

具有进化效应的SIR模型的稳定性和Hopf分支分析

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摘 要: 主要研究一类具有进化效应的SIR模型的动力学行为. 首先, 建立数学模型, 并证明系统解的存在性, 正性, 有界性等基本性质. 其次, 分析系统正平衡点的稳定性, 并给出健康个体的防御成本和已感染个体的感染能力引起健康个体和已感染个体数量振荡的充分条件. 然后, 利用中心流形定理, 讨论Hopf分支的性质. 最后, 利用合适的参数值, 对系统进行数值模拟.

关键词: SIR模型; 进化效应; 稳定性; Hopf分支

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§1 引 言

一直以来, 许多学者都利用SIR模型研究传染病的传播规律^[1-7]. 本文研究以健康个体, 已感染个体和已恢复个体为研究对象的SIR模型, 是基于文献[8]中研究的模型, 进一步, 考虑健康个体自身的防御成本和已感染个体感染能力的适应性变化对系统动力学行为的进化效应影响^[9-11]. 由于健康个体往往通过自身免疫系统产生防御作用来驱动健康个体适应环境, 同时健康个体防御成本的降低可能导致病毒的变异, 进而增加攻击能力. 于是这类问题的研究更贴近现实生活, 更有实际意义.

与现有文献中的SIR模型相比, 考虑到健康个体的内禀增长率取决于自身对病毒的防御成本, 同时, 感染率由健康人群防御成本和已感染者感染能力共同决定. 于是考虑的模型为

$$\begin{cases} \frac{dS}{dt} = r(u) \left(1 - \frac{S+I}{K} \right) S - \frac{\beta(u,v)SI}{S+I}, \\ \frac{dI}{dt} = \frac{\beta(u,v)SI}{S+I} - (\delta + \gamma)I, \\ \frac{dR}{dt} = \gamma I - \mu R. \end{cases} \quad (1)$$

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模型参数的生物学意义见表1, 其中 S 表示健康个体的数量, I 表示已感染个体的数量, R 表示已恢复个体的数量. 函数 $r(u)$, $\beta(u, v)$ 的假设如下.

(1) 防御成本 u 和健康个体的内禀增长率 r 之间的关系假设为 $r(u) = r_0 e^{-c_1 u^2}$ (参见[9]). 因此随着人群总数接近环境承载能力, r 对动力学的影响减小. 当健康人群占有率低时, 相同水平的防御成本较高, 而当健康人群占有率高时, 相同水平的防御成本较低.

(2) 易感人群的感染率为 $\beta(u, v) = \frac{\beta_0}{1 + e^{\theta(u-v)}}$, 这是由防御成本和感染能力决定的^[10]. 感染率随着防御成本的增加而减小, 随着感染能力的增强而增加.

表1模型(1)符号的生物意义

参数	描述
u	健康个体的防御成本
v	已感染个体的感染能力
$r(u)$	内禀增长率
K	环境最大容纳量
δ	已感染个体的死亡率
$\beta(u, v)$	感染率
γ	个体的恢复率
μ	已恢复个体的死亡率
r_0	最大增长率
c_1	防御成本
β_0	最大感染率
θ	感染率在不同人群之间差异的敏感度

这里系统(1)的初始条件为 $S(0) = S_0 \geq 0, I(0) = I_0 \geq 0, R(0) = R_0 \geq 0$. 并且为满足实际的生物意义, 假设 $S(0) + I(0) < K$.

由于系统(1)中的第三个方程与前两个方程解耦, 因此可以将所研究的系统简化为

$$\begin{cases} \frac{dS}{dt} = r(u) \left(1 - \frac{S+I}{K}\right) S - \frac{\beta(u, v)SI}{S+I}, \\ \frac{dI}{dt} = \frac{\beta(u, v)SI}{S+I} - (\delta + \gamma)I. \end{cases} \quad (2)$$

§2 模型的基本性质

在这一部分中, 主要是给出所研究系统(2)的解的存在唯一性, 正性和有界性.

定理2.1 在初始条件下, 系统(2)的解对任意 $t > 0$ 是存在且唯一的, 并且保持非负有界.

证 在初始条件下, 系统(2)关于 (S, I) 满足局部Lipschitz条件, 说明存在某个 $c > 0$, 使得其解在区间 $[0, c)$ 内存在且唯一. 为了证明正性, 首先证明对于任意 $t > 0$, S 和 I 的有界性. 由于 $S(0) = 0$, 意味着 $S(t) \equiv 0$. 所以假设 $S(0) > 0$, 对于系统(2)的第一个方程来说,

$$S'(t) = S(t) \left[r(u) \left(1 - \frac{S(t) + I(t)}{K}\right) - \frac{\beta(u, v)I(t)}{S(t) + I(t)} \right],$$

对于任意 $t > 0$ 得到 $S(t) = S(0)e^{\int_0^t [r(u)(1 - \frac{S(x) + I(x)}{K}) - \frac{\beta(u, v)I(x)}{S(x) + I(x)}] dx}$. 由此可得 S 的正性. 同理可得 I 的正性.

