

B-矩阵线性互补问题解的误差界新估计式

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摘要: 根据 P -矩阵的子类 B -矩阵的定义和性质, 利用严格对角占优 M -矩阵的逆的无穷大范数的上界估计式和不等式的放缩技巧, 得到了 B -矩阵线性互补问题解的误差界新估计式. 理论分析和数值例子说明, 新估计式改进了现有文献的相关结果.

关键词: 误差界; 线性互补问题; B -矩阵

中图分类号: O241.1

文献标识码: A **文章编号:** 1000-4424(2022)02-0142-09

§1 引言

线性互补问题应用广泛, 如力学中的接触力学问题, 经济学中的期权定价问题, 工程学中的流体弹性动态润滑问题等都可以转化为线性互补问题来进行求解(见文献[1-2]). 线性互补问题的数学模型为找到解 $\mathbf{x} \in \mathbf{R}^n$, 使得

$$\mathbf{M}\mathbf{x} + \mathbf{q} \geq \mathbf{0}, \mathbf{x} \geq \mathbf{0}, (\mathbf{M}\mathbf{x} + \mathbf{q})^T \mathbf{x} = 0,$$

记为 $\text{LCP}(\mathbf{M}, \mathbf{q})$, 其中 $\mathbf{M} = (m_{ij}) \in \mathbf{R}^{n \times n}$ 为给定实矩阵, $\mathbf{q} \in \mathbf{R}^n$ 为给定实向量.

2006年, 文献[3]给出当矩阵 $\mathbf{M}$ 为 P -矩阵时, 其相应线性互补问题解的误差界

$$\|\mathbf{x} - \mathbf{x}^*\|_{\infty} \leq \max_{\mathbf{d} \in [0, 1]^n} \|(I - \mathbf{D} + \mathbf{D}\mathbf{M})^{-1}\|_{\infty} \|\mathbf{r}(\mathbf{x})\|_{\infty},$$

其中 $\mathbf{x}^*$ 是 $\text{LCP}(\mathbf{M}, \mathbf{q})$ 的解, $\mathbf{r}(\mathbf{x}) = \min\{\mathbf{x}, \mathbf{M}\mathbf{x} + \mathbf{q}\}$, $\mathbf{d} \in [0, 1]^n$ 表示 $\mathbf{d}$ 为分量 $d_i \in [0, 1]$ 的区间向量, $\mathbf{D} = \text{diag}(d_1, d_2, \dots, d_n)$. 近年来, 国内外学者基于文献[3]的研究, 对 P -矩阵的子类矩阵线性互补问题解的误差界估计进行了系列研究, 获得了一些较好的误差界估计式(见文献[4-10]). 本文利用文献[11]中的结果, 对 P -矩阵的子类矩阵 B -矩阵线性互补问题解的误差界进行了进一步研究, 得到了 B -矩阵线性互补问题解的误差界新估计式. 理论分析证明新估计式优于文献[6-7]中的已有结果, 数值算例也验证了结果的有效性.

首先介绍定义和有关结论.

定义1.1^[12] 设 $\mathbf{A} = (a_{ij}) \in \mathbf{R}^{n \times n}$, 若

$$|a_{ii}| > \sum_{j=1, j \neq i}^n |a_{ij}|, \forall i \in \mathbf{N},$$

收稿日期: 2021-04-10 修回日期: 2021-06-27

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基金项目: 国家自然科学基金(11461027)

则称 A 为严格对角占优矩阵. 若 $a_{ij} \leq 0 (i \neq j), i, j \in \mathbf{N}$, 则称 A 为 Z -矩阵. 若 A 为 Z -矩阵且 $A^{-1} \geq 0$, 则称 A 为 M -矩阵.

定义1.2^[4] 设 $M = (m_{ij}) \in \mathbf{R}^{n \times n}$, 若对任意的 $i, j \in \mathbf{N}, j \neq i$, 有

$$\sum_{k \in \mathbf{N}} m_{ik} > 0, \frac{1}{n} \left(\sum_{k \in \mathbf{N}} m_{ik} \right) > m_{ij},$$

则称 M 为 B -矩阵.

命题1.1^[5] 设 $M = (m_{ij}) \in \mathbf{R}^{n \times n}$, 且 $M = B^+ + C$, 其中

$$B^+ = (b_{ij}) = \begin{pmatrix} m_{11} - r_1^+ & \cdots & m_{1n} - r_1^+ \\ \vdots & \ddots & \vdots \\ m_{n1} - r_n^+ & \cdots & m_{nn} - r_n^+ \end{pmatrix}, C = \begin{pmatrix} r_1^+ & \cdots & r_1^+ \\ \vdots & \ddots & \vdots \\ r_n^+ & \cdots & r_n^+ \end{pmatrix}, \quad (1)$$

$r_i^+ = \max\{0, m_{ij} | j \neq i\}$, 则 M 为 B -矩阵当且仅当 B^+ 为具有正对角元的严格对角占优 M -矩阵.

