{\overline{iA}}\mathrm{e}_{\overline{iA}}\frac{\partial(\Delta^m f_A)}{\partial x_i}\\ &= \sum_A \mathrm{e}_A\frac{\partial(\Delta^m f_A)}{\partial x_0}-\sum_B\sum_{i=1}^{2m+1}\delta_{\overline{iB}}\mathrm{e}_B\frac{\partial(\Delta^m f_{\overline{iB}})}{\partial x_i},\end{aligned}$$

又因为$\overline{iA}=B$, $\overline{iB}=A(1\leq i\leq 2m+1)$, 易知所有$A$的集合与所有$B$的集合相同, 故可将上式中第二项中的$B$换成$A$有

$$D\Delta^m f = \sum_A \mathrm{e}_A\frac{\partial(\Delta^m f_A)}{\partial x_0}-\sum_A\sum_{i=1}^{2m+1}\delta_{\overline{iA}}\mathrm{e}_A\frac{\partial(\Delta^m f_{\overline{iA}})}{\partial x_i}.$$

由$D\Delta^m f=0$, 故有$\dfrac{\partial(\Delta^m f_A)}{\partial x_0}=\sum\limits_{i=1}^{2m+1}\delta_{\overline{iA}}\dfrac{\partial(\Delta^m f_{\overline{iA}})}{\partial x_i}$. 从而定理得证.

定理3.3 (全纯Cliffordian函数的充要条件) $D\Delta^m f(x)=0$的充要条件是

$$\sum_{i=0}^{2m+1}\delta_{\overline{iB}}\frac{\partial(\Delta^m f_B)}{\partial x_i}=0.$$

证 由定理3.2以及$\delta_{\overline{iA}}=-\delta_{\overline{iB}}$,$(i\neq 0$时, $\overline{iA}=B,\overline{iB}=A)$容易证得定理结论.

定理3.4 (有界域上全纯Cliffordian函数的开拓定理) 设$\Omega,\partial\Omega$如上所述, $f\in F^{(2m+1)}_{\mathbf{R}^{2m+2}}$, 若对任意的$x\in\mathbf{R}^{2m+2}/\partial\Omega$, 有$D\Delta^m f(x)=0$, 且当$x\in\partial\Omega$时, 有$f^+(x)=f^-(x)$, 且$f^+(x)$, $f^-(x)\in F^{(2m+1)}_{\partial\Omega}$, 则$D\Delta^m f(x)=0$,$x\in\mathbf{R}^{2m+2}$.

证 首先定义$f(x)=f^+(x)=f^-(x)$, $x\in\partial\Omega$. 对任意给定的$x_0\in\partial\Omega$, 取一常数$\delta>0$, 可以做一个以x_0为心, δ为半径的超球B_*, $\partial\Omega$可将B_*分为两部分, 记为B_{*1}和B_{*2}, 其中B_{*1}在$\partial\Omega$内部, B_{*2}在$\partial\Omega$外部, 且$\partial\Omega$被B_*分成两部分, 分别记球内, 球外的部分为$\partial\Omega_1,\partial\Omega_2$, 则$x_0\in\partial\Omega_1$. 由引理2.2, 当$x_1\in\Omega$时,

$$\begin{aligned}f(x_1) &= \int_{\partial\Omega}(\Delta^m N(y-x_1))\,\mathrm{d}\sigma_y f(y)-\sum_{k=1}^m\int_{\partial\Omega}\left(\frac{\partial}{\partial n}\Delta^{m-k}N(y-x_1)\right)D\Delta^{k-1}f(y)\mathrm{d}S_y\\ &\quad+\sum_{k=1}^m\int_{\partial\Omega}\left(\Delta^{m-k}N(y-x_1)\right)\frac{\partial}{\partial n}D\Delta^{k-1}f(y)\mathrm{d}S_y.\end{aligned}$$

当$x_2\in B_*/\overline{\Omega}$时,

$$\begin{aligned}f(x_2) &= \int_{\partial B_{*2}}(\Delta^m N(y-x_2))\,\mathrm{d}\sigma_y f(y)-\sum_{k=1}^m\int_{\partial B_{*2}}\left(\frac{\partial}{\partial n}\Delta^{m-k}N(y-x_2)\right)D\Delta^{k-1}f(y)\mathrm{d}S_y\\ &\quad+\sum_{k=1}^m\int_{\partial B_{*2}}\left(\Delta^{m-k}N(y-x_2)\right)\frac{\partial}{\partial n}D\Delta^{k-1}f(y)\mathrm{d}S_y.\end{aligned}$$