于是 $(S+I)' = r(u) \left(1 - \frac{S+I}{K}\right) - (\delta + \gamma)I \leq -\frac{r_0}{K}(S+I-K)S$, 在 $S(0) + I(0) < K$ 的条件下, 可得出 $S(t) + I(t) \leq K + [S(0) + I(0) - K]e^{-\frac{r_0}{K} \int_0^t S(z)dz} \leq K$. 即证得 S 和 I 的有界性.

另外对任意 $t > 0$, 由系统(2)的解的正性可得

$$\dot{S}(t) \leq r(u) \left(1 - \frac{S+I}{K}\right) S,$$

从而 $\lim_{t \rightarrow \infty} S(t) \leq K$. 令 $V(t) = S(t) + I(t)$, 计算 $V(t)$ 关于时间 t 的导数得

$$\begin{aligned} \dot{V}(t) &= \dot{S}(t) + \dot{I}(t) \\ &= r(u) \left(1 - \frac{S(t) + I(t)}{K}\right) S(t) - (\delta + \gamma)I(t). \end{aligned}$$

对于一个非常小的正数 m , 不妨设 $m < (\delta + \gamma)$, 有

$$\dot{V}(t) + mV(t) \leq r(u) \left(1 - \frac{S(t)}{K}\right) S(t) + mS(t).$$

因此存在一个正常数, 使得 $\dot{V}(t) + mV(t) \leq L$. 因此可以得到

$$V(t) \leq \left(V(0) - \frac{L}{m}\right)e^{-mt} + \frac{L}{m},$$

则 $\lim_{t \rightarrow \infty} [S(t) + I(t)] \leq \frac{L}{m}$. 因此, 系统(2)中解是最终一致有界的.

§3 稳定性与分支分析

下面对系统(2)在平衡点处进行稳定性和分支分析, 并给出分支周期解存在的充分条件.

首先, 研究系统平衡点的稳定性. 显然系统(2)存在唯一的无病平衡点 $E_1 = (K, 0)$.

定理3.1 当 $\beta(u, v) < (\delta + \gamma)$ 时, 无病平衡点 E_1 是局部渐近稳定的. 反之 E_1 是不稳定的.

证 系统(2)在无病平衡点 $E_1 = (K, 0)$ 处的 Jacobian 矩阵是

$$J_{E_1} = \begin{pmatrix} -r(u) & -r(u) - \beta(u, v) \\ 0 & \beta(u, v) - (\delta + \gamma) \end{pmatrix},$$

在 E_1 处的特征方程为

$$(-r(u) - \lambda)(\beta(u, v) - (\delta + \gamma) - \lambda) = 0.$$

显然只需 $\beta(u, v) < (\delta + \gamma)$, 就有 $\lambda_1 < 0, \lambda_2 < 0$. 故该系统的无病平衡点 E_1 是局部渐近稳定的. 反之, 当 $\beta(u, v) > (\delta + \gamma)$ 时, E_1 是不稳定的.

定理3.2 当 $\beta(u, v) \leq (\delta + \gamma)$ 时, 无病平衡点 E_1 是全局渐近稳定的.

证 构造 Lyapunov 函数 $V(t) = I(t)$, 于是

$$\dot{V} = \dot{I} = \frac{\beta(u, v)SI}{S+I} - (\delta + \gamma)I < \frac{\beta(u, v)S - (\delta + \gamma)S}{S}I = (\beta(u, v) - (\delta + \gamma))S.$$

当 $\beta(u, v) \leq (\delta + \gamma)$ 时, $\dot{V} \leq 0$. 当 $\dot{V} = 0$ 时, 当且仅当 $S = K, I = 0$, 故依据 LaSalle 不变集原理^[12]可得, 无病平衡点 E_1 是全局渐近稳定的.

下面分析正平衡点 $E_* = (S_*, I_*)$ 的存在性和稳定性. 当系统参数满足下面条件

$$(\delta + \gamma) < \beta(u, v) < r(u) + (\delta + \gamma)$$

时, 系统(2)的平衡点 $E_* = (S_*, I_*)$ 唯一存在, 为

$$\begin{cases} S_* = \frac{(r(u) - \beta(u, v) + (\delta + \gamma))(\delta + \gamma)K}{r(u)\beta(u, v)}, \\ I_* = \frac{S_*(\beta(u, v) - (\delta + \gamma))}{\delta + \gamma}. \end{cases}$$

于是 E_* 处的Jacobian矩阵为

$$J_{E_*} = \begin{pmatrix} a_{11} & b_{11} \\ a_{22} & b_{22} \end{pmatrix},$$

其中

$$a_{11} = r(u) \left(1 - \frac{2S_* + I_*}{K} \right) - \frac{\beta(u, v) I_*^2}{(S_* + I_*)^2}, \quad a_{22} = \frac{\beta(u, v) I_*^2}{(S_* + I_*)^2},$$

$$b_{11} = -\frac{r(u) S_*}{K} - \frac{\beta(u, v) S_*^2}{(S_* + I_*)^2}, \quad b_{22} = \frac{\beta(u, v) S_*^2}{(S_* + I_*)^2} - (\delta + \gamma).$$

