高校应用数学学报
2024, 39(1): 73-88

Banach空间中可数算子族的不动点集和广义混合均衡问题的公共解的强收敛性

倪仁兴*, 徐亚军

(绍兴文理学院 数学系, 浙江绍兴 312000)

摘 要: 在自反Banach空间框架中, 提出一类新的收缩投影算法, 来逼近一闭Bregman拟渐近非扩张可数算子族的不动点集和广义混合均衡问题的公共解, 建立了一闭Bregman拟渐近非扩张可数算子族的不动点集和广义混合均衡问题的公共解的强收敛性结果, 并用一数值例子来支撑文中所得的结果. 无论算子族从有限个到可数个, 还是算子从Bregman相对非扩张到Bregman拟渐近非扩张等方面, 均对Jantakarn K和Kaewcharoen A(2021)的结果进行了的拓展和补充.

关键词: 广义混合均衡问题; Bregman相对非扩张映像; Bregman拟渐进非扩张映射; 收缩投影法; 强收敛性

中图分类号: O177.91

文献标识码: A　　**文章编号**: 1000-4424(2024)01-0073-16

§1 引 言

1967年, Bregman[1]在Banach空间中引入Bregman距离函数并讨论了Bregman非扩张映射不动点及其相关问题. 1994年, Blum和Oetti[2]讨论了均衡问题(EP). 2008年, Peng和Yao[3]引入并研究了广义混合均衡问题(GMEP), 它是均衡问题(EP)的推广. 研究表明, 均衡问题为非线性分析, 优化, 博弈论中出现的问题提供了一个自然, 新颖和统一的框架.

2009年, Takahashi和Zembayashi[4]提出一种迭代算法, 此方法用于在Banach空间中找均衡问题(EP)和相对非扩张映射T的不动点问题, 并证明了该迭代方法的一个强收敛定理, 相关研究结果也可见参考文献[5-7]; 2018年, Kazmi和Ali[8]在Banach空间中提出混合迭代方法以找广义均衡问题(GEP)和Bregman相对非扩张映射不动点的公共解, 在以上学者的基础上, 2021年, Jantakarn K和Kaewcharoen A[9]提出了一种新的迭代方法, 用于求解自反Banach空间中一有限族Bregman相对非扩张映射不动点和混合平衡问题((MEP)的公共解. 并证明了其所提出的迭代算法生成的序列强收敛到上述问题的一公共解(具体可参见本文的定理2.1).

收稿日期: 2022-08-04　　修回日期: 2023-11-13
*通讯作者, E-mail: nrx1964@163.com
基金项目: 国家自然科学基金(12171435)

受文献[4, 8, 9]的研究工作的启发, 在自反Banach框架下, 提出一种新的收缩投影算法, 并证明了由该迭代算法生成的序列强收敛到广义混合均衡问题(GMEP)和一Bregman拟渐近非扩张可数算子族的不动点问题的公共解的结果. 所得的结果是文献[9]中主要结果–定理2.1的多方面拓展和推广.

§2 预备知识

本文中若无特别说明, 记$\mathbf{Z}^+$和$\mathbf{R}$分别为正整数集和实数集. $\rightarrow$和$\rightharpoonup$分别表示强, 弱收敛. E是Banach空间, E^*是E的对偶空间, C是E的非空间凸子集, $f: E \rightarrow (-\infty, +\infty]$ 是一真的下半连续凸函数. 记dom$f$为f的定义域, 即dom$f$=$\{x \in E : f(x) < +\infty\}$. f在$x(\in \mathrm{int}(\mathrm{dom} f))$处的次微分定义为$\partial f(x) = \{x^* \in E^* : f(x) + \langle x^*, y - x\rangle \le f(y), \forall y \in E\}$; f的共轭函数$f^* : E^* \rightarrow (-\infty, +\infty]$定义为$f^*(x^*) = \sup\{\langle x^*, x\rangle - f(x) : x \in E\}$, f在x处关于方向y的右导数定义为

$$f^0(x, y) := \lim_{t \longrightarrow 0^+} \frac{f(x + ty) - f(x)}{t}. \tag{2.1}$$

称函数f在x处是Gateâux可微的, 如果(2.1)中极限$t \rightarrow 0^+$时存在. 这时f在x处的梯度是线性函数$\nabla f(x)$, 这样$\langle y, \nabla f(x)\rangle := f^0(x, y), \forall y \in E$. 称函数Gateâux是可微的, 如果(2.1)中的极限当$t \rightarrow 0^+$时关于$\|y\| = 1$ 一致成立. 称f在E的子集C上是一致Frêchet可微的, 如果(2.1)中的极限, 当$x \in C, \|y\| = 1$时一致成立.

定义2.1[10] 设E是一自反Banach空间, 称函数$f : E \rightarrow (-\infty, +\infty]$为Legendre函数, 若$f$满足下列两个条件.

(C1) 函数f是Gateâux可微的, $\mathrm{int}(\mathrm{dom}(f)) \neq \varnothing, \mathrm{dom}(f) = \mathrm{int}(\nabla f)$.

(C2) 函数f^*是Gateâux可微的, $\mathrm{int}(\mathrm{dom}(f^*)) \neq \varnothing, \mathrm{dom}(f^*) = \mathrm{int}(\nabla f^*)$.

定义在自反Banach空间上的Legendre函数具下列性质.

(1) f是Legendre函数当且仅当f^*是Legendre函数;

(2) $(\partial f)^{-1} = \partial f^*$;

(3) $\nabla f = (\nabla f^*)^{-1}, ran\nabla f = \mathrm{dom}\nabla f^*, ran\nabla f^* = \mathrm{dom}\nabla f = \mathrm{int}(\mathrm{dom} f)$;

(4) f, f^*分别在$\mathrm{int}(\mathrm{dom} f)$和$\mathrm{int}(\mathrm{dom} f^*)$上是严格凸的.

注2.1[10] 特别地, 若E是光滑和严格凸Banach空间, 一个重要和有趣的Legendre函数是

$$f(x) = \frac{1}{p}\|x\|^p (1 < p < \infty), \forall x \in E.$$

这时f的梯度∇f等于E的广义对偶映射, 即$\nabla f = J_p (1 < p < \infty)$. 特别地, 在Hilbert空间中, $\nabla f = I$, 其中I是恒等映射.

定义2.2[11] 设C是Banach空间E的一个非空闭凸子集, 广义混合均衡问题(GMEP)是指找$x \in C$, s.t.

$$g(x, y) + \langle Ax, y - x\rangle\varphi(y) - \varphi(x) \ge 0, \forall y \in C, \tag{2.2}$$

其中$g : C \times C \rightarrow \mathbf{R}$是二元函数, $\varphi : C \rightarrow \mathbf{R}$一实值函数, $A : C \rightarrow E^*$是一非线性算子. GMEP(2.2)的解集记为GMEP(g, φ), 即

$$\mathrm{GMEP}(g, \varphi) = \{x \in C | g(x, y) + \langle Ax, y - x\rangle + \varphi(y) - \varphi(x) \ge 0, \forall y \in C\}.$$

定义2.3[1,12] 设$f : E \rightarrow (-\infty, +\infty]$是一Gateâux可微的凸函数. 关于$f$的Bregman距离函数$D_f : \mathrm{dom} f \times \mathrm{int}(\mathrm{dom} f) \rightarrow [0, +\infty)$定义为

$$D_f(y, x) = f(y) - f(x) - \langle\nabla f(x), y - x\rangle. \tag{2.3}$$

Bregman函数距离D_f一般不是真正意义的距离, 因为D_f是不对称函数, 显然$D_f(x,x)=0$, 但是$D_f(y,x)=0$并不能得出$y=x$, 由文献[10]知, 当f是Legendre函数时, $D_f(y,x)=0$, 当且仅当$y=x$, D_f一般不满足三点不等式, 但$\forall y,z\in \mathrm{int}(\mathrm{dom}f), x\in \mathrm{dom}f$, D_f满足三点恒等式

$$D_f(x,z)=D_f(x,y)+D_f(y,z)+\langle \nabla f(y)-\nabla f(z), x-y\rangle. \tag{2.4}$$

定义2.4[1] 设$f:E\to(-\infty,+\infty]$是一Gateâux可微的凸函数, C时$\mathrm{int}(\mathrm{dom}f)$非空闭凸子集, 关于$x\in(\mathrm{int}(\mathrm{dom}f))$到$C$的$f$Bregman投影是指存在唯一元$\Pi_C x\in C$满足

$$D_f(\Pi_C(x),x)=\inf\{D_f(y,x):y\in C\}.$$

定义2.5[13] 设C是E的子集, 算子$T:C\to C, F(T)$为T的不动点集, 即

$$F(T)=\{x\in C|Tx=x\}.$$

$\hat{F}(T)$为T的渐近不动点集, 即

$$\hat{F}(T)=\{x\in C|\exists\{x_n\}\subset C, x_n\rightharpoonup x, \text{s.t.} \lim_{n\to+\infty}\|x_n-Tx_n\|=0\}.$$