由引理2.2, 引理2.4, 相对于区域Ω, 有

$$f(x_0)=f^+(x_0)$$

$$=\frac{1}{2}f(x_0)+\int_{\partial\Omega}\left(\Delta^m N(y-x_0)\right)\mathrm{d}\sigma_y f(y)-\sum_{k=1}^{m}\int_{\partial\Omega}\left(\frac{\partial}{\partial n}\Delta^{m-k}N(y-x_0)\right)D\Delta^{k-1}f(y)\mathrm{d}S_y$$

$$+\sum_{k=1}^{m}\int_{\partial\Omega}\left(\Delta^{m-k}N(y-x_0)\right)\frac{\partial}{\partial n}D\Delta^{k-1}f(y)\mathrm{d}S_y.$$

相对于区域B_{*2}, 有

$$f(x_0)=f^-(x_0)=\widetilde{f^+(x_0)}=\frac{1}{2}f(x_0)+\int_{\partial B_{*2}}\left(\Delta^m N(y-x_0)\right)\mathrm{d}\sigma_y f(y)$$

$$-\sum_{k=1}^{m}\int_{\partial B_{*2}}\left(\frac{\partial}{\partial n}\Delta^{m-k}N(y-x_0)\right)D\Delta^{k-1}f(y)\mathrm{d}S_y$$

$$+\sum_{k=1}^{m}\int_{\partial B_{*2}}\left(\Delta^{m-k}N(y-x_0)\right)\frac{\partial}{\partial n}D\Delta^{k-1}f(y)\mathrm{d}S_y,$$

其中$\widetilde{f^+(x_0)}$为$f(x)$当$x_0\in B_{*2}, x\to x_0$时的正边界值. 将上面两式相加得

$$\begin{aligned} f(x_0) \quad = \quad & \int_{\partial(\Omega\bigcup B_{*2})}\left(\Delta^m N(y-x_0)\right)\mathrm{d}\sigma_y f(y) \\ & -\sum_{k=1}^{m}\int_{\partial(\Omega\bigcup B_{*2})}\left(\frac{\partial}{\partial n}\Delta^{m-k}N(y-x_0)\right)D\Delta^{k-1}f(y)\mathrm{d}S_y \\ & +\sum_{k=1}^{m}\int_{\partial(\Omega\bigcup B_{*2})}\left(\Delta^{m-k}N(y-x_0)\right)\frac{\partial}{\partial n}D\Delta^{k-1}f(y)\mathrm{d}S_y. \end{aligned}$$

由引理2.3及$x_0\in\partial\Omega$的任意性得证定理结论.

§4 无界域上全纯Cliffordian函数Cauchy型积分的边值特性

以下讨论了由Cliffordian函数的Cauchy积分公式所引入的一类奇异积分的性质, 得到了在无界域上这类积分的收敛性和Plemelj公式. 由于在无界域上的积分无穷远点是它的一个自然奇点, 所以处理无穷远点的收敛问题是重点也是难点.

假设将要讨论的$\mathbf{R}^{2m+2}$中的无界域U具有Lipschitz连续的边界∂U, 且U的余集中包含非空开集, $0\in\partial U$, 且对任意点$t\in\partial U$, 0不在t处的切平面上.

称积分

$$\begin{aligned} \Phi(x) \quad = \quad & \int_{\partial U}\left(\Delta^m N_*(y,x)\right)\mathrm{d}\sigma_y f(y)-\sum_{k=1}^{m}\int_{\partial U}\left(\frac{\partial}{\partial n}\Delta^{m-k}N_*(y,x)\right)D\Delta^{k-1}f(y)\mathrm{d}S_y \\ & +\sum_{k=1}^{m}\int_{\partial U}\left(\Delta^{m-k}N_*(y,x)\right)\frac{\partial}{\partial n}D\Delta^{k-1}f(y)\mathrm{d}S_y \end{aligned} \qquad (4.1)$$

为全纯Cliffordian函数的Cauchy型积分, 其中$U,\partial U$如上所述. 并且有$N(x)=\varepsilon_m x^{-1}=\varepsilon_m\dfrac{\overline{x}}{|x|^2}$, $N_*(y,x)=N(y-x)-N(y-x_0)$, $x\in\mathbf{R}^{2m+2}/\partial U$, $x_0\in\mathbf{R}^{2m+2}/\overline{U}$,$x_0$为固定点, 且$f\in F_{\partial U}^{(2m+1)}$, 显然此积分是有意义的.