这里很显然可知 $a_{22} > 0$, $b_{11} < 0$, $b_{22} < 0$. 计算 E_* 处Jacobian矩阵对应的特征方程为

$$\lambda^2 - \text{Tr}(J_{E_*})\lambda + \text{Det}(J_{E_*}) = 0,$$

其中

$$\text{Tr}(J_{E_*}) = a_{11} + b_{22}, \quad \text{Det}(J_{E_*}) = a_{11}b_{22} - a_{22}b_{11}.$$

于是特征方程的特征根为

$$\lambda_{1,2} = \alpha(\beta(u, v)) \pm iw(\beta(u, v)) = \frac{\text{Tr}(J_{E_*}) \pm \sqrt{\text{Tr}(J_{E_*})^2 - 4\text{Det}(J_{E_*})}}{2}.$$

那么特征方程的根有负实部当且仅当

$$a_{11} + b_{22} < 0, \quad \left(\beta(u, v) < \left(\delta + \gamma - r(u) \left(1 - \frac{2S_* + I_*}{K} \right) \right) \cdot \left(\frac{S_* + I_*}{S_* - I_*} \right) \right)$$

且

$$a_{11}b_{22} - a_{22}b_{11} > 0 \text{ (H1)}.$$

下面主要考虑 $\beta(u, v)$ 的变化所引起对系统动力学性质的影响. 在分析过程中以 $\beta(u, v)$ 作为分支参数. 并且, 当 $(\delta + \gamma) < \beta(u, v) < 2(\delta + \gamma)$ 随着 $\beta(u, v)$ 的变化, 特征根自左向右穿过纯虚轴时, 可以得到

$$\left\{ \frac{d(\text{Re } \lambda(\beta(u, v)))}{d\beta(u, v)} \right\}_{\lambda=iw, \beta(u, v)=\beta} = \frac{S_* - I_*}{S_* + I_*} > 0.$$

定理3.3 假设(H1)成立, 对于系统(1)有如下结果.

(1) 如果 $a_{11} < 0$ 且 $\beta(u, v) < \frac{(\delta + \gamma)(S_* + I_*)^2}{S_*^2}$ 成立, 则平衡点 E_* 局部渐近稳定.

(2) 如果 $a_{11} > 0$ 且 $\beta(u, v) < \left((\delta + \gamma) - r(u) \left(1 - \frac{2S_* + I_*}{K} \right) \right) \left(\frac{S_* + I_*}{S_* - I_*} \right)$ 成立, 则平衡点 E_* 局部渐近稳定.

(3) 如果 $a_{11} > 0$ 且 $b_{22} < 0$, 当 $\beta(u, v) = \left((\delta + \gamma) - r(u) \left(1 - \frac{2S_* + I_*}{K} \right) \right) \left(\frac{S_* + I_*}{S_* - I_*} \right) < 2(\delta + \gamma)$ 时, 则系统的平衡点 E_* 处产生Hopf 分支.

接下来给出分支周期解的稳定性. 为此通过平移 $\bar{x} = S - S_*$, $\bar{y} = I - I_*$, 将 E_* 平移到原点. 因此系统(2)变为

$$\begin{cases} \dot{\bar{x}} = r(u) \left(1 - \frac{(\bar{x} + S_*) + (\bar{y} + I_*)}{K} \right) (\bar{x} + S_*) - \frac{\beta(u, v) (\bar{x} + S_*) (\bar{y} + I_*)}{(\bar{x} + S_*) + (\bar{y} + I_*)}, \\ \dot{\bar{y}} = \frac{\beta(u, v) (\bar{x} + S_*) (\bar{y} + I_*)}{(\bar{x} + S_*) + (\bar{y} + I_*)} - (\delta + \gamma) (\bar{y} + I_*). \end{cases}$$

可写成

$$\begin{pmatrix} \dot{\bar{x}} \\ \dot{\bar{y}} \end{pmatrix} = J \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} + \begin{pmatrix} f(\bar{x}, \bar{y}, \beta(u, v)) \\ h(\bar{x}, \bar{y}, \beta(u, v)) \end{pmatrix}.$$