在文献[5]中, Peña等给出 B -矩阵线性互补问题解的误差界估计式: 设 $M = (m_{ij}) \in \mathbf{R}^{n \times n}$ 为 B -矩阵, 且 $M = B^+ + C$, 其中 $B^+ = (b_{ij})$ 如(1)式所示, 则

$$\max_{d \in [0,1]^n} \|(I - D + DM)^{-1}\|_{\infty} \leq \frac{n-1}{\min\{\beta, 1\}}, \quad (2)$$

其中 $\beta = \min_{i \in \mathbf{N}} \{\beta_i\}$, $\beta_i = b_{ii} - \sum_{j=1, j \neq i}^n |b_{ij}|$.

当矩阵 B^+ 的对角占优性很弱时, β 为一个很小的值, 则相应解的误差界(2)就会是一个很大的值. 为了改进(2)式, Li等在文献[6]中得到以下新误差界估计式: 设 $M = (m_{ij}) \in \mathbf{R}^{n \times n}$ 为 B -矩阵, 且 $M = B^+ + C$, 其中 $B^+ = (b_{ij})$ 如(1)式所示, 则

$$\max_{d \in [0,1]^n} \|(I - D + DM)^{-1}\|_{\infty} \leq \frac{n-1}{\min\{\bar{\beta}_1, 1\}} + \sum_{i=2}^n \frac{n-1}{\min\{\bar{\beta}_i, 1\}} \prod_{j=1}^{i-1} \left(1 + \frac{1}{\bar{\beta}_j} \sum_{k=j+1}^n |b_{jk}| \right), \quad (3)$$

其中

$$\bar{\beta}_i = b_{ii} - \sum_{j=i+1}^n |b_{ij}| l_i(B^+), l_k(B^+) = \max_{k \leq i \leq n} \left\{ \frac{1}{|b_{ii}|} \sum_{j=k, j \neq i}^n |b_{ij}| \right\}.$$

在文献[7]中, Li等给出: 若 $M = (m_{ij}) \in \mathbf{R}^{n \times n}$ 为 B -矩阵, 则矩阵 M 是弱链对角占优 B -矩阵, 并给出以下误差界新估计式: 设 $M = (m_{ij}) \in \mathbf{R}^{n \times n}$ 为 B -矩阵, 且 $M = B^+ + C$, 其中 $B^+ = (b_{ij})$ 如(1)式所示, 则

$$\max_{d \in [0,1]^n} \|(I - D + DM)^{-1}\|_{\infty} \leq \sum_{i=1}^n \left(\frac{n-1}{\min\{\tilde{\beta}_i, 1\}} \prod_{j=1}^{i-1} \frac{b_{jj}}{\tilde{\beta}_j} \right), \quad (4)$$

其中

$$\tilde{\beta}_i = b_{ii} - \sum_{j=i+1}^n |b_{ij}| > 0,$$

且规定 $\prod_{j=1}^0 \frac{b_{jj}}{\tilde{\beta}_j} = 1$.

下面继续对 B -矩阵线性互补问题解的误差界估计进行研究, 给出 B -矩阵线性互补问题解的误差界新估计式.

§2 主要结果

为研究需要, 引入以下符号: 设 $\mathbf{A} = (a_{ij}) \in \mathbf{R}^{n \times n}$, 记

$$u_i(\mathbf{A}) = \frac{\sum_{j=i+1}^n |a_{ij}|}{|a_{ii}|}; \quad b_k(\mathbf{A}) = \max \left\{ \frac{\sum_{j \neq i+k, k \leq j \leq n} |a_{i+k, j}|}{|a_{i+k, i+k}|}, i = 1, \dots, n-k \right\}, k = 1, \dots, n;$$

$$p_k(\mathbf{A}) = \max \left\{ \frac{|a_{i+k, k}| + \sum_{h=k+1, h \neq i+k}^n |a_{i+k, h}| b_k(\mathbf{A})}{|a_{i+k, i+k}|}, i = 1, \dots, n-k \right\}, k = 1, \dots, n;$$

$$r_{li}(\mathbf{A}) = \frac{|a_{li}|}{|a_{ll}| - \sum_{k \neq l, i}^n |a_{lk}|}; \quad r_i(\mathbf{A}) = \max_{l \neq i} \{r_{li}(\mathbf{A})\}, l \neq i, i \in \mathbf{N};$$

$$m_{ji}(\mathbf{A}) = \frac{|a_{ji}| + \sum_{k \neq j, i}^n |a_{jk}| r_i(\mathbf{A})}{|a_{jj}|}; \quad w_{ji}(\mathbf{A}) = \frac{|a_{ji}| + \sum_{k \neq j, i}^n |a_{jk}| m_{ki}(\mathbf{A})}{|a_{jj}|}, j \neq i, i, j \in \mathbf{N}.$$