称算子T为

(1) Bregman拟非扩张的是指如果$F(T)\neq\varnothing, D_f(p,Tx)\le D_f(p,x), \forall x\in C, p\in F(T)$;

(2) Bregman相对非扩张的是指$F(T)=\hat{F}(T)\neq\varnothing, D_f(p,Tx)\le D_f(p,x), \forall x\in C, p\in F(T)$;

(3) Bregman拟渐近非扩张的是指若$F(T)\neq\varnothing$, 存在实序列$\{k_n\}\subset[1,+\infty)$, $\lim\limits_{n\to+\infty}k_n=1$, 使得

$$D_f(p,T^nx)\le k_nD_f(p,x), \forall x\in C, p\in F(T), \forall n\ge 1. \tag{2.5}$$

(4) 闭的是指如果对于C中的序列$x_n\to x$和$Tx_n\to y$, 其中$x,y\in C$,有$Tx=y$;

(5) 在C上是一致渐近正则的, 如果

$$\lim_{n\to\infty}\sup_{x\in C}\|T^{n+1}x-T^nx\|=0. \tag{2.6}$$

Bregman相对非扩张算子须要求$F(T)=\hat{F}(T)\neq\varnothing$, 其条件较强, 易见Bregman相对非扩张算子是Bregman拟非扩张算子, 每个Bregman拟非扩张算子都是系数序列$\{k_n\}$满足$k_n=1$的Bregman拟渐近非扩张算子. 因此可见Bregman拟渐近非扩张算子适用范围更广泛.

定义2.6[14] 设$f:E\to(-\infty,+\infty]$是一Gateâux可微的凸函数, f在x处的全凸性模系数为

$$v_f(x,t):=\inf\{D_f(y,x):y\in \mathrm{dom}f, \|y-x\|=t\},$$

其中$v_f:\mathrm{int}(\mathrm{dom}f)\times[0,+\infty)\to[0,+\infty)$.

函数f称为

(a) $x(\in \mathrm{int}(\mathrm{dom}f))$是全凸的是指若$f$在$x$处的全凸性模系数$v_f(x,t)>0, \forall t>0$;

(b) 全凸的是指若f在$\forall x\in \mathrm{int}(\mathrm{dom}f)$处都是全凸的;

(c) 在有界集上全凸的是指若f在集合B上的全凸性模系数$v_f(B,t)>0, \forall t>0, \forall E$的有界集$B$. 其中$f$在$B$上的全凸性模系数为

$$v_f(B,t):=\inf\{v_f(x,t):x\in B\cap \mathrm{dom}f\}.$$

定义2.7[14-15] 函数$f:E\to(-\infty,+\infty]$称为

(a) 强制的是指f满足$\lim\limits_{\|x\|\to+\infty}\frac{f(x)}{\|x\|}=+\infty$;

(b) 序列一致的是指若对任意两个序列$\{x_n\}$, $\{y_n\}\subset E$使得$\{x_n\}$是有界序列, 且

$$\lim_{n\to+\infty}D_f(y_n,x_n)=0,$$

必有

$$\lim_{n\to\infty}\|y_n-x_n\|=0. \tag{2.7}$$

定义2.8[16] 设$f:E\to\mathbf{R}$是一Gateâux可微的Legendre函数, 关于f的函数$V_f:E\times E^*\to[0,+\infty)$定义为

$$V_f(x,x^*):=f(x)-\langle x,x^*\rangle+f^*(x^*),\forall x\in E,x^*\in E^*.$$

易见V_f是非负的, 且

$$V_f(x,x^*):=D_f(x,\nabla f(x^*)),\forall x\in E,x^*\in E^*.$$

假设2.1[13] 二元函数$g:C\times C\to\mathbf{R}$满足下列假设.

(1) $g(x,x)=0,\forall x\in C$;

(2) g是单调的$\Leftrightarrow g(x,y)+g(y,x)\le,\forall x,y\in C$;

(3) $\mathrm{limsup}_{t\to 0^+}g(tz+(1-t)x,y)\le g(x,y),\forall x,y,z\in C$;

(4) $\forall x\in C,g(x,.)$是下半连续凸函数.

假设2.2[4] 设$\phi:C\times C\to\mathbf{R}$是一二元函数, 满足下列假设.

(1) ϕ是反对称的$\Leftrightarrow\phi(x,x)-\phi(x,y)-\phi(y,x)+\phi(y,y)\ge 0,\forall x,y\in C$;

(2) ϕ关于第二个参数是凸的;

(3) ϕ是连续的.

引理2.1 设C是Banach空间E的非空闭凸子集, $f:E\to(-\infty,+\infty]$是一Legendre函数, 且其在E的有界子集上是全凸的, 设$T:C\to C$是一闭的Bregman拟渐近非扩张算子, $\{k_n\}\subset[1,+\infty)$, s.t. $\lim\limits_{n\to\infty}k_n=1$, 则$F(T)$是闭凸集.

证 设$\{x_n\}$是$F(T)$中的序列使得$\lim\limits_{n\to\infty}x_n=x^*$,由$T:C\to C$是一Bregman拟渐近非扩张算子得$\exists\{k_n\}\subset[1,+\infty)$且$\lim\limits_{n\to\infty}k_n=1$均有$0\le D_f(x_n,T^nx^*)\le k_nD_f(x_n,x^*)$得

$$\lim_{n\to\infty}D_f(x_n,T^nx^*)=0.$$

而$\{x_n\}$有界得$\lim\limits_{n\to\infty}\|x_n-T^nx^*\|=0$. 注意到$0\le\|x^*-T^nx^*\|\le\|x_n-x^*\|+\|x_n-T^nx^*\|$, 得

$$\lim_{n\to\infty}\|x^*-T^nx^*\|=0.$$

即$T^nx^*\to x^*(n\to\infty)$, 从而$T(T^nx^*)=T^{n+1}x^*\to x^*$, 由$T$的闭性得$Tx^*=x^*$, 即$x^*\in F(T)$, 这意味着$F(T)$是闭集.

下证$F(T)$是凸集. 设$\forall a,b\in F(T),t\in(0,1)$. 令$c=ta+(1-t)b$, 下证$c\in F(T)$. 由Bregman距离函数定义知

$$\begin{aligned}D_f(c,T^nc)&=f(c)-f(T^nc)-\langle\nabla f(T^nc),c-T^nc\rangle\\&=f(c)-f(T^nc)-\langle\nabla f(T^nc),ta+(1-t)b-T^nc\rangle. \qquad (*)\\&=f(c)+tD_f(a,T^nc)+(1-t)D_f(b,T^n)-tf(a)-(1-t)f(b).\end{aligned}$$

由(2.5)知

$$\begin{aligned}&tD_f(a,T^nc)+(1-t)D_f(b,T^nc)\\&\le k_nD_f(a,c)+(1-t)k_nD_f(b,c)\\&=tk_n[f(a)-f(c)-\langle\nabla f(c),a-c\rangle]+(1-t)k_n[f(b)-f(c)-\langle\nabla f(c),b-c\rangle]\\&=tk_nf(a)+(1-t)k_nf(b)-k_nf(c),\end{aligned}$$

将上式子代入(*)的右边, 化简得

$$0 \le D_f(c, T^n c) \le (k_n - 1)[tf(a) + (1-t)f(b) - f(c)]. \tag{2.8}$$

因此对(2.8)取极限可得 $\lim\limits_{n\to\infty} D_f(c, T^n c) = 0$, 由$f$是Legendre函数, 有$T^n c \to c(n \to \infty)$. 意味着$T(T^n c) = T^{n+1}c \to c(n \to \infty)$, 因为$T$是闭的, 可得$c = Tc$,即$c \in F(T)$. 得$F(T)$是$C$中的凸集, 这样$F(T)$是闭凸集.

引理2.2[17] 设$f : E \to (-\infty, +\infty]$是一致Frêchet可微的, 且在$E$的有界子集上是有界的. 则$f$在$E$的有界子集上是一直连续的, 且$\nabla f$在$E$的有界集上是从$E$的强拓扑到$E^*$的强拓扑上一致连续的.

引理2.3[10] 设E是一自反Banach空间, $f : E \to \mathbf{R}$是凸函数且在E的有界子集上有界的. 则下列命题等价.

(1) f在E的有界子集上是强制且一致凸的;

(2) $\mathrm{dom} f^* = E^*, f^*$在$E^*$的有界子集上有界且一致光滑的;

(3) $\mathrm{dom} f^* = E^*, f^*$和$\nabla f^*$在$E^*$的有界子集上分别是Frêchet可微和范数到范数一致连续的;

(4) f, f^*在各自定义域内部都是严格凸函数.