其中

$$\begin{aligned} f(\bar{x}, \bar{y}, \beta(u, v)) &= \bar{x}^2 \left(\frac{I_*^2(\delta + \gamma)^3}{\beta(u, v)^2 S_*^3} - \frac{r(u)}{K} \right) - \bar{x}\bar{y} \left(\frac{r(u)}{K} + \frac{2I_*(\delta + \gamma)^3}{(\beta(u, v)S_*)^2} \right) + \bar{y}^2 \left(\frac{(\delta + \gamma)^3}{\beta(u, v)^2 S_*^3} \right) \\ &\quad - \bar{x}^3 \left(\frac{I_*^2(\delta + \gamma)^4}{\beta(u, v)^3 S_*^4} \right) + \bar{x}^2\bar{y} \left(\frac{(2S_*I_* - I_*^2)(\delta + \gamma)^4}{\beta(u, v)^3 S_*^4} \right) + \bar{x}\bar{y}^2 \left(\frac{(2I_* - S_*)(\delta + \gamma)^4}{(\beta(u, v)S_*)^3} \right) \\ &\quad - \bar{y}^3 \frac{(\delta + \gamma)^4}{\beta(u, v)^3 S_*^2} + O(|(\bar{x}, \bar{y})|^4), \end{aligned}$$

$$\begin{aligned} h(\bar{x}, \bar{y}, \beta(u, v)) &= -\bar{x}^2 \left(\frac{I_*^2(\delta + \gamma)^3}{\beta(u, v)^2 S_*^3} \right) + \bar{x}\bar{y} \left(\frac{2I_*(\delta + \gamma)^3}{(\beta(u, v)S_*)^2} \right) - \bar{y}^2 \left(\frac{(\delta + \gamma)^3}{\beta(u, v)^2 S_*^3} \right) \\ &\quad + \bar{x}^3 \left(\frac{I_*^2(\delta + \gamma)^4}{\beta(u, v)^3 S_*^4} \right) + \bar{x}^2\bar{y} \left(\frac{(I_*^2 - 2S_*I_*)(\delta + \gamma)^4}{\beta(u, v)^3 S_*^4} \right) + \bar{x}\bar{y}^2 \left(\frac{(S_* - 2I_*)(\delta + \gamma)^4}{(\beta(u, v)S_*)^3} \right) \\ &\quad + \bar{y}^3 \frac{(\delta + \gamma)^4}{\beta(u, v)^3 S_*^2} + O(|(\bar{x}, \bar{y})|^4). \end{aligned}$$

记 $\beta(u, v) = \beta$, 则 $\lambda_{1,2}(\beta) = \pm iw$, $w = \sqrt{a_{11}b_{22} - a_{22}b_{11}}$ 时, 对应于特征值 $\lambda_{1,2}(\beta)$ 的一个特征向量是 $\zeta = (-b_{11}, a_{11} - iw)^T$, 有

$$B = \begin{pmatrix} -b_{11} & 0 \\ a_{11} & -w \end{pmatrix} \quad B^{-1} = \begin{pmatrix} -\frac{1}{b_{11}} & 0 \\ -\frac{a_{11}}{wb_{11}} & -\frac{1}{w} \end{pmatrix}.$$

通过转换 $\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = B \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix}$, 得到

$$\begin{pmatrix} \dot{\tilde{x}} \\ \dot{\tilde{y}} \end{pmatrix} = \begin{pmatrix} 0 & -w \\ w & 0 \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} + \begin{pmatrix} f_1(\tilde{x}, \tilde{y}) \\ h_1(\tilde{x}, \tilde{y}) \end{pmatrix},$$

即

$$f_1(\tilde{x}, \tilde{y}) = -\frac{1}{b_{11}} f(-b_{11}\tilde{x}, a_{11}\tilde{x} - w\tilde{y}, \beta(u, v)),$$

$$h_1(\tilde{x}, \tilde{y}) = -\frac{1}{w} h(-b_{11}\tilde{x}, a_{11}\tilde{x} - w\tilde{y}, \beta(u, v)) - \frac{a_{11}}{wb_{11}} f(-b_{11}\tilde{x}, a_{11}\tilde{x} - w\tilde{y}, \beta(u, v)),$$

其中

$$\begin{aligned} &f(-b_{11}\tilde{x}, a_{11}\tilde{x} - w\tilde{y}, \beta(u, v)) \\ &= b_{11}^2 \tilde{x}^2 \left(\frac{I_*^2(\delta + \gamma)^3}{\beta(u, v)^2 S_*^3} - \frac{r(u)}{K} \right) + b_{11}\tilde{x} (a_{11}\tilde{x} - w\tilde{y}) \left(\frac{r(u)}{K} + \frac{2I_*(\delta + \gamma)^3}{(\beta(u, v)S_*)^2} \right) \\ &\quad + (a_{11}\tilde{x} - w\tilde{y})^2 \left(\frac{(\delta + \gamma)^3}{\beta(u, v)^2 S_*^3} \right) + (b_{11}\tilde{x})^3 \left(\frac{I_*^2(\delta + \gamma)^4}{\beta(u, v)^3 S_*^4} \right) \\ &\quad + (b_{11}\tilde{x})^2 (a_{11}\tilde{x} - w\tilde{y}) \left(\frac{(2S_*I_* - I_*^2)(\delta + \gamma)^4}{\beta(u, v)^3 S_*^4} \right) - b_{11}\tilde{x} (a_{11}\tilde{x} - w\tilde{y})^2 \left(\frac{(2I_* - S_*)(\delta + \gamma)^4}{(\beta(u, v)S_*)^3} \right) \\ &\quad - (a_{11}\tilde{x} - w\tilde{y})^3 \frac{(\delta + \gamma)^4}{\beta(u, v)^3 S_*^2} + O(|(-b_{11}\tilde{x}, a_{11}\tilde{x} - w\tilde{y})|^4), \end{aligned}$$