这里引入了修正的核$N_*(y,x)$, 它是在原来Cliffordian函数的Cauchy型核的基础上减去$N(y-x_0)$得到的. 比较原来的核$N(y-x)$, 修正核$N_*(y,x)$在有限部分的积分与$N(y-x)$一样. 因为$N(y-x_0)$为一全纯Cliffordian函数, 但在无穷远点$N_*(y,x)$比$N(y-x)$少了一阶奇性, 从而使积分对于更多的密度函数有意义.

若$x\in\partial U$, 上述定义中的积分为奇异积分, 则构造一个以x为心, δ为半径的小球交∂U于λ_δ, 记

$$\begin{aligned}\Phi_\delta(x) &= \int_{\partial U/\lambda_\delta}(\Delta^m N_*(y,x))\,\mathrm{d}\sigma_y f(y)\\ &-\sum_{k=1}^{m}\int_{\partial U/\lambda_\delta}\left(\frac{\partial}{\partial n}\Delta^{m-k}N_*(y,x)\right)D\Delta^{k-1}f(y)\mathrm{d}S_y\\ &+\sum_{k=1}^{m}\int_{\partial U/\lambda_\delta}\left(\Delta^{m-k}N_*(y,x)\right)\frac{\partial}{\partial n}D\Delta^{k-1}f(y)\mathrm{d}S_y.\end{aligned}\tag{4.2}$$

若$\lim\limits_{\delta\to 0}\Phi_\delta(x)=I$, 则称上述奇异积分$\Phi(x)$在Cauchy主值意义下收敛, I为$\Phi(x)$的Cauchy主值, 记$I=\Phi(x)$.

定理4.1　设$f\in F_{\partial U}^{(2m+1)}$, 若存在正常数$C_1,C_2$及$s\in(0,1)$, 使得

$$|D\Delta^{k-1}f(y)|\le C_1|y|^{1+s-2k},\ \left|\frac{\partial}{\partial n}D\Delta^{k-1}f(y)\right|\le C_2|y|^{s-2k},\ k=1,2,\cdots,m.$$

则对任意的$x\in\partial U$, $\Phi(x)$作为Cauchy主值意义下的积分收敛.

证　因为

$$\begin{aligned}\Phi(x) &= \int_{\partial U}(\Delta^m N_*(y,x))\,\mathrm{d}\sigma_y f(y)\\ &-\sum_{k=1}^{m}\int_{\partial U}\left(\frac{\partial}{\partial n}\Delta^{m-k}N_*(y,x)\right)D\Delta^{k-1}f(y)\mathrm{d}S_y\\ &+\sum_{k=1}^{m}\int_{\partial U}\left(\Delta^{m-k}N_*(y,x)\right)\frac{\partial}{\partial n}D\Delta^{k-1}f(y)\mathrm{d}S_y.\end{aligned}$$

首先证明$\int_{\partial U}(\Delta^m N_*(y,x))\,\mathrm{d}\sigma_y f(y)$ 在Cauchy主值意义下的积分收敛, 事实上由引理2.7及$\Delta^m N_*(y,x)=E_*(y,x)=E(y-x)-E(y-x_0)$容易证明.

其次考虑

$$I_1=\sum_{k=1}^{m}\int_{\partial U}\left(\frac{\partial}{\partial n}\Delta^{m-k}N_*(y,x)\right)D\Delta^{k-1}f(y)\mathrm{d}S_y,$$
$$I_2=\sum_{k=1}^{m}\int_{\partial U}\left(\Delta^{m-k}N_*(y,x)\right)\frac{\partial}{\partial n}D\Delta^{k-1}f(y)\mathrm{d}S_y$$