引理2.4[18] 设$f : E \to (-\infty, +\infty]$在$\mathrm{int}(\mathrm{dom} f)$上是一Gâteaux可微且全凸函数. 设$x \in \mathrm{int}(\mathrm{dom} f), C(\subset \mathrm{int}(\mathrm{dom} f))$是一非空有界闭凸集, 若$z \in C$, 则下列命题等价.

(i) $z \in C$是x到C关于函数f的Bregman投影$\Leftrightarrow z = \Pi_C(x)$;

(ii) $z \in C$是变分不等式

$$\langle \nabla f(x) - \nabla f(z), z - y\rangle \ge 0, \forall y \in C \tag{2.9}$$

的唯一解;

(iii) $z \in C$是不等式

$$D_f(y, z) + D_f(z, x) \ge D_f(y, x), \forall y \in C \tag{2.10}$$

的唯一解.

引理2.5[18] 设$f : E \to (-\infty, +\infty]$是一凸函数且定义域$E$至少包含两点, 则下列陈述等价.

(1) f是序列一致的当且仅当f在有界集上是全凸的;

(2) 如果f是下半连续的, 则f是序列一致的当且仅当f在有界集上是一致凸的.

引理2.6[19] 设$f : E \to (-\infty, +\infty]$是一Legendre函数, 使得$\nabla f^*$在$\mathrm{int}(\mathrm{dom} f^*)$ 的有界子集上是有界的. 若$x_1 \in E$且$\{D_f(x_1, x_n)\}$是有界的, 则序列$\{x_n\}$是有界的.

引理2.7[19] 设$f : E \to (-\infty, +\infty]$是一真的下半连续凸函数, 则$f^* : E^* \to (-\infty, +\infty]$是一真的弱*下半连续凸函数. 这样$\forall x \in E, V_f(x, \cdot)$是凸函数. 从而$\forall z \in E$有

$$D_f(z, \nabla f^*(\sum_{i=1}^{N} t_i \nabla f(x_i))) \le \sum_{i=1}^{N} t_i D_f(z, x_i), \tag{2.11}$$

其中$\{x_i\}_{i=1}^N \subset E, \{t_i\}_{i=1}^N \subset (0, 1), \sum_{i=1}^N t_i \le 1$.

引理2.8[13] 设E是一自反Banach空间, $f : E \to (-\infty, +\infty]$是连续强制Legendre凸函数, 在$E$的有界子集上是有界且一致凸的, C是E中一非空闭凸子集. 设$g : C \times C \to \mathbf{R}$是满足假设2.1的二元函数, $EP(G) \ne \varnothing, \varphi : C \to \mathbf{R}$是一下半连续凸函数. $A : C \to E^*$是一连续单调算子. 对$r > 0, x \in E$, 算子$T_r^G : E \to 2^C$定义如下$T_r^G = \{z \in C : G(z, y) + \frac{1}{r}\langle \nabla f(z) - \nabla f(x), y - z\rangle \ge 0, \forall y \in C\}$, 其中$G(x, y) = g(x, y) + \varphi(y) - \varphi(x) + \langle Ax, y - x\rangle, \forall x, y \in E$. 则下列命题成立.

(1)T_r^G是单值的, $\mathrm{dom}(T_r^G)=E$;

(2)T_r^G是Bregman坚定非扩张算子;

(3)$F(T_r^G)=\mathrm{GMEP}(g,\varphi)$是$C$的闭凸集;

(4)$D_f(q,T_r^Gx)+D_f(T_r^Gx,x)\le D_f(q,x),\forall q\in F(T_r^G)$;

(5)T_r^G是Bregman拟非扩张的.

2009年, Takahashi W, Zembayashi K[4]在自反Banach空间中, 研究寻找EP的相对非扩张映射的不动点的公共解, 构造了迭代算法

$$\begin{cases} x_0=x\in C,\\ y_n=J^{-1}(\alpha_nJx_n+(1-\alpha_n)JTx_n),\\ u_n\in C,\text{s.t.}g(u_n,y)+\frac{1}{r_n}\langle y-u_n,Ju_n-Jy_n\rangle\ge 0,\forall y\in C,\\ H_n=\{\omega\in C:\phi(\omega,u_n)\le\phi(\omega,x_n)\},\\ W_n=\{\omega\in C:\langle x_n-\omega,Jx-Jx_n\rangle\ge 0\},\\ x_{n+1}=P_{H_n\cap W_n}x_0,\forall n\ge 1. \end{cases} \tag{2.12}$$

$P_{H_n\cap W_n}$表示从E到$H_n\cap W_n$的广义投影, $\{r_n\}\subset[a,+\infty),a>0,\phi(x,y)=\|x\|^n-2\langle x,Jy\rangle+\|y\|^2$, 二元函数$g$满足假设2.1, J为正规对偶映射, 他们证明了算法(2.12)生成的序列$\{x_n\}$强收敛到$P_{F(T)\cap EP(g)}x_0$.

2018年, Kazmi K R, Ali R和Yousuf S[8]在自反Banach空间中, 引进并研究了一种混合迭代算法用来求GEP与Bregman相对非扩张映射T不动点的公共解, 构造了迭代算法

$$\begin{cases} x_0,z_0\in C,\\ u_n=\nabla f^*(\alpha_n\nabla f(z_n)+(1-\alpha_n)\nabla f(Tx_n)),\\ z_{n+1}=\mathrm{res}^f_{g,\phi}u_n,\\ C_n=\{\omega\in C:D_f(\omega,z_{n+1})\le\alpha_nD_f(\omega,z_n)+(1-\alpha_n)D_f(z,x_n)\},\\ Q_n=\{\omega\in C:\langle\nabla f(x_0)-\nabla f(x_n),\omega-x_n\rangle\le 0\},\\ x_{n+1}=\Pi_{C_n\cap Q_n}x_0,\forall n\ge 0. \end{cases} \tag{2.13}$$

其中$\Pi_{C_n\cap Q_n}$表示E到$C_n\cap Q_n$的Bregman投影. 对于$\forall x\in E$, 算子$\mathrm{res}^f_{g,\phi}:E\to 2^C$定义为

$$\mathrm{res}^f_{g,\phi}(x)=\{z\in C,g(z,y)+\phi(z,y)-\phi(z,z)+\langle\nabla f(z)-\nabla f(x),y-z\rangle\ge 0,\forall y\in C\}.$$

二元函数g,ϕ分别满足假设2.1和假设2.2, 他们证明了由算法(2.13)生成的序列$\{x_n\}$强收敛到$\Pi_{F(T)\cap GEP(g,\phi)}x_0$.

2021年, Jantakarn. K和Kaewcharoen. A在文献[9]中, 得到如下主要结果.

定理2.1[9] 设E是一自反Banach空间, C是E中一非空闭凸子集使得$C\subset\mathrm{int}(\mathrm{dom}f)$. $f:E\to(-\infty,+\infty]$是一强制的Legendre函数, 且在E的有界子集上是有界的, 及一致Frêchet可微全凸函数. 设$g:C\times C\to\mathbf{R}$是满足假设2.1的二元函数, $\varphi:C\to\mathbf{R}$是一下半连续凸函数. 设$\{T_i:C\to C\}_{i=1}^N$是一Bregman相对非扩张有限算子族. 假设$\Omega:=\cap_{i=1}^NF(T_i)\cap\mathrm{MEP}(g,\phi)\ne\varnothing$,

设$\{x_n\}$是由迭代算法

$$\begin{cases} x_1 \in C, T_i x_1 = z_1^i \in C;, \\ u_n^i = \nabla f^*(\alpha_n \nabla f(z_n^i) + (1-\alpha_n)\nabla f(T_i x_n)), \\ z_{n+1}^i = \mathrm{Res}_{g,\varphi}^f u_n^i, \\ C_n^i = \{z \in C : D_f(z, z_{n+1}^i) \le \alpha_n D_f(z, z_n^i) + (1-\alpha_n) D_f(z, x_n)\}, \\ C_n = \cap_{i=1}^N C_n^i; \\ Q_n = \{z \in C : \langle \nabla f(x_1) - \nabla f(x_n), z - x_n \rangle \le 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x_1, \forall n \ge 1 \end{cases} \tag{2.14}$$

生成的序列. 其中$\{\alpha_n\} \subset [0,1]$, s.t. $\lim\limits_{n \longrightarrow +\infty} \alpha_n = 0$. 对于$\forall x \in E$, 算子$\mathrm{Res}_{g,\varphi}^f : E \to 2^C$定义为

$$\mathrm{Res}_{g,\varphi}^f(x) = \{z \in C : g(z,y) + \varphi(y) - \varphi(z) + \langle \nabla f(z) - \nabla f(x), y - z \rangle \ge 0, \forall y \in C\}.$$

则$x_n \to \Pi_\Omega x_1 (n \to +\infty)$.