$$\begin{aligned}
& h(-b_{11}\tilde{x}, a_{11}\tilde{x} - w\tilde{y}, \beta(u, v)) \\
&= -b_{11}^2\tilde{x}^2 \left(\frac{I_*^2(\delta + \gamma)^3}{\beta(u, v)^2 S_*^3} \right) - b_{11}\tilde{x}(a_{11}\tilde{x} - w\tilde{y}) \left(\frac{2I_*(\delta + \gamma)^3}{(\beta(u, v)S_*)^2} \right) \\
&- (a_{11}\tilde{x} - w\tilde{y})^2 \left(\frac{(\delta + \gamma)^3}{\beta(u, v)^2 S_*} \right) - (b_{11}\tilde{x})^3 \left(\frac{I_*^2(\delta + \gamma)^4}{\beta(u, v)^3 S_*^4} \right) \\
&+ (b_{11}\tilde{x})^2 (a_{11}\tilde{x} - w\tilde{y}) \left(\frac{(I_*^2 - 2S_*I_*)(\delta + \gamma)^4}{\beta(u, v)^3 S_*^4} \right) - b_{11}\tilde{x}(a_{11}\tilde{x} - w\tilde{y})^2 \left(\frac{(S_* - 2I_*)(\delta + \gamma)^4}{(\beta(u, v)S_*)^3} \right) \\
&+ (a_{11}\tilde{x} - w\tilde{y})^3 \frac{(\delta + \gamma)^4}{\beta(u, v)^3 S_*^2} + O(|(-b_{11}\tilde{x}, a_{11}\tilde{x} - w\tilde{y})|^4).
\end{aligned}$$

将关于 $\tilde{x}$ 和 $\tilde{y}$ 的方程改写为以下极坐标形式

$$\begin{cases} \dot{r} = \alpha(\beta(u, v))r + \sigma(\beta(u, v))r^3 + O(|r|^4), \\ \dot{\theta} = w(\beta(u, v)) + c(\beta(u, v))r^2 + O(|r|^4), \end{cases}$$

在 $\beta(u, v) = \beta$ 处进行泰勒展开得

$$\begin{cases} \dot{r} = \alpha'(\beta)(\beta(u, v) - \beta)r + \sigma(\beta)r^3 + O(|r|^4), \\ \dot{\theta} = w(\beta) + w'(\beta)(\beta(u, v) - \beta) + c(\beta)r^2 + O(|r|^4). \end{cases}$$

周期解的稳定性是由 $\sigma(\beta)$ 决定的, 所以有

$$\begin{aligned}
\sigma(\beta) &= \frac{1}{16} (f_{1\tilde{x}\tilde{x}\tilde{x}} + f_{1\tilde{x}\tilde{y}\tilde{y}} + h_{1\tilde{x}\tilde{x}\tilde{y}} + h_{1\tilde{y}\tilde{y}\tilde{y}}) \\
&+ \frac{1}{16w(\beta)} [f_{1\tilde{x}\tilde{y}}(f_{1\tilde{x}\tilde{x}} + f_{1\tilde{y}\tilde{y}}) - h_{1\tilde{y}\tilde{x}}(h_{1\tilde{x}\tilde{x}} + h_{1\tilde{y}\tilde{y}}) - f_{1\tilde{x}\tilde{x}}h_{1\tilde{x}\tilde{x}} + f_{1\tilde{y}\tilde{y}}h_{1\tilde{y}\tilde{y}}],
\end{aligned}$$