引理2.1(见[11, p11]) 设 $\mathbf{A} = (a_{ij}) \in \mathbf{R}^{n \times n}$ 是严格对角占优 M -矩阵, 则

$$\|\mathbf{A}^{-1}\|_{\infty} \leq \max \left\{ \frac{1}{a_{11} - \sum_{j \neq 1}^n |a_{1j}| w_{j1}(\mathbf{A})} + \sum_{i=2}^n \left(\frac{1}{a_{ii}(1 - u_i(\mathbf{A}) p_i(\mathbf{A}))} \prod_{j=1}^{i-1} \frac{u_j(\mathbf{A})}{1 - u_j(\mathbf{A}) p_j(\mathbf{A})} \right), \right.$$

$$\left. \frac{p_1(\mathbf{A})}{a_{11} - \sum_{j \neq 1}^n |a_{1j}| w_{j1}(\mathbf{A})} + \sum_{i=2}^n \left(\frac{p_i(\mathbf{A})}{a_{ii}(1 - u_i(\mathbf{A}) p_i(\mathbf{A}))} \prod_{j=1}^{i-1} \frac{1}{1 - u_j(\mathbf{A}) p_j(\mathbf{A})} \right) \right\}.$$

引理2.2(见[8, p2]) 设 $\gamma > 0$ 且 $\eta \geq 0$, 则对任意的 $x \in [0, 1]$, 有

$$\frac{1}{1 - x + \gamma x} \leq \frac{1}{\min\{\gamma, 1\}}, \quad \frac{\eta x}{1 - x + \gamma x} \leq \frac{\eta}{\gamma}.$$

引理2.3(见[9, p4]) 设 $\mathbf{A} = (a_{ij}) \in \mathbf{R}^{n \times n}$, 且

$$a_{ii} > \sum_{k=i+1}^n |a_{ik}|, \quad \forall i \in \mathbf{N},$$

则对任意的 $x_i \in [0, 1]$, $i \in \mathbf{N}$, 有

$$\frac{1 - x_i + a_{ii} x_i}{1 - x_i + a_{ii} x_i - \sum_{k=i+1}^n |a_{ik}| x_i} \leq \frac{a_{ii}}{a_{ii} - \sum_{k=i+1}^n |a_{ik}|}.$$

定理2.1 设 $\mathbf{M} = (m_{ij}) \in \mathbf{R}^{n \times n}$ 为 B -矩阵, 且 $\mathbf{M} = \mathbf{B}^+ + \mathbf{C}$, 其中 $\mathbf{B}^+ = (b_{ij})$ 如(1)式所示, 则

$$\max_{d \in [0, 1]^n} \|(I - D + DM)^{-1}\|_{\infty}$$

$$\leq \max \left\{ \frac{(n-1)}{\min\{\alpha_1(\mathbf{B}^+), 1\}} + \sum_{i=2}^n \left(\frac{(n-1)}{\min\{\hat{\beta}_i, 1\}} \prod_{j=1}^{i-1} \frac{\sum_{k=j+1}^n |b_{jk}|}{\hat{\beta}_j} \right) \right\},$$

$$\frac{(n-1)p_1(\mathbf{B}^+)}{\min\{\alpha_1(\mathbf{B}^+), 1\}} + \sum_{i=2}^n \left(\frac{(n-1)p_i(\mathbf{B}^+)}{\min\left\{\frac{\hat{\beta}_i}{\beta_i}, 1\right\}} \prod_{j=1}^{i-1} \frac{b_{jj}}{\hat{\beta}_j} \right) \Bigg\}, \quad (5)$$

其中 $\hat{\beta}_i = b_{ii} - \sum_{j=i+1}^n |b_{ij}| p_i(\mathbf{B}^+)$, $\alpha_1(\mathbf{B}^+) = b_{11} - \sum_{j \neq 1}^n |b_{1j}| w_{j1}(\mathbf{B}^+)$.