显然, Bregman相对非扩张算子所需条件较Bregman拟渐近非扩张算子强, 广义混合均衡问题的应用比混合均衡问题更为广泛, 可数族比有限族更为常见. 于是有一个很自然的问题A: 定理2.1[9]中从研究混合均衡问题和一有限Bregman相对非扩张算子族不动点的公共解的强收敛性拓广到研究广义混合均衡问题和一可数Bregman拟渐近非扩张算子族(或一可数Bregman拟非扩张算子族)不动点的公共解的强收敛性, 通过对算法(2.14)适当的修正, 上述定理2.1是否还成立? 本文的目的是给出问题A的一个肯定回答. 主要得到下面的定理3.1和定理3.2, 并给出一具体的数值算例来支撑所得的主要结果.

§3 主要结果

定理3.1 设E是一自反Banach空间, C是E中一非空闭凸子集. $f : E \to (-\infty, +\infty]$是一强制的Legendre函数且在$E$的有界子集上是有界的, 及一致Frêchet可微且全凸函数. 其中$C \subset \mathrm{int}(\mathrm{dom} f)$. 设$g : C \times C \to \mathbf{R}$为满足假设2.1的二元函数, $\varphi : C \to \mathbf{R}$是一下半连续凸函数, 设$T_i : C \to C (\forall i \ge 1)$是一闭的具序列系数$\{k_n^i\}$的Bregman拟渐近非扩张可数算子族, $\{k_n^i\} \subset [1, +\infty)$, $\lim\limits_{n \longrightarrow +\infty} k_n^i = 1$. 假设$\Omega := \cap_{i=1}^\infty F(T_i) \cap \mathrm{GMEP}(g, \varphi) \ne \varnothing$. $T_i (\forall i \ge 1)$在E上是一致渐近正则的, $C_1^i = C, \forall i \ge 1$和$C_1 = \cap_{i=1}^\infty C_1^i = C$, 设$\{x_n\}$是由迭代算法

$$\begin{cases} T_i^n x_1 = z_i^i \in C, x_1 \in C, \\ u_n^i = \nabla f^*[(1-(1-\alpha_n)k_n^i)\nabla f(z_n^i) + (1-\alpha_n)\nabla f(T_i^n x_n)], \\ z_{n+1}^i = T_r^G u_n^i, \\ C_{n+1}^i = \{z \in C_n : D_f(z, z_{n+1}^i) \le [1-(1-\alpha_n)k_n^i] D_f(z, z_n^i) + (1-\alpha_n) k_n^i D_f(z, x_n)\}, \\ C_{n+1} = \cap_{i=1}^\infty C_{n+1}^i, \\ x_{n+1} = \Pi_{C_{n+1}}(x_1), \forall n \ge 1 \end{cases} \tag{3.1}$$

生成的序列. 其中$\{\alpha_n\} \subset [0,1]$, s.t. $\lim\limits_{n \longrightarrow \infty} \alpha_n = 0, r \in (0, +\infty), T_r^G$和$G(x,y)$如引理2.8中所述, $(1-\alpha_n)k_n^i \le 1 (\forall n \ge 1, \forall i \ge 1)$, 则$x_n \to \Pi_\Omega(x_1) (n \to \infty)$.

证 用引理2.8可证$G(x,y) = g(x,y) + \varphi(y) - \varphi(x) + \langle Ax, y - x \rangle$满足假设2.1, 所以广义

混合均衡问题GMEP(g,φ)等价于下列均衡问题EP(G): 找$x\in C$, s.t. $G(x,y)\geq 0,\forall y\in C$, 所以GMEP$(g,\varphi)$=EP(G). 由引理2.1和引理2.8可得$\Omega$是闭凸集. 下面分七步来证明定理3.1.

步骤一 证明$C_n(\forall n\geq 1)$是闭凸集.

因$C_1^i=C$, 有$C_1=\cap_{i=1}^{\infty}C_1^i=C$是闭凸集, 下证$C_n(\forall n\geq 2)$是闭凸集, 由$C_{n+1}^i$的定义, 对任意$z\in C_{n+1}^i$, 可得

$$D_f(z,z_{n+1}^i)\leq[1-(1-\alpha_n)k_n^i]D_f(z,z_n^i)+(1-\alpha_n)k_n^iD_f(z,x_n),$$

上式等价于下式子

$$\begin{aligned}&\langle z-z_{n+1}^i,(1-\alpha_n)k_n^i\nabla f(x_n)-\nabla f(z_{n+1}^i)+[1-(1-\alpha_n)k_n^i]\nabla f(z_n^i)\rangle\\&\qquad\leq(1-\alpha_n)k_n^iD_f(z_{n+1}^i,x_n)+[1-(1-\alpha_n)k_n^i]D_f(z_{n+1}^i,z_n^i).\end{aligned}\tag{3.2}$$

因为(3.2)左边关于z的函数$\langle\cdot-z_{n+1}^i,(1-\alpha_n)k_n^i\nabla f(x_n)-\nabla f(z_{n+1}^i)-[1-(1-\alpha_n)k_n^i-1]\nabla f(z_n^i)\rangle$是线性和连续的, 所以$C_{n+1}^i$是闭凸集, 这样$C_{n+1}=\cap_{i=1}^{\infty}C_{n+1}^i(\forall n\geq 1)$是闭凸集. 综上证得$C_n(\forall n\geq 1)$是闭凸集.

步骤二 证明$\Omega\subset C_n,\forall\geq 1$.

事实上$\Omega\subset C_1=C$, 假设$p\in\Omega$, 因为$T_i:C\to C(\forall i\geq 1)$是一闭Bregman拟渐近非扩张可数算子族, 由定义(2.5)和(2.11)有

$$\begin{aligned}D_f(p,u_n^i)&=D_f(p,\nabla f^*([1-(1-\alpha_n)k_n^i]\nabla f(z_n^i)+(1-\alpha_n)\nabla f(T_i^nx_n)))\\&\leq[1-(1-\alpha_n)k_n^i]D_f(p,z_n^i)+(1-\alpha_n)D_f(p,T_i^nx_n)\\&\leq[1-(1-\alpha_n)k_n^i]D_f(p,z_n^i)+(1-\alpha_n)k_n^iD_f(p,x_n).\end{aligned}\tag{3.3}$$

根据引理2.8的(5), (2.5), (2.11)和(3.3)知$\forall p\in\Omega$有

$$\begin{aligned}&D_f(p,z_{n+1}^i)=D_f(p,T_r^Gu_n^i)\\&\qquad\leq D_f(p,u_n^i)\leq[1-(1-\alpha_n)k_n^i]D_f(p,z_n^i)+(1-\alpha_n)k_n^iD_f(p,x_n),\end{aligned}\tag{3.4}$$

上式说明$p\in C_{n+1}^i,\forall i\geq 1,n\geq 1$, 证得$p\in C_{n+1}=\cap_{i=1}^{\infty}C_{n+1}^i$, 故$\Omega\subset C_n,\forall n\geq 1$.

步骤三 证明$\{x_n\}$, $\{z_n^i\}$, $\{u_n^i\}$和$\{T_n^ix_n\}(\forall i\geq 1,n\geq 1)$是有界序列.

设$p\in\Omega$, 由步骤二可知$p\in C_n(\forall n\geq 1)$. 由$x_n=\Pi_{C_n}x_1$和(2.10)知

$$D_f(p,x_n)=D_f(p,\Pi_{C_n}x_1)\leq D_f(p,x_1)-D_f(p,\Pi_{C_n}x_1)\leq D_f(p,x_1).$$

因此序列$\{D_f(p,x_{n+1})\}$是有界的, 因为f是强制函数, $f^*,\nabla f^*$在E^*的有界子集上是有界的, 由引理2.6知序列$\{x_n\}$是有界的, 因为$D_f(p,T_i^nx_n)\leq k_n^iD_f(p,x_n),\forall p\in\Omega,i\geq 1$. 由$\{k_n^i\}\subset[1,+\infty)$, $\lim\limits_{n\to+\infty}k_n^i=1$. 知序列$\{D_f(p,T_i^nx_n)\}$是有界的, 再一次使用引理2.6知序列$\{T_i^nx_n\}$是有界的. 注意到$\{D_f(p,x_n)\}$是有界的, 所以存在$M>0$使得$D_f(p,x_n)\leq M$, 由(3.4)知

$$D_f(p,z_{n+1}^i)\leq[1-(1-\alpha_n)k_n^i]D_f(p,z_n^i)+(1-\alpha_n)k_n^iM.\tag{3.5}$$

而由$D_f(p,z_1^i)=D_f(p,T_1^nx_1)\leq k_n^1D_f(p,x_1)$和$\lim\limits_{n\to\infty}k_n^1=1$ 得$\{D_f(p,z_1^i)\}$有界.