在Cauchy主值意义下的积分收敛. 构造一个以0为心, R为半径的球, 记其为$D(0,R)$, 其中$R=$

$2\max\{|x|,|x_0|\}$, 并令$U_R=D(0,R)\bigcap\partial U$, 则

$$\begin{aligned}I_1&=\sum_{k=1}^{m}\int_{\partial U}\left(\frac{\partial}{\partial n}\varDelta^{m-k}N_*(y,x)\right)D\varDelta^{k-1}f(y)\mathrm{d}S_y\\&=\sum_{k=1}^{m}\int_{\partial U/U_R}\left(\frac{\partial}{\partial n}\varDelta^{m-k}N_*(y,x)\right)D\varDelta^{k-1}f(y)\mathrm{d}S_y\\&\quad+\sum_{k=1}^{m}\int_{U_R}\left(\frac{\partial}{\partial n}\varDelta^{m-k}N_*(y,x)\right)D\varDelta^{k-1}f(y)\mathrm{d}S_y=I_{11}+I_{12}.\end{aligned}$$

对于I_{11}, 由于

$$|I_{11}|\leq\sum_{k=1}^{m}\int_{\partial U/U_R}J\left|\frac{\partial}{\partial n}\varDelta^{m-k}N_*(y,x)\right||D\varDelta^{k-1}f(y)||\mathrm{d}S_y|,$$

其中

$$\left|\frac{\partial}{\partial n}\varDelta^{m-k}N_*(y,x)\right|=\left|\frac{\partial}{\partial n}\varDelta^{m-k}N(y-x)-\frac{\partial}{\partial n}\varDelta^{m-k}N(y-x_0)\right|,$$

因题设$x_0\in\mathbf{R}^{2m+2}/\overline{U}$, 故$\dfrac{|y-x|}{|y-x_0|}$为有界量以及引理2.1, 引理2.9, 则存在$M>0$, 使得

$$\left|\frac{\partial}{\partial n}\varDelta^{m-k}N_*(y,x)\right|\leq\frac{M|x-x_0|}{|y-x_0|^{2(m-k)+3}}.$$

因为$R=2\max\{|x|,|x_0|\}$, 且$y\in\partial U/U_R$, 所以$|x-x_0|\leq R,|y|\geq R,|y|\geq 2|x|,|y|\geq 2|x_0|$. 依三角不等式得$|y-x|\geq|y|-|x|\geq\dfrac{|y|}{2},|y-x_0|\geq|y|-|x_0|\geq\dfrac{|y|}{2}$, 故存在常数$J_1>0$, 使得上式有

$$\left|\frac{\partial}{\partial n}\varDelta^{m-k}N_*(y,x)\right|\leq\frac{J_1|x-x_0|}{|y|^{2(m-k)+3}}.$$

故

$$\begin{aligned}|I_{11}|&\leq\sum_{k=1}^{m}\int_{\partial U/U_R}J\left|\frac{\partial}{\partial n}\varDelta^{m-k}N_*(y,x)\right|\left|D\varDelta^{k-1}f(y)\right||\mathrm{d}S_y|\\&\leq\sum_{k=1}^{m}\int_{\partial U/U_R}\frac{JJ_1|x-x_0|}{|y|^{2(m-k)+3}}\left|D\varDelta^{k-1}f(y)\right||\mathrm{d}S_y|,\end{aligned}$$

又由假设$|D\varDelta^{k-1}f(y)|\leq C_1|y|^{1+s-2k}$,$|x-x_0|\leq R$, 因此有

$$\begin{aligned}|I_{11}|&\leq\sum_{k=1}^{m}\int_{\partial U/U_R}\frac{JJ_1|x-x_0|}{|y|^{2(m-k)+3}}C_1|y|^{1+s-2k}|\mathrm{d}S_y|\\&\leq\sum_{k=1}^{m}JJ_1RC_1\int_{\partial U/U_R}\frac{1}{|y|^{2m+2-s}}|\mathrm{d}S_y|,\end{aligned}$$

由文献[27]以及$2m+2-s>2m+1$, 所以上式最后一个积分收敛.

下面处理$I_{12}=\displaystyle\sum_{k=1}^{m}\int_{U_R}\left(\frac{\partial}{\partial n}\varDelta^{m-k}N_*(y,x)\right)D\varDelta^{k-1}f(y)\mathrm{d}S_y$.