证 设 $M_D = I - D + DM$, 则有

$$M_D = I - D + DM$$

$$= I - D + D(\mathbf{B}^+ + C) = (I - D + D\mathbf{B}^+) + DC = \mathbf{B}_D^+ + C_D,$$

其中 $\mathbf{B}_D^+ = I - D + D\mathbf{B}^+$, $C_D = DC$. 根据文献[5]中定理2.2的证明, 知 $\mathbf{B}_D^+$ 为严格对角占优 M -矩阵, 具有正的主对角元, 且

$$\|M_D^{-1}\|_\infty \leq \left\| \left(I + (\mathbf{B}_D^+)^{-1} C_D \right)^{-1} \right\|_\infty \|(\mathbf{B}_D^+)^{-1}\|_\infty \leq (n-1) \|(\mathbf{B}_D^+)^{-1}\|_\infty. \quad (6)$$

下面求 $\|(\mathbf{B}_D^+)^{-1}\|_\infty$ 的上界. 由引理2.1有

$$\begin{aligned} \|(\mathbf{B}_D^+)^{-1}\|_\infty &\leq \max \left\{ \frac{1}{1 - d_1 + b_{11}d_1 - \sum_{j \neq 1}^n d_1 |b_{1j}| w_{j1}(\mathbf{B}_D^+)} + \right. \\ &\quad \sum_{i=2}^n \left(\frac{1}{(1 - d_i + b_{ii}d_i)(1 - u_i(\mathbf{B}_D^+)p_i(\mathbf{B}_D^+))} \prod_{j=1}^{i-1} \frac{u_j(\mathbf{B}_D^+)}{1 - u_j(\mathbf{B}_D^+)p_j(\mathbf{B}_D^+)} \right) \\ &\quad \frac{p_1(\mathbf{B}_D^+)}{1 - d_1 + b_{11}d_1 - \sum_{j \neq 1}^n d_1 |b_{1j}| w_{j1}(\mathbf{B}_D^+)} + \\ &\quad \left. \sum_{i=2}^n \left(\frac{p_i(\mathbf{B}_D^+)}{(1 - d_i + b_{ii}d_i)(1 - u_i(\mathbf{B}_D^+)p_i(\mathbf{B}_D^+))} \prod_{j=1}^{i-1} \frac{1}{1 - u_j(\mathbf{B}_D^+)p_j(\mathbf{B}_D^+)} \right) \right\}, \quad (7) \end{aligned}$$

根据引理2.2, 对 $\forall i, j, k \in \mathbf{N}$, 有

$$b_k(\mathbf{B}_D^+) = \max \left\{ \frac{\sum_{j \neq i+k, k \leq j \leq n} |b_{i+k,j}| d_{i+k}}{1 - d_{i+k} + b_{i+k,i+k}d_{i+k}} \right\} \leq \max \left\{ \frac{\sum_{j \neq i+k, k \leq j \leq n} |b_{i+k,j}|}{b_{i+k,i+k}} \right\} = b_k(\mathbf{B}^+), \quad (8)$$

$$\begin{aligned} p_k(\mathbf{B}_D^+) &= \max \left\{ \frac{|b_{i+k,k}| d_{i+k} + \sum_{h=k+1, h \neq i+k}^n |b_{i+k,h}| d_{i+k} b_k(\mathbf{B}_D^+)}{1 - d_{i+k} + b_{i+k,i+k}d_{i+k}} \right\} \\ &\leq \max \left\{ \frac{|b_{i+k,k}| + \sum_{h=k+1, h \neq i+k}^n |b_{i+k,h}| b_k(\mathbf{B}^+)}{b_{i+k,i+k}} \right\} = p_k(\mathbf{B}^+), \quad (9) \end{aligned}$$

$$r_i(\mathbf{B}_D^+) = \max_{l \neq i} \left\{ \frac{|b_{li}| d_l}{(1 - d_l + b_{ll}d_l) - \sum_{k \neq l, i}^n |b_{lk}| d_l} \right\} \leq \max_{l \neq i} \left\{ \frac{|b_{li}|}{b_{ll} - \sum_{k \neq l, i}^n |b_{lk}|} \right\} = r_i(\mathbf{B}^+), \quad (10)$$

$$m_{ji}(\mathbf{B}_D^+) = \frac{|b_{ji}|d_j + \sum_{k \neq j,i}^n |b_{jk}|d_j r_i(\mathbf{B}_D^+)}{1 - d_j + b_{jj}d_j} \leq \frac{|b_{ji}| + \sum_{k \neq j,i}^n |b_{jk}|r_i(\mathbf{B}^+)}{b_{jj}} = m_{ji}(\mathbf{B}^+), \quad (11)$$

$$w_{ji}(\mathbf{B}_D^+) = \frac{|b_{ji}|d_j + \sum_{k \neq j,i}^n |b_{jk}|d_j m_{ki}(\mathbf{B}_D^+)}{1 - d_j + b_{jj}d_j} \leq \frac{|b_{ji}| + \sum_{k \neq j,i}^n |b_{jk}|m_{ki}(\mathbf{B}^+)}{b_{jj}} = w_{ji}(\mathbf{B}^+). \quad (12)$$

又根据引理2.2和(12)式有

$$\begin{aligned} \frac{1}{1 - d_1 + b_{11}d_1 - \sum_{j \neq 1}^n |b_{1j}|d_1 w_{j1}(\mathbf{B}_D^+)} &\leq \frac{1}{\min \left\{ b_{