记$K=\max\{D_f(p,z_1^i),M\}$. 显然$D_f(p,z_1^i)\leq K,\forall i\geq 1$. 设对某一个$n$有$D_f(p,z_n^i)\leq K$, 从(3.5)式得

$$D_f(p,z_{n+1}^i)\leq[1-(1-\alpha_n)k_n^i]K+(1-\alpha_n)k_n^iK=K,\forall i\geq 1.$$

因此$\{D_f(p,z_n^i)\}_{n=1}^{\infty},\forall i\geq 1$是有界的, 再一次利用引理2.6知$\{z_n^i\}_{i=1}^{\infty},\forall i\geq 1$是有界的. 由(2.5),

引理2.7中的(2.11), $D_f(p,z_n^i)\le K, D_f(p,x_n)\le M, M\le K$和(2.6)知

$$\begin{aligned}D_f(p,u_n^i)&=D_f(p,\nabla f^*([1-(1-\alpha_n)k_n^i]\nabla f(z_n^i)+(1-\alpha_n)\nabla f(T_i^n x_n)))\\&\le[1-(1-\alpha_n)k_n^i]D_f(p,z_n^i)+(1-\alpha_n)D_f(p,T_i^n x_n)\\&\le[1-(1-\alpha_n)k_n^i]D_f(p,z_n^i)+(1-\alpha_n)k_n^iD_f(p,x_n)\\&\le\max\{D_f(p,z_n^i),D_f(p,x_n)\}\le K.\end{aligned}$$

故序列$\{D_f(p,u_n^i)\}$是有界的, 再由引理2.6知序列$\{u_n^i\}$也是有界的.

步骤四 证明$\{x_n\}$收敛于C中一元.

由$x_1\in C=C_1$, 所以$x_n=\Pi_{C_n}x_1,\forall n\ge 1, x_{n+1}=\Pi_{C_{n+1}}x_1\in C_{n+1}\subset C_n,\forall n\ge 1$. 由(2.10)知

$$D_f(x_n,x_1)=D_f(\Pi_{C_n}x_1,x_1)\le D_f(x_{n+1},x_1)-D_f(x_{n+1},x_n),$$

进而得

$$0\le D_f(x_{n+1},x_n)\le D_f(x_{n+1},x_1),\forall n\ge 1,\tag{3.6}$$

说明序列$\{D_f(x_n,x_1)\}$是单调非减的, 设$p\in\varOmega$, 知$p\in C_n,\forall n\ge 1$. 由$x_n=\Pi_{C_n}x_1$和引理2.4知

$$D_f(x_n,x_1)=D_f(\Pi_{C_n}x_1,x_1)\le D_f(p,x_1)-D_f(p,\Pi_{C_n}x_1)\le D_f(p,x_1),$$

因此序列$\{D_f(x_n,x_1)\}$是有界的, 所以极限$\lim\limits_{n\longrightarrow\infty}D_f(x_n,x_1)$存在. 这样对(3.6)取极限得

$$\lim_{n\longrightarrow\infty}D_f(x_{n+1},x_n)=0,\forall n\ge 1,$$

对于$\forall m\in\mathbf{Z}^+, m\ge 1, x_{n+m}=\Pi_{C_{n+m}}x_1\in C_{n+m}\subset C_{n+1}$, 由(2.10)知

$$\begin{aligned}0\le D_f(x_{n+m},x_{n+1})=D_f(x_{n+m},\Pi_{C_{n+1}}x_1)&\le D_f(x_{n+m},x_1)-D_f(\Pi_{C_{n+1}}x_1,x_1)\\&=D_f(x_{n+m},x_1)-D_f(x_{n+1},x_1).\end{aligned}$$

对上式取极限得$\lim\limits_{m,n\longrightarrow\infty}D_f(x_{m+n},x_{n+1})=0$. 由$f$在$E$的有界子集上是全凸函数, 由引理2.5的(1)知$f$是序列一致的, 又因为由步骤三知$\{x_{n+1}\}$是有界的, 根据定义(2.7)得

$$\lim_{m,n\longrightarrow\infty}\|x_{m+n}-x_{n+1}\|=0.$$

因此序列$\{x_n\}$是C中的Cauchy列, 因为E是Banach空间和C是E中闭凸集, 所以存在$x^*\in C$使得$x_n\to x^*$. (其中$n\to\infty$)

步骤五 证明$\|x_n-z_{n+1}^i\|\to 0, \|x_n-u_n^i\|\to 0$和$\|x_n-T_i^n x_n\|\to 0$. (其中$n\to\infty,\forall i\ge 1$)

由(2.7)和序列$\{x_n\}$是Cauchy列得

$$\lim_{n\longrightarrow\infty}\|x_{n+1}-x_n\|=0,\tag{3.7}$$

由(2.4), 设$p\in\varOmega$得

$$D_f(x_{n+1},z_n^i)=D_f(x_{n+1},p)+D_f(p,z_n^i)+\langle\nabla f(p)-\nabla f(z_n^i),x_{n+1}-p\rangle.\tag{3.8}$$

因为f在E的有界子集上是有界的, 所以∇f在E的有界子集上也是有界的. 由$\{x_n\},\{z_n^i\},\{T_i^n x_n\}$的有界性可知$\{\nabla f(x_n)\},\{\nabla f(z_n^i)\},\{\nabla f(T_i^n x_n)\}$在$E^*$上也是有界的. 由(3.8)知$\{D_f(x_{n+1},z_n^i)\}$是有界的. 由$x_{n+1}=\Pi_{C_{n+1}}(x_1)\in C_{n+1}=\cap_{i=1}^\infty C_{n+1}^i\subset C_n^i$和$C_n^i$定义, 有

$$\begin{aligned}0&\le D_f(x_{n+1},z_{n+1}^i)\le[1-(1-\alpha_n)k_n^i]D_f(x_{n+1},z_n^i)+(1-\alpha_n)k_n^iD_f(x_{n+1},x_n)\\&=[1-\alpha_n-(1-\alpha_n)k_n^i+\alpha_n]D_f(x_{n+1},z_n^i)+(1-\alpha_n)k_n^iD_f(x_{n+1},x_n)\\&=[(1-\alpha_n)(1-k_n^i)+\alpha_n]D_f(x_{n+1},z_n^i)+(1-\alpha_n)k_n^iD_f(x_{n+1},x_n)\\&=[(1-\alpha_n)(1-k_n^i)]D_f(x_{n+1},z_n^i)+\alpha_nD_f(x_{n+1},z_n^i)+(1-\alpha_n)k_n^iD_f(x_{n+1},x_n).\end{aligned}$$

因为$\{D_f(x_{n+1},z_n^i)\}$是有界的, $\lim\limits_{n\to\infty}[(1-\alpha_n)(1-k_n^i)]=0$, $\lim\limits_{n\to\infty}\alpha_n=0$和$\lim\limits_{n\to\infty}D_f(x_{n+1},x_n)=0$可得

$$\lim_{n\to\infty}D_f(x_{n+1},z_{n+1}^i)=0,\forall i\geq 1.$$

因为f在有界子集上是全凸的, 由引理2.5的(1)知f是序列一致的, 有因序列$\{z_{n+1}^i\}_{n=0}^{\infty}$是有界的, 故由定义2.7中(2.7)式得

$$\lim_{n\to\infty}\|x_{n+1}-z_{n+1}^i\|=0. \tag{3.9}$$

考虑到

$$\|x_n-z_{n+1}^i\|\leq\|x_n-x_{n+1}\|+\|x_{n+1}-z_{n+1}^i\|,$$

由(3.7), (3.9)可得

$$\lim_{n\to\infty}\|x_n-z_{n+1}^i\|=0. \tag{3.10}$$

因为f在E得有界子集上一致Frêchet可微的, 故由引理2.2知f和∇f在E的有界子集上是一致连续的, 所以由

$$\lim_{n\to\infty}\|f(x_n)-f(z_{n+1}^i)\|=0,\forall i\geq 1 \tag{3.11}$$

和

$$\lim_{n\to\infty}\|\nabla f(x_n)-\nabla f(z_{n+1}^i)\|=0,\forall i\geq 1, \tag{3.12}$$

以及(2.3)有

$$\begin{aligned}D_f(p,x_n)-D_f(p,z_{n+1}^i)&=f(p)-f(x_n)-\langle\nabla f(x_n),p-x_n\rangle\\&\quad-(f(p)-f(z_{n+1}^i)-\langle\nabla f(z_{n+1}^i),p-z_{n+1}^i\rangle)\\&=f(z_{n+1}^i)-f(x_n)+\langle\nabla f(z_{n+1}^i),p-x_n\rangle\\&\quad+\langle\nabla f(z_{n+1}^i),x_n-z_{n+1}^i\rangle-\langle\nabla f(x_n),p-x_n\rangle\\&=f(z_{n+1}^i)-f(x_n)+\langle\nabla f(z_{n+1}^i)-\nabla f(x_n),p-x_n\rangle\\&\quad+\langle\nabla f(z_{n+1}^i),x_n-z_{n+1}^i\rangle.\end{aligned} \tag{3.13}$$