由于

$$
\begin{aligned}
I_{12} &= \sum_{k=1}^{m}\int_{U_R}\left(\frac{\partial}{\partial n}\varDelta^{m-k}N_*(y,x)\right)D\varDelta^{k-1}f(y)\mathrm{d}S_y\\
&= \sum_{k=1}^{m}\int_{U_R}\left(\frac{\partial}{\partial n}\varDelta^{m-k}N(y-x)\right)D\varDelta^{k-1}f(y)\mathrm{d}S_y\\
&\quad -\sum_{k=1}^{m}\int_{U_R}\left(\frac{\partial}{\partial n}\varDelta^{m-k}N(y-x_0)\right)D\varDelta^{k-1}f(y)\mathrm{d}S_y = I_{121}+I_{122}.
\end{aligned}
$$

由$x_0\in\mathbf{R}^{2m+2}/\overline{U}$可知$I_{122}$是正常意义下的积分. 对于积分$I_{121}$, 因为$U_R$为有界区域, 由文献[23]知其在Cauchy主值意义下收敛, 故$I_{12}=I_{121}+I_{122}$在Cauchy主值意义下收敛. 因而$I_1=I_{11}+I_{12}$在Cauchy主值意义下收敛.

最后类似处理I_1的方法同理可知I_2在Cauchy主值意义下收敛. 综上可知: 对任意的$x\in\partial U,\varPhi(x)$在Cauchy主值意义下收敛.

记$\varPhi(x)=F_0(x)-F(x)+G(x)$, 其中

$$
F_0(x)=\int_{\partial U}\left(\varDelta^m N_*(y,x)\right)\mathrm{d}\sigma_y f(y),
$$

$$
F(x)=\sum_{k=1}^{m}\int_{\partial U}\left(\frac{\partial}{\partial n}\varDelta^{m-k}N_*(y,x)\right)D\varDelta^{k-1}f(y)\mathrm{d}S_y,
$$

$$
G(x)=\sum_{k=1}^{m}\int_{\partial U}\left(\varDelta^{m-k}N_*(y,x)\right)\frac{\partial}{\partial n}D\varDelta^{k-1}f(y)\mathrm{d}S_y.
$$

定理4.2 设$U,\partial U$如上所述, $f\in F_{\partial U}^{(2m+1)}$, $t\in\partial U$, 并且假设存在正常数C_1及$s\in(0,1)$, 使得有$|D\varDelta^{k-1}f(y)|\le C_1|y|^{1+s-2k}$, $k=1,2,\cdots,m$. 则有

$$
\lim_{x\to t,x\in U^{\pm}}F(x)=F(t),
$$

其中$U^+=U$, $U^-=\mathbf{R}^{2m+2}/\overline{U}$.

证 因为

$$
F(x)-F(t)=\sum_{k=1}^{m}\int_{\partial U}\left[\left(\frac{\partial}{\partial n}\varDelta^{m-k}N_*(y,x)\right)-\left(\frac{\partial}{\partial n}\varDelta^{m-k}N_*(y,t)\right)\right]D\varDelta^{k-1}f(y)\mathrm{d}S_y,
$$

其中

$$
\frac{\partial}{\partial n}\varDelta^{m-k}N_*(y,x)-\frac{\partial}{\partial n}\varDelta^{m-k}N_*(y,t)=\left(\frac{\partial}{\partial n}\varDelta^{m-k}N(y-x)\right)-\left(\frac{\partial}{\partial n}\varDelta^{m-k}N(y-t)\right).
$$

故

$$
F(x)-F(t)=\sum_{k=1}^{m}\int_{\partial U}\left[\left(\frac{\partial}{\partial n}\varDelta^{m-k}N(y-x)\right)-\left(\frac{\partial}{\partial n}\varDelta^{m-k}N(y-t)\right)\right]D\varDelta^{k-1}f(y)\mathrm{d}S_y.
$$

对任意的$x\in U^{\pm},t\in\partial U$, 取充分大的$r$, 作以0为心, r为半径的球$D(0,r)$, 使得$x,t\in U^{\pm}\bigcap D(0,r)$, 且$y\in\partial U/D(0,r)$, 并满足$|y|\ge 2\max\{|x|,|t|\}$. 故当$y\in\partial U/D(0,r)$时, 有$|y-x|\ge|y|-|x|\ge\dfrac{|y|}{2}$, 令$U_r=\partial U\bigcap D(0,r)$, 记

$$
F_r^1(x)-F_r^1(t)=\sum_{k=1}^{m}\int_{\partial U/U_r}\left[\left(\frac{\partial}{\partial n}\varDelta^{m-k}N(y-x)\right)-\left(\frac{\partial}{\partial n}\varDelta^{m-k}N(y-t)\right)\right]D\varDelta^{k-1}f(y)\mathrm{d}S_y,
$$