其中

$$\begin{aligned}
f_{1\tilde{x}\tilde{x}\tilde{x}} &= -\frac{6b_{11}^2 I_*^2(\delta + \gamma)^4}{\beta(u, v)^3 S_*^4} - \frac{6a_{11}b_{11}(2S_*I_* - I_*^2)(\delta + \gamma)^4}{\beta(u, v)^3 S_*^4} + \frac{6a_{11}^2(2I_* - S_*)(\delta + \gamma)^4}{(\beta(u, v)S_*)^3} + \frac{6a_{11}^3(\delta + \gamma)^4}{b_{11}\beta(u, v)^3 S_*^2}, \\
f_{1\tilde{x}\tilde{y}\tilde{y}} &= \frac{2w^2(2I_* - S_*)(\delta + \gamma)^4}{(\beta(u, v)S_*)^3} - \frac{6a_{11}w^2(\delta + \gamma)^4}{b_{11}\beta(u, v)^3 S_*^2}, \\
h_{1\tilde{x}\tilde{x}\tilde{y}} &= \left[\frac{2b_{11}^2(I_*^2 - 2S_*I_*)(\delta + \gamma)^4}{\beta(u, v)^3 S_*^4} - \frac{4a_{11}b_{11}(S_* - 2I_*)(\delta + \gamma)^4}{(\beta(u, v)S_*)^3} + \frac{6a_{11}^2(\delta + \gamma)^4}{\beta(u, v)^3 S_*^2} \right] \\
&+ \left[\frac{2a_{11}b_{11}(2S_*I_* - I_*^2)(\delta + \gamma)^4}{\beta(u, v)^3 S_*^4} - \frac{4a_{11}^2(2I_* - S_*)(\delta + \gamma)^4}{(\beta(u, v)S_*)^3} + \frac{6a_{11}^3(\delta + \gamma)^4}{b_{11}\beta(u, v)^3 S_*^2} \right], \\
h_{1\tilde{y}\tilde{y}\tilde{y}} &= \frac{6w^2(\delta + \gamma)^4}{\beta(u, v)^3 S_*^2} + \frac{6a_{11}w^2(\delta + \gamma)^4}{b_{11}\beta(u, v)^3 S_*^2}, \\
f_{1\tilde{x}\tilde{y}} &= w \left[\frac{r(u)}{K} + \frac{2I_*(\delta + \gamma)^3}{(\beta(u, v)S_*)^2} \right] + \frac{2wa_{11}(\delta + \gamma)^3}{b_{11}\beta(u, v)^2 S_*} + \frac{2b_{11}w\tilde{x}(2S_*I_* - I_*^2)(\delta + \gamma)^4}{\beta(u, v)^3 S_*^4} \\
&- 2w(2a_{11}\tilde{x} - w\tilde{y}) \left(\frac{(2I_* - S_*)(\delta + \gamma)^4}{(\beta(u, v)S_*)^3} \right) + \frac{6a_{11}w(a_{11}\tilde{x} - w\tilde{y})(\delta + \gamma)^4}{b_{11}\beta(u, v)^3 S_*^2},
\end{aligned}$$

$$\begin{aligned}
f_{1\tilde{x}\tilde{x}} &= 2b_{11} \left(\frac{r(u)}{K} - \frac{I_*^2(\delta + \gamma)^3}{\beta(u, v)^2 S_*^3} \right) \\
&\quad - 2a_{11} \left(\frac{r(u)}{K} + \frac{2I_*(\delta + \gamma)^3}{(\beta(u, v)S_*)^2} \right) - \frac{2a_{11}^2(\delta + \gamma)^3}{b_{11}\beta(u, v)^2 S_*} - \frac{6b_{11}^2 I_*^2(\delta + \gamma)^4 \tilde{x}}{\beta(u, v)^3 S_*^4} \\
&\quad - b_{11} (6a_{11}\tilde{x} - 2w\tilde{y}) \left(\frac{(2S_* I_* - I_*^2)(\delta + \gamma)^4}{\beta(u, v)^3 S_*^4} \right) + (6a_{11}^2 \tilde{x} - 4a_{11}w\tilde{y}) \left(\frac{(2I_* - S_*)(\delta + \gamma)^4}{(\beta(u, v)S_*)^3} \right) \\
&\quad - \frac{6a_{11}^2 (a_{11}\tilde{x} - w\tilde{y})(\delta + \gamma)^4}{b_{11}\beta(u, v)^3 S_*^2}, \\
f_{1\tilde{y}\tilde{y}} &= -\frac{2w^2(\delta + \gamma)^3}{b_{11}\beta(u, v)^2 S_*} + \frac{2w^2 \tilde{x} (2I_* - S_*)(\delta + \gamma)^4}{(\beta(u, v)S_*)^3} - \frac{6w^2 (a_{11}\tilde{x} - w\tilde{y})(\delta + \gamma)^4}{b_{11}\beta(u, v)^3 S_*^2}, \\
h_{1\tilde{x}\tilde{y}} &= -\frac{2b_{11}I_*(\delta + \gamma)^3}{(\beta(u, v)S_*)^2} - \frac{2a_{11}(\delta + \gamma)^3}{\beta(u, v)^2 S_*} + \frac{2b_{11}^2 \tilde{x} (I_*^2 - 2S_* I_*)(\delta + \gamma)^4}{\beta(u, v)^3 S_*^4} \\
&\quad - b_{11} (2a_{11}\tilde{x} - w\tilde{y}) \left(\frac{(S_* - 2I_*)(\delta + \gamma)^4}{(\beta(u, v)S_*)^3} \right) + 6a_{11} (a_{11}\tilde{x} - w\tilde{y}) \frac{(\delta + \gamma)^4}{\beta(u, v)^3 S_*^2} + \frac{a_{11}}{w} f_{1\tilde{x}\tilde{y}}, \\
h_{1\tilde{y}\tilde{y}} &= \frac{2w(\delta + \gamma)^3}{\beta(u, v)^2 S_*} + \frac{wb_{11}\tilde{x} (S_* - 2I_*)(\delta + \gamma)^4}{(\beta(u, v)S_*)^3} - 6w (a_{11}\tilde{x} - w\tilde{y}) \frac{(\delta + \gamma)^4}{\beta(u, v)^3 S_*^2} + \frac{a_{11}}{w} f_{1\tilde{y}\tilde{y}}, \\
h_{1\tilde{x}\tilde{x}} &= -\frac{1}{w} \left[-\frac{2b_{11}^2 I_*^2(\delta + \gamma)^3}{\beta(u, v)^2 S_*^3} - \frac{4a_{11}b_{11}I_*(\delta + \gamma)^3}{(\beta(u, v)S_*)^2} - \frac{2a_{11}^2(\delta + \gamma)^3}{\beta(u, v)^2 S_*} - \frac{6b_{11}^3 \tilde{x} I_*^2(\delta + \gamma)^4}{\beta(u, v)^3 S_*^4} \right. \\
&\quad \left. + b_{11}^2 (6a_{11}\tilde{x} - 2w\tilde{y}) - b_{11} (6a_{11}^2 \tilde{x} - 4a_{11}w\tilde{y}) \left(\frac{(S_* - 2I_*)(\delta + \gamma)^4}{(\beta(u, v)S_*)^3} \right) \right. \\
&\quad \left. + 6a_{11}^2 (a_{11}\tilde{x} - w\tilde{y}) \frac{(\delta + \gamma)^4}{\beta(u, v)^3 S_*^2} \right] + \frac{a_{11}}{w} f_{1\tilde{x}\tilde{x}}.
\end{aligned}$$