由(3.10)-(3.12), 对(3.13)取极限得

$$\lim_{n\to\infty}[D_f(p,x_n)-D_f(p,z_{n+1}^i)]=0. \tag{3.14}$$

此外由$z_{n+1}^i=T_r^G u_n^i$, (2.5), 引理2.8的(4)和(2.11)可得, $\forall p\in\Omega\subset\mathrm{GMEP}(g,\varphi)=F(T_r^G)$. 有

$$\begin{aligned}D_f(z_{n+1}^i,u_n^i)&\leq D_f(p,u_n^i)-D_f(p,z_{n+1}^i)\\&=D_f(p,\nabla f^*([1-(1-\alpha_n)k_n^i]\nabla f(z_n^i)+(1-\alpha_n)\nabla f(T_i^n)))-D_f(p,z_{n+1}^i)\\&\leq[1-(1-\alpha_n)k_n^i]D_f(p,z_n^i)+(1-\alpha_n)D_f(p,T_i^n x_n)-D_f(p,z_n^i)\\&\leq[1-(1-\alpha_n)k_n^i]D_f(p,z_n^i)+(1-\alpha_n)k_n^i D_f(p,x_n)-D_f(p,z_n^i)\\&=[1-(1-\alpha_n)k_n^i](D_f(p,z_n^i)-D_f(p,x_n))+D_f(p,x_n)-D_f(p,z_n^i)\\&=[(1-\alpha_n)(1-k_n^i)+\alpha_n](D_f(p,z_n^i)-D_f(p,x_n))+D_f(p,x_n)-D_f(p,z_n^i)\\&=(1-\alpha_n)(1-k_n^i)(D_f(p,z_n^i)-D_f(p,x_n))+\alpha_n(D_f(p,z_n^i)-D_f(p,x_n))\\&\quad+D_f(p,x_n)-D_f(p,z_{n+1}^i).\end{aligned} \tag{3.15}$$

因为$\{D_f(p,x_n)\}$和$\{D_f(p,z_n^i)\}$是有界的, 由$\lim\limits_{n\to\infty}\alpha_n=0$和$\lim\limits_{n\to+\infty}k_n^i=1$, 以及(3.14), 可对(3.15)取极限得

$$\lim_{n\to\infty} D_f(z_{n+1}^i, u_n^i) = 0.$$

由序列$\{u_n^i\}$是有界的, 故由定义2.7中的(2.7)式知

$$\lim_{n\to\infty} \|z_{n+1}^i - u_n^i\| = 0. \tag{3.16}$$

因为

$$\|x_n - u_n^i\| = \|x_n - z_{n+1}^i + z_{n+1}^i - u_n^i\| \le \|x_n - z_{n+1}^i\| + \|z_{n+1}^i - u_n^i\|,$$

利用(3.10), (3.16)有

$$\lim_{n\to\infty} \|x_n - u_n^i\| = 0. \tag{3.17}$$

因为f是一致Frêchet可微且有界的, 由引理2.2知, ∇f在E的有界集上是一致连续的. 从(3.16), (3.17)分别可得

$$\lim_{n\to\infty} \|\nabla f(z_{n+1}^i) - \nabla f(u_n^i)\| = 0 \tag{3.18}$$

和

$$\lim_{n\to\infty} \|\nabla f(x_n) - \nabla f(u_n^i)\| = 0. \tag{3.19}$$

进一步, 因为f是Legendre函数, 由定义2.1的(3)知$\nabla f = (\nabla f^*)^{-1}$, 考虑下列不等式

$$\begin{aligned}
&\|\nabla f(x_n) - \nabla f(u_n^i)\| = \|\nabla f(x_n) - \nabla f(\nabla f^*([1-(1-\alpha_n)k_n^i]\nabla f(z_n^i) + (1-\alpha_n)\nabla f(T_i^n x_n)))\| \\
&= \|[1-(1-\alpha_n)k_n^i](\nabla f(x_n) - \nabla f(z_n^i)) + (1-\alpha_n)k_n^i(\nabla f(x_n) - \nabla f(T_i^n x_n)) \\
&+ (1-\alpha_n)(k_n^i - 1)\nabla f(T_i^n x_n\| \\
&= \|[1-\alpha_n-(1-\alpha_n)k_n^i+\alpha_n](\nabla f(x_n) - \nabla f(z_n^i)) \\
&+ (1-\alpha_n)k_n^i(\nabla f(x_n) - \nabla f(T_i^n x_n)) + (1-\alpha_n)(k_n^i - 1)\nabla f(T_i^n x_n)\| \\
&= \|(1-\alpha_n)(1-k_n^i)(\nabla f(x_n) - \nabla f(z_n^i)) + \alpha_n(\nabla f(x_n) - \nabla f(z_n^i)) \\
&+ (1-\alpha_n)k_n^i(\nabla f(x_n) - \nabla f(T_i^n x_n)) + (1-\alpha_n)(k_n^i - 1)\nabla f(T_i^n x_n)\| \\
&\ge (1-\alpha_n)k_n^i\|\nabla f(x_n) - \nabla f(T_i^n x_n)\| - \alpha\|\nabla f(x_n) - \nabla f(z_n^i)\| \\
&- (1-\alpha_n)(1-k_n^i)\|\nabla f(x_n) - \nabla f(z_n^i)\| - (1-\alpha_n)(k_n^i - 1)\|\nabla f(T_i^n x_n)\|,
\end{aligned}$$

从而有

$$\begin{aligned}
(1-\alpha_n)k_n^i\|\nabla f(x_n) - \nabla f(T_i^n x_n)\| \le &\|\nabla f(x_n) - \nabla f(u_n^i)\| + \alpha_n\|\nabla f(x_n) - \nabla f(z_n^i)\| \\
&+(1-\alpha_n)(1-k_n^i)\|\nabla f(x_n) - \nabla f(z_n^i)\| \\
&+(1-\alpha_n)(k_n^i - 1)\|\nabla f(T_i^n x_n)\|,
\end{aligned} \tag{3.20}$$

因为$\{\nabla f(x_n)\}, \{f(z_n^i)\}, \{f(T_i^n x_n)\}(\forall i \ge 1)$是有界的, 由(3.19)和$\lim\limits_{n\to\infty} \alpha_n = 0, \{\alpha_n\} \subset [0,1]$, $\lim\limits_{n\to\infty} k_n^i = 1$. 对(3.20)取极限得

$$\lim_{n\to\infty} \|\nabla f(x_n) - \nabla f(T_i^n x_n)\| = 0.$$

由定义2.1的(3)知$(\nabla f)^{-1} = \nabla f^*$, 因为$f$是全凸函数, 由引理2.5的(1)知$f$是序列一致的, 又因为$f$是下半连续的, 所以由引理2.5的(2)知$f$在$E$的有界集上是强制且一致凸的, 故由引理2.3的(1)和(3)知$\nabla f^* = (\nabla f)^{-1}$在$E^*$的有界子集上是范数到范数一致连续的, 故有

$$\lim_{n\to\infty} \|x_n - T_i^n x_n\| = 0. \tag{3.21}$$

由步骤四知$\{x_n\}$是C的Cauchy列, 有$x^*\in C$ s.t. $\lim\limits_{n\to\infty}x_n=x^*$. 故从(3.10), (3.17), (3.21)可得

$$\begin{cases} z_{n+1}^i\to x^*, n\to\infty,\\ u_n^i\to x^*, n\to\infty,\\ T_i^n x_n\to x^*.n\to\infty, \forall i\geq 1.\end{cases}$$

步骤六 证明$x^*\in \Omega$.

先证$x^*\in\cap_{i=1}^{\infty}F(T_i)$. 事实上

$$\|T_i^n x_n-x^*\|=\|T_i^n x_n-x_n+x_n-x^*\|\leq\|T_i^n x_n-x_n\|+\|x_n-x^*\|,$$

由(3.21)和$x_n\to x^*$得

$$\lim_{n\to\infty}\|T_i^n x_n-x^*\|=0. \tag{3.22}$$

注意到

$$\|T_i^{n+1}x_n-x^*\|=\|T_i^{n+1}x_n-T_i^n x_n+T_i^n x_n-x^*\|\leq\|T_i^{n+1}x_n-T_i^n x_n\|+\|T_i^n x_n-x^*\|,$$

由(2.6)和(3.22)知$\lim\limits_{n\to\infty}\|T_i^{n+1}x_n-x^*\|=0$. 即$T_i(T_i^n x_n)\to x^*, n\to\infty$, 由$T_i(\forall i\geq 1)$的闭性得$T_i x^*=x^*(\forall i\geq 1)$. 即得证$x^*\in\cap_{i=1}^{\infty}F(T_i)$.