及

$$F_r(x)-F_r(t)=\sum_{k=1}^{m}\int_{U_r}\left[\left(\frac{\partial}{\partial n}\Delta^{m-k}N(y-x)\right)-\left(\frac{\partial}{\partial n}\Delta^{m-k}N(y-t)\right)\right]D\Delta^{k-1}f(y)\mathrm{d}S_y.$$

因为

$$\begin{aligned}&\left|F_r^1(x)-F_r^1(t)\right|\\ \leq&\sum_{k=1}^{m}\int_{\partial U/U_r}J\left|\left(\frac{\partial}{\partial n}\Delta^{m-k}N(y-x)\right)-\left(\frac{\partial}{\partial n}\Delta^{m-k}N(y-t)\right)\right|\left|D\Delta^{k-1}f(y)\right||\mathrm{d}S_y|,\end{aligned}$$

其中由于$(\partial U/U_r)\bigcap(U^{\pm}\bigcap D(0,r))=\varnothing$, 故当$y\in\partial U/U_r,t\in U^{\pm}\bigcap D(0,r)$时, 可知$\dfrac{|y-x|}{|y-t|}$有界, 再由引理2.9及定理4.1证明知存在常数$J_1>0$, 使得

$$\left|\left(\frac{\partial}{\partial n}\Delta^{m-k}N(y-x)\right)-\left(\frac{\partial}{\partial n}\Delta^{m-k}N(y-t)\right)\right|\leq\frac{J_2|x-t|}{|y|^{2(m-k)+3}},$$

又由题设$|D\Delta^{k-1}f(y)|\leq C_1|y|^{1+s-2k}$, 故

$$\begin{aligned}\left|F_r^1(x)-F_r^1(t)\right|&\leq\sum_{k=1}^{m}\int_{\partial U/U_r}\frac{JJ_2|x-t|C_1|y|^{1+s-2k}}{|y|^{2(m-k)+3}}|\mathrm{d}S_y|\\&=\sum_{k=1}^{m}JJ_2C_1|x-t|\int_{\partial U/U_r}\frac{1}{|y|^{2m+2-s}}|\mathrm{d}S_y|.\end{aligned}$$

由文献[27]以及$2m+2-s>2m+1$, 可知上式最后一个积分收敛, 所以对任意的$\varepsilon>0$, 存在$\delta>0$, 当$|x-t|<\delta$时, 有$\left|F_r^1(x)-F_r^1(t)\right|\leq\varepsilon$, 即$\lim\limits_{x\to t,x\in\partial U/U_r}[F_r^1(x)-F_r^1(t)]=0$.

又由文献[23], 对于有界域U_r上的积分

$$F_r(x)-F_r(t)=\sum_{k=1}^{m}\int_{U_r}\left[\left(\frac{\partial}{\partial n}\Delta^{m-k}N(y-x)\right)-\left(\frac{\partial}{\partial n}\Delta^{m-k}N(y-t)\right)\right]D\Delta^{k-1}f(y)\mathrm{d}S_y,$$

有$\lim\limits_{x\to t,x\in U_r}[F_r(x)-F_r(t)]=0$. 综上即得$\lim\limits_{x\to t,x\in U^{\pm}}F(x)=F(t)$.

定理4.3 设$U,\partial U,U^+,U^-$如上所述, $f\in F_{\partial U}^{(2m+1)}$,$t\in\partial U$, 并且假设存在正常数$C_2$及$s\in(0,1)$, 使得有$\left|\dfrac{\partial}{\partial n}D\Delta^{k-1}f(y)\right|\leq C_2|y|^{s-2k}$, $k=1,2,\cdots,m$. 则有$\lim\limits_{x\to t,\ x\in U^{\pm}}G(x)=G(t)$.

证 类似定理4.2的证明容易证得定理结论.