根据参考文献[13-15]的方法, 可知如果 $\alpha(\beta) < 0$, 分支周期解是稳定的, 相反如果 $\alpha(\beta) > 0$, 分支周期解是不稳定的.

§4 数值模拟

在本节中, 通过考虑以下参数值进行一些数值模拟来验证本文的理论结果. 接下来考虑四组参数值:

- (1) $r_0 = 0.95, c_1 = 0.4, K = 6, \beta_0 = 1.85, \theta = 5, u = 0.2, v = 0.3, (\delta + \gamma) = 1.2$;
- (2) $r_0 = 0.95, c_1 = 0.4, K = 6, \beta_0 = 1.85, \theta = 5, u = 0.2, v = 0.3, (\delta + \gamma) = 0.95$;
- (3) $r_0 = 0.95, c_1 = 0.4, K = 6, \beta_0 = 1.85, \theta = 5, u = 0.2, v = 0.28, (\delta + \gamma) = 0.95$;
- (4) $r_0 = 0.95, c_1 = 0.4, K = 6, \beta_0 = 1.85, \theta = 5, u = 0.16, v = 0.3, (\delta + \gamma) = 0.95$.

通过以上图1-图4可以看出进化效应对系统的影响, 在其他参数不变的情况下, 随着 v 的减小, $\beta(u, v)$ 减小, I^* 减小, 而随着 u 的减小, $r(u)$ 增大, I^* 减小. 也就是说, 已感染个体的传染能力减弱, 使得个体间接触的感染率降低, 则平衡态已感染个体数量减少, 健康个体的防御能力减弱, 则平衡态健康个体数量减少.

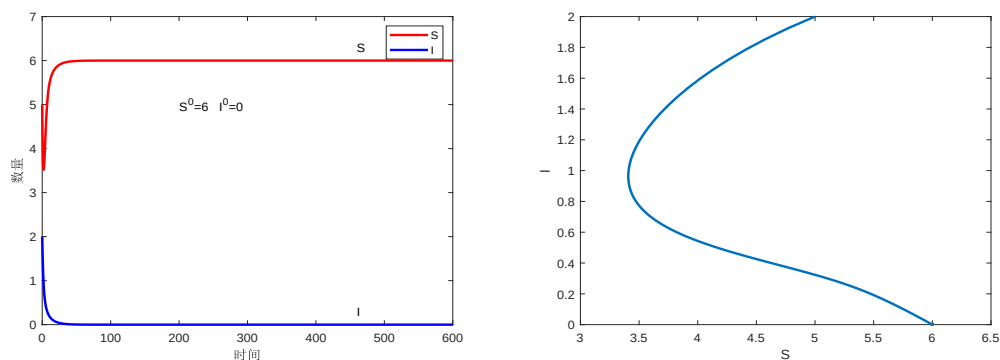


图1 系统(2)在 $u = 0.2, v = 0.3, r(u) = 0.9349, \beta(u, v) = 1.1515, K = 6, (\delta + \gamma) = 1.2$ 和初始条件为 $(5, 2)$ 时的数值模拟,此时满足 $\beta(u, v) < (\delta + \gamma)$,无病平衡点 $E_0 = (6, 0)$ 全局渐近稳定