接下来证明$x^*\in \mathrm{GMEP}(g,\varphi)=\mathrm{EP}(G)=F(T_r^G)$.

由(3.18), $\forall r>0$有

$$\lim_{n\to\infty}\frac{\|\nabla f(z_{n+1}^i)-\nabla f(u_n^i)\|}{r}=0. \tag{3.23}$$

因为$z_{n+1}^i=T_r^G u_n^i(\forall i\geq 1)$, 所以由引理2.8知

$$G(z_{n+1}^i,y)+\frac{1}{r}\langle\nabla f(z_{n+1}^i)-\nabla f(u_n^i), y-z_{n+1}^i\rangle\neq 0, \forall y\in C, i\geq 1. \tag{3.24}$$

由(3.24)和$G(x,y)$满足假设2.1的(2), 用n_k代替(3.24)中的n得

$$\begin{aligned}&\|y-z_{n_k+1}^i\|\frac{\|\nabla f(z_{n_k+1}^i)-\nabla f(u_{n_k}^i)\|}{r}\\ &\geq\frac{1}{r}\langle\nabla f(z_{n_k+1}^i)-\nabla f(u_{n_k}^i), y-z_{n_k+1}^i\rangle\geq -G(z_{n_k+1}^i,y)\geq G(y,z_{n_k+1}^i).\end{aligned} \tag{3.25}$$

由$z_{n+1}^i\to x^*, n\to+\infty$, 知$z_{n_k+1}^i\to x^*, k\to\infty$. $G(x,.)$是下半连续函数和(3.23), 令$k\to\infty$, 对(3.25)取极限得

$$G(y,x^*)\leq 0, \forall y\in C.$$

对$\forall y\in C, t\in(0,1]$, 定义$y_t=ty+(1-t)x^*$, 由$x^*, y\in C$, 注意到$C$是闭凸集得$y_t\in C$, 于是有$G(y_t,x^*)\leq 0$. 由$G(x,y)$满足假设2.1的(1)和(4)知

$$0=G(y_t,y_t)\leq tG(y_t,y)+(1-t)G(y_t,x^*)\leq tG(y_t,y),$$

即

$$G(y_t,y)\geq 0,$$

令$t\to 0^+$, 由$G(x,.)$满足假设2.1的(3)知

$$\limsup\nolimits_{t\to 0^+}G(tz+(1-t)x^*,y)\leq G(x^*,y),$$

上式说明

$$G(x^*,y)\geq 0, \forall y\in C.$$

这说明$x^*\in\mathrm{EP}(G)$, 即$x^*\in\mathrm{GMEP}(g,\varphi)=\mathrm{EP}(G)$. 综上有$x^*\in\Omega$.

步骤七 证明$x^*=\Pi_{\Omega}(x_1)$.

由(2.9)和$x_n=\Pi_{C_n}x_1$知

$$\langle x_n-z,\nabla f(x_1)-\nabla f(x_n)\rangle\geq 0,\forall z\in C_n,$$

因为$\Omega\subset C_n,\forall n\geq 1$, 得

$$\langle x_n-q,\nabla f(x_1)-\nabla f(x_n)\rangle\geq 0,\forall q\in \Omega.$$

对上式两边取极限得

$$\langle x^*-q,\nabla f(x_1)-\nabla f(x_n)\rangle\geq 0,\forall q\in \Omega.$$

由引理2.4知$x^*=\Pi_{\Omega}(x_1)$.

定理3.2 设E是一自反Banach空间, C是E中一非空闭凸子集. $f:E\to(-\infty,+\infty]$是一强制的Legendre函数且在E的有界子集上是有界的, 及一致Frêchet可微且全凸函数, 其中$C\subset\mathrm{int}(\mathrm{dom}f)$. 设$g:C\times C\to\mathbf{R}$为满足假设2.1的二元函数, $\varphi:C\to\mathbf{R}$是一下半连续凸函数, 设$T_i:C\to C(\forall i\geq 1)$是一闭Bregman拟非扩张可数算子族, 假设$\Omega:=\cap_{i=1}^{\infty}F(T_i)\cap\mathrm{GMEP}(g,\varphi)\neq\varnothing$. 设$C_1^i=C(\forall i\geq 1)$和$C_1=\cap_{i=1}^{\infty}C_1^i=C$, 设$\{x_n\}$是由迭代算法

$$\begin{cases}x_1\in C,T_i^nx_1=z_1^i\in C,\\ u_n^i=\nabla f^*(\alpha_n\nabla f(z_n^i)+(1-\alpha_n)\nabla f(T_ix_n)),\\ z_{n+1}^i=T_r^Gu_n^i,\\ C_{n+1}^i=\{z\in C_n:D_f(z,z_{n+1}^i)\leq\alpha_nD_f(z,z_n^i)\}+(1-\alpha_n)D_f(z,x_n),\\ C_{n+1}=\cap_{i=1}^{\infty}C_{n+1}^i,\\ x_{n+1}=\Pi_{C_{n+1}}(x_1),\forall n\geq 1\end{cases}\tag{3.26}$$

生成的序列, 其中$\{\alpha_n\}\subset[0,1]$, s.t. $\lim\limits_{n\longrightarrow\infty}\alpha_n=0,r\in(0,\infty),T_r^G,G(x,y)$如引理2.8中所述, 则$x_n\to\Pi_{\Omega}(x_1)(n\to\infty)$.

证 注意到Bregman拟非扩张算子族是系数序列$\{k_n^i\}$满足$k_n^i\equiv 1(\forall n,i\in\mathbf{N}^+)$的Bregman拟渐近非扩张算子族(这时$\lim\limits_{n\longrightarrow\infty}k_n^i=1$), 这样由定理3.1即得定理3.2.

注3.1 一方面, 定理3.1和3.2说明了定理2.1[9]对一闭Bregman拟非扩张可数算子族或一闭Bregman拟渐近非扩张可数算子族也成立. 因此, 本文的主要结果–定理3.1和3.2是对Jantakarn K和Kaewcharoen A中定理2.1[9]的有益拓展和补充; 另一方面, 收缩投影算法(3.1)和(3.26)比定理2.1[9]中的C-Q算法(2.14)更为简单且算法所需条件更具一般性.

下面给出一个具体数值例子来支撑所得的主要结果–定理3.1和定理3.2.

例3.1 设$E=\mathbf{R}$即实数集并赋通常的绝对值范数, 则E是一自反实Banach空间.

令

$$C=(-\infty,0],f(x)=\frac{2}{3}x^2,$$

则$f:E\to(-\infty,+\infty]$满足定理3.1和3.2的条件且$\nabla f(x)=\dfrac{4}{3}x$. 由定义$f^*(x^*)=\sup\{\langle x^*,x\rangle-f(x)|x\in E\}$得