定理4.4 (无界域上全纯Cliffordian函数的Plemelj公式)设$U,\partial U,U^+,U^-$如上所述, $f\in F_{\partial U}^{(2m+1)}$,$t\in\partial U$, 并且假设存在正常数$C_1,C_2$及$s\in(0,1)$ 使得

$$|D\Delta^{k-1}f(y)|\leq C_1|y|^{1+s-2k},\left|\frac{\partial}{\partial n}D\Delta^{k-1}f(y)\right|\leq C_2|y|^{s-2k},\ k=1,2,\cdots,m.$$

其中

$$\begin{aligned}\Phi(t)&=\int_{\partial U}\left(\Delta^m N_*(y,t)\right)\mathrm{d}\sigma_y f(y)-\sum_{k=1}^{m}\int_{\partial U}\left(\frac{\partial}{\partial n}\Delta^{m-k}N_*(y,t)\right)D\Delta^{k-1}f(y)\mathrm{d}S_y\\&\quad+\sum_{k=1}^{m}\int_{\partial U}\left(\Delta^{m-k}N_*(y,t)\right)\frac{\partial}{\partial n}D\Delta^{k-1}f(y)\mathrm{d}S_y\\&=F_0(t)-F(t)+G(t).\end{aligned}\tag{4.3}$$

分别记$\Phi^+(t), \Phi^-(t)$为$x \to t, x \in U^+, x \in U^-$时的边界值, 则有

$$\begin{cases} \Phi^+(t) = \frac{1}{2}f(t) + \Phi(t), \\ \Phi^-(t) = -\frac{1}{2}f(t) + \Phi(t). \end{cases} \tag{4.4}$$

或

$$\begin{cases} \Phi^+(t) + \Phi^-(t) = 2\Phi(t), \\ \Phi^+(t) - \Phi^-(t) = f(t). \end{cases} \tag{4.4}$$

证　由于$\Phi(x) = F_0(x) - F(x) + G(x), F_0(x), F(x), G(x)$如上所述, 由引理2.5知

$$\lim_{x \to t, x \in U^+} F_0(x) = \frac{1}{2}f(t) + F_0(t), \quad \lim_{x \to t, x \in U^-} F_0(x) = -\frac{1}{2}f(t) + F_0(t).$$

再由定理4.2, 定理4.3得证.

参考文献:

[1] Clifford W K. Applications of Grassmann's extensive algebra[J]. American Journal of Mathematics, 1978, 1(4): 350-358.

[2] Brackx F, Delanghe R, Sommen F. Clifford Analysis[M]. Boston: Pinman Book Limited, 1982.

[3] 黄沙. 拟置换与实Clifford分析中的广义正则函数[J]. 系统科学与数学, 1998, 18(3): 380-384.

[4] Huang Sha, Qiao Yuying, Wen Guochun. Real and Complex Clifford Analysis[M]. New York: Springer, 2006.

[5] Du Jinyuan, Xu Na, Zhang Zhongxiang. Boundary behavior of Cauchy-type integrals in Clifford analysis[J]. Acta Mathematica Scientia, Series B, 2009, 29(1): 210-224.

[6] Ren Guangbin, Wang Xieping, Xu Zhenghua. Slice regular functions on regular quadratic cones of real alternative algebras[A]. Mordern Trends in Hypercomplex Analysis[C]. 2016, 227-245.

[7] Qiao Yuying, Bernstein S, Eriksson S L, et al. Function theory for Laplace and Dirac-Hodge operators on hyperbolic space[J]. Journal D'Analyse Mathematique, 2006, 98(1): 43-64.

[8] Xie Yonghong. Boundary properties of hypergenic-Cauchy type integrals in real Clifford analysis[J]. Complex Variables and Elliptic Equations, 2014, 59(5): 599-615.

[9] Wang Liping. Some properties of a kind of generalized Teodorescu operator in Clifford analysis[J]. Journal of Inequalities and Applications, 2016, 102(2016): 1-11.

[10] Yang Heju, Xie Yonghong, Qiao Yuying. Cauchy integral formula for k-monogenic function with α-weight[J]. Advances in Applied Clifford Algebras, 2018, 28: 2, 11 pages.

[11] Li Zunfeng, Yang Heju, Qiao Yuying. A new Cauchy integral formula in the complex Clifford analysis[J]. Advances in Applied Clifford Algebras, 2018, 28: 75, 12 pages.