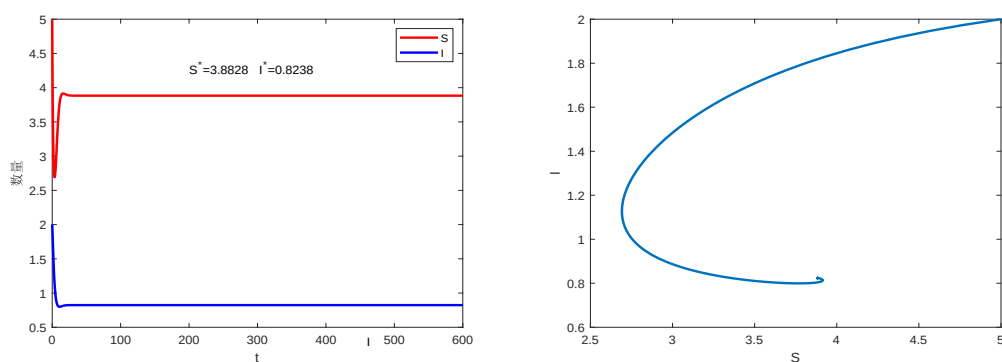


图2 系统(2)在 $u = 0.2, v = 0.3, r(u) = 0.9349, \beta(u, v) = 1.1515, K = 6, (\delta + \gamma) = 0.95$ 和初始条件为 $(5, 2)$ 时的数值模拟,此时满足 $(\delta + \gamma) < \beta(u, v) < (\delta + \gamma) + r(u)$,正平衡点为 $E_* = (3.8828, 0.8238)$ 是局部渐近稳定的

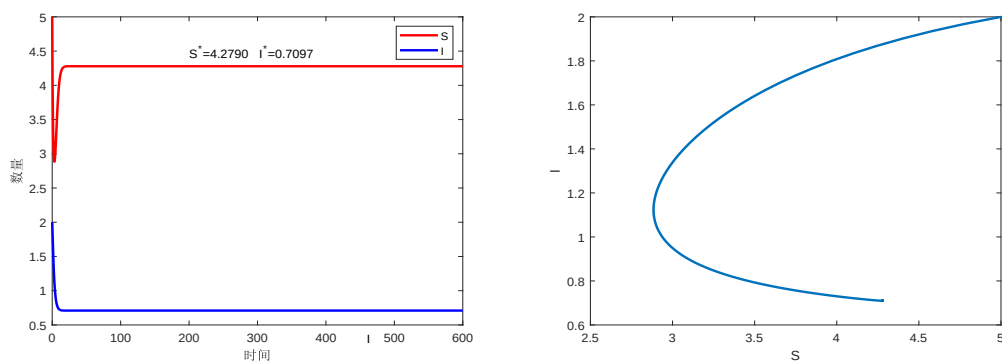


图3 系统(2)在 $u = 0.2, v = 0.28, r(u) = 0.9349, \beta(u, v) = 1.1076, K = 6, (\delta + \gamma) = 0.95$ 和初始条件为 $(5, 2)$ 时的数值模拟,此时满足 $(\delta + \gamma) < \beta(u, v) < (\delta + \gamma) + r(u)$,正平衡点为 $E_* = (4.2790, 0.7097)$ 是局部渐近稳定的

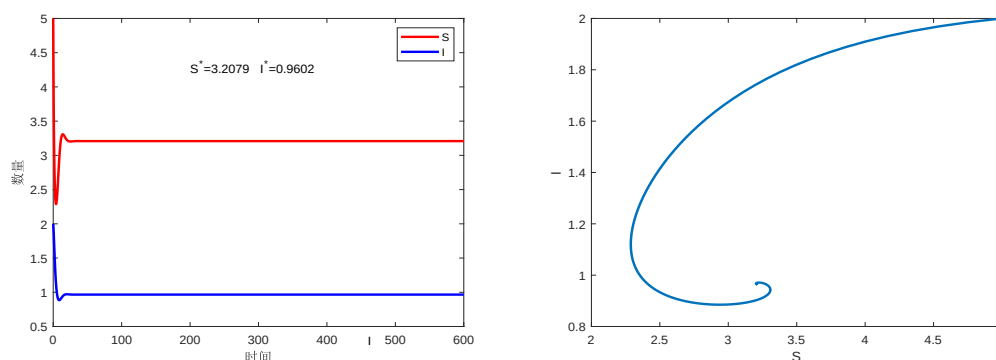


图 4 系统(2)在 $u = 0.16$, $v = 0.3$, $r(u) = 0.9403$, $\beta(u, v) = 1.2361$, $K = 6$, $(\delta + \gamma) = 0.95$ 和初始条件为 $(5, 2)$ 时的数值模拟, 此时满足 $(\delta + \gamma) < \beta(u, v) < (\delta + \gamma) + r(u)$, 正平衡点为 $E_* = (3.2079, 0.9602)$ 是局部渐近稳定的

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Stability and Hopf bifurcation analysis of a SIR model with evolutionary effects

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Abstract: In this paper, the dynamics of a SIR model with evolutionary effects is studied. Firstly, the mathematical model is established and the existence, positivity, boundedness of the system solution are proved. Secondly, the stability of the positive equilibrium of the system is analyzed, and sufficient conditions are given for the oscillation of the number of healthy and infected individuals, which are caused by the defense ability of healthy individuals and the infection ability of infected individuals. Then, by using the central manifold theorem, the properties of Hopf bifurcation are discussed. Finally, the system is numerically simulated with appropriate parameter values.

Keywords: SIR model; evolutionary effect; stability; Hopf bifurcation

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