$$f^*(z)=\frac{3}{8}z^2,\nabla f^*(z)=\frac{3}{4}z.$$

对$T_i: C \to C(\forall i \in N^+)$定义$T_i x = \dfrac{1}{i+1}x, \forall x \in C$可得
$$T_i^n x = \frac{1}{(i+1)^n}x(\forall n, i \in N^+).$$
对$g: C \times C \to \mathbf{R}$定义$g(x,y) = x - y, \forall x, y \in C; \varphi: C \to \mathbf{R}$, 定义$\varphi(x) = x^2, \forall x \in C$.
注意到
$$\begin{aligned} D_f(0, T_i^n x) &= f(0) - f(T_i^n x) - \langle \nabla f(T_i^n x), 0 - T_i^n x\rangle \\ &= 0 - \frac{2}{3}[\frac{1}{(i+1)^n}x]^2 + \langle \frac{4}{3}\frac{1}{(i+1)^n}x, \frac{1}{(i+1)^n}x\rangle \\ &= \frac{4}{3}\frac{1}{(i+1)^{2n}}x^2 - \frac{2}{3}\frac{1}{(i+1)^{2n}}x^2 = \frac{2}{3}\frac{1}{(i+1)^{2n}}x^2 \end{aligned}$$
和
$$D_f(0,x) = f(0) - f(x) - \langle \nabla f(x), 0 - x\rangle = 0 - \frac{2}{3}x^2 + \langle \frac{4}{3}x, x\rangle = \frac{2}{3}x^2,$$
这样有
$$D_f(0, T_i^n x) = \frac{2}{3}\frac{1}{(i+1)^{2n}}x^2 \le \frac{2}{3}x^2 = D_f(0,x).$$
进而得$T_i: C \to C$是一具序列系数$\{k_n^i\} = \{1\}$的Bregman拟渐近非扩张可数算子族. 注意到$\cap_{i=1}^{\infty} F(T_i) = \{0\}$. 对$r > 0, A: C \to E^* = \mathbf{R}$. 定义$Ax = \dfrac{4}{3r}x, \forall c \in C$. 则$A$是一连续单调算子. 又
$$\begin{aligned} G(0,y) &= g(0,y) + \varphi(y) - \varphi(0) + \langle A0, y - 0\rangle \\ &= (0 - y) + y^2 - 0^2 + 0 \\ &= y(y-1) \ge 0, \forall y \in C. \end{aligned}$$
从而$0 \in \mathrm{GMEP}(g,\varphi)$, 故$\Omega = \cap_{i=1}^{\infty} F(T_i) \cap \mathrm{GMEP}(g,\varphi) = \{0\}$, 从而由$x_1 \in C$得$\Pi_{\Omega}(x_1) = \{0\}$. 令$\alpha_n = \dfrac{1}{n}(n = 1, 2, \cdots)$, 注意到
$$T_r^G(x) = \frac{2x}{4+3r}, \forall x \in E.$$
对定理3.1中式(3.1)和定理3.2中式(3.26)生成的迭代序列$\{x_n\}$有
$$\begin{cases} T_i^n x_1 = \dfrac{1}{(i+1)^n}x_1 = z_1^i \in C, x_1 \in C, \\ u_n^i = \nabla f^*[(1 - (1-\alpha_n)k_n^i)\nabla f(z_n^i) + (1-\alpha_n)k_n^i \nabla f(T_i^n x_n)] \\ \qquad = \alpha_n z_n^i + (1-\alpha_n)\dfrac{1}{(1+i)^n}x_n = \dfrac{1}{n}z_n^i + (1 - \dfrac{1}{n})\dfrac{1}{(i+1)^n}x_n, \\ z_{n+1}^i = T_r^G u_n^i = \dfrac{2u_n^i}{4+3r}, \\ C_{n+1}^i = [e_n^i, +\infty) \cap C, \text{其中} e_n^i = \dfrac{(z_{n+1}^i)^2 - \alpha_n (z_n^i)^2 - (1-\alpha_n)x_n^2}{2[z_{n+1}^i - \alpha_n z_n^i - (1-\alpha_n)x_n]}, \\ C_{n+1} = \cap_{i=1}^{\infty} C_{n+1}^i, \\ x_{n+1} = \Pi_{C_{n+1}}(x_1). \end{cases} \tag{3.27}$$
由上面的(3.27)可得
$$\frac{4+3r}{2}z_{n+1}^i = \frac{1}{n}z_n^i + (1 - \frac{1}{n})\frac{1}{(i+1)^2}x_n, \tag{3.28}$$
注意到

$$x_{n+1} \in C_{n+1} = \cap_{i=1}^{\infty}[e_n^i, +\infty) \cap C,$$

并结合数学归纳法可得$\{x_n\}$和$\{z_n^i\}$有界, 结合(3.27)和(3.28)有 $\lim\limits_{n\to\infty} z_n^i = 0(\forall i \in N^+)$和$\{x_n\}$强收敛于$0(n \to \infty)$, 其中$\Pi_{\Omega}(x_1) = \{0\}$. 得定理3.1和3.2成立.

参考文献:

[1] Bregman L M. The relaxation method of finding the common point of convex sets and its application to the solution of problems in convex programming[J]. USSR Computational Mathematics and Mathematical Physics, 1967, 7(3): 200-217.

[2] Blum E,Oettli W.From optimization and variational inequalities to equilibrium problems[J]. The Mathematical Student, 1994, 63: 123-145.

[3] Peng Jianwen, Yao Jenchih. A new hybrid-extragradient method for generalized mixed equilibrium problems, fixed point problems and variational inequality problems[J]. Taiwanese Journal of Mathematics, 2008, 12(6): 1401-1432.

[4] Takahashi W, Zembayashi K. Strong and weak convergence theorems for equilibrium problems and relatively nonexpansive mappings in Banach spaces[J]. Nonlinear Analysis: Theory, Methods Applications, 2009, 70(1): 45-57.

[5] Darvish V, Jantakarn K, Kaewcharoen A, et al. A convergence theorem for solving generalized mixed equilibrium problems and finding fixed points of a weak Bregman relatively nonexpansive mappings in Banach spaces[J]. Acta Mathematica Vietnamica, 2022, 47(2): 553-569.

[6] Alakoya T O, Mewomo O T. Viscosity S-iteration method with inertial technique and self-adaptive step size for split variational inclusion, equilibrium and fixed point problems[J]. Computational and Applied Mathematics, 2022, 41(1): 1-31.

[7] Taiwo A, Mewomo O T. Inertial-viscosity-type algorithms for solving generalized equilibrium and fixed point problems in Hilbert spaces[J]. Vietnam Journal of Mathematics, 2022, 50(1): 125-149.

[8] Kazmi K R, Ali R. Common solution to an equilibrium problem and a fixed point problem for an asymptotically quasi- Φ -nonexpansive mapping in intermediate sense[J]. Revista de la Real Academia de Ciencias Exactas, Fisicas y Naturales-Serie A: Matematicas, 2017, 111(3): 877-889.

[9] Jantakarn K, Kaewcharoen A. Strong convergence theorems for mixed equilibrium problems and Bregman relatively nonexpansive mappings in reflexive Banach spaces[J]. Journal of Nonlinear Sciences and Applications, 2021, 14(2): 63-79.

[10] Bauschke H H, Borwein J M, Combettes P L. Essential smoothness, essential strict convexity, and Legendre functions in Banach spaces[J]. Communications in Contemporary Mathematics, 2001, 3(4): 615-647.

[11] Siwaporn S, Poom K, Kriengsak W. Convergence theorem based on a new hybrid projection method for finding a common solution of generalized equilibrium and variational inequality problems in Banach spaces[J]. Abstract Applied Analysis, 2014, 2010(2): 11-34.

[12] Hanjing A, Suantai S. Hybrid inertial accelerated algorithms for split fixed point problems of demicontractive mappings and equilibrium problems[J]. Numerical Algorithms, 2020, 85(3): 1051-1073.

[13] Ni Renxing, Yao Jenchih. The modified Ishikawa iterative algorithm with errors for a countable family of Bregman totally quasi-D-asymptotically nonexpansive mappings in reflexive Banach spaces[J]. Fixed Point Theory and Applications, 2015, 2015(1): 35-44.

[14] Butnariu D, Iusem A N. Totally Convex Functions for Fixed Points Computation and Infinite Dimensional Optimization[M]. Berlin: Springer Science Business Media, 2000.

[15] Eskandani G Z, Raeisi M. Solving a general split equality problem without prior knowledge of operator norms in Banach spaces[J]. Results in Mathematics, 2021, 76(1): 1-25.

[16] Homidan S A, Ali B, Suleiman Y I. Generalized split feasibility problem for multi-valued Bregman quasi-nonexpansive mappings in Banach spaces[J]. Applied Numerical Mathematics, 2020, 161(1): 437-451.

[17] Reich S, Sabach S. A strong convergence theorem for a proximal-type algorithm in reflexive Banach spaces[J]. Journal of Nonlinear and Convex Analysis, 2009, 10(3): 471-485.

[18] Butnariu D, Resmerita E. Bregman distances, totally convex functions, and a method for solving operator equations in Banach spaces[J]. Abstract Applied Analysis, 2006, 2006(2): 99-121.

[19] Sabach S. Products of finitely many resolvents of maximal monotone mappings in reflexive Banach spaces[J]. SIAM Journal on Optimization, 2011, 21(4): 1289-1308.

[20] Izuchukwu C, Ogbuisi F U, Mewomo O T. A common solution of split equality monotone inclusion problem and split equality fixed point problem in real Banach spaces[J]. Advances in Operator Theory, 2020, 6(1): 11-19.

Strong convergence of common solutions of the fixed point sets of countable operators family and the generalized mixed equilibrium problems in Banach Spaces

NI Ren-xing, XU Ya-jun

(Department of Mathematical, Shaoxing University, Shaoxing 312000, China)

Abstract: In the framework of reflexive Banach space, a new shrinking projection algorithm is proposed to approximate the common solutions of the solution of the generalized mixed equilibrium problems and the fixed point sets of a closed Bregman quasi-asymptotically nonexpansive countable operators family, and the strong convergence results of the common solutions of the solution of the generalized mixed equilibrium problems and the fixed point sets of a closed Bregman quasi-asymptotically nonexpansive countable operators family are established. A numerical example of the iterative algorithm supporting our main results is presented. The results of Jantakarn K & Kaewcharoen A(2021) are extended and supplemented, whether the operator family is from finite to countable, or from Bregman relatively nonexpansive to Bregman quasi-asymptotically nonexpansive.

Keywords: generalized mixed equilibrium problem; Bregman relatively nonexpansive mapping; Bregman quasi asymptotically nonexpansive mapping; shrinking projection method; strong convergence

MR Subject Classification: 47H09; 47H10; 47J20; 49J40