[12] Shi Haipan, Yang Heju, Li Zunfeng, et al. Two-sided Fourier transform in Clifford analysis and its application[J]. Advances in Applied Clifford Algebras, 2020, 30: 67, 23 pages.

[13] Shi Haipan, Yang Heju, Li Zunfeng, et al. Fractional Clifford-Fourier transform and its application[J]. Advances in Applied Clifford Algebras, 2020, 30: 68, 17 pages.

[14] Li Zunfeng, Shi Haipan, Qiao Yuying. Two-sided fractional quaternion Fourier transform and its application[J]. Journal of Inequalities and Applications, 2021, 121(2021): 1-15.

[15] Dinh D C, Tuyet L T. Representation of Weinstein k-monogenic functions by differential operators[J]. Complex Analysis and Operator Theory, 2020, 14: 20, 11 pages.

[16] Dinh D C. Generalized (ki)-monogenic functions[J]. Advances in Applied Clifford Algebras, 2020, 30: 58, 16 pages.

[17] Blaya R, Reyes J, Gareía A, Gareía T. A Cauchy integral formula for infrapolymonogenic functions in Clifford analysis[J]. Advances in Applied Clifford Algebras, 2020, 30: 21, 17 pages.
[18] Xu Na, Li Zunfeng, Yang Heju. Cauchy integral formula on the distinguished boundary with values in complex universal Clifford algebra[J]. Advances in Applied Clifford Algebras, 2021, 31: 72, 19 pages.
[19] Laville G, Ramadanoff I. Holomorphic Cliffordian function[J]. Advances in Applied Clifford Algebras, 1998, 8(2): 323-340.
[20] Laville G, Lehman E. Analytic Cliffordian functions[J]. Annales Academiæ Scientiarum Fennicæ Mathematica, 2004, 29(2): 251-268.
[21] Laville G, Ramadanoff I. Elliptic Cliffordian functions[J]. Complex Variables and Elliptic Equations, 2001, 45(4): 297-318.
[22] Laville G, Ramadanoff I. Jaccobi elliptic Cliffordian functions[J]. Complex Variables and Elliptic Equations, 2002, 47(9): 787-802.
[23] 扈玮玮. 全纯Cliffordian函数的一些性质和Cauchy型积分的边值特性[D]. 石家庄: 河北师范大学, 2008.
[24] Brackx F, Pincket W. Two Hartogs theorems for nullsolutions of overdetermined systems in Euclidean space[J]. Complex Variables and Elliptic Equations, 1985, 4(3): 205-222.
[25] Iftimie V. Functions hypercomplexes[J]. Bulletin Mathématique de la Société des Sciences Mathématiques de Roumanie, 1965, 9(57): 279-332.
[26] Franks E, Ryan J. Bounded monogenic funtions on unbounded domains[J]. Contemporary Mathematics, 1998, 212(1): 71-79.
[27] Gürlebeck K, Kähler U, Ryan J, Sprössig W. Clifford analysis over unbounded domains[J]. Advances in applied mathematics, 1997, 19(2): 216 - 239.

Some properties of holomorphic Cliffordian functions and boundary value properties of the Cauchy type integral in unbounded domain

ZHANG Kun[1], GAO Long[2], QIAO Yu-ying[2]
(1. Cangzhou Jiaotong College, Cangzhou 061199, China
2. School of Mathematical Science, Hebei Normal University, Shijiazhuang 050024, China)

Abstract: In the first place, the left and right holomorphic Cliffordian functions are defined in Euclidean space and with values in real Clifford algebra. Then some spacial properties of holomorphic Cliffordian functions are discussed by way of the properties of regular functions. With the help of the first class of Quasi-Permutation, the equal conditions are proved from the angle of regular functions, which build the relations between regular functions and holomorphic Cliffordian functions. And then, the extension theorem is discussed based on the Cauchy type integral formula and the Plemelj formula in the bounded domain using some small techniques. Finally, the Cauchy type integral is defined on unbounded domains, and it is proved be convergent under the meaning of Cauchy principal value. And the Plemelj formula is discussed by way of some significant integral estimation and some methods above.

Keywords: holomorphic Cliffordian functions; Cauchy type integral; Cauchy principal value; Plemelj formula

MR Subject Classification: 47H05; 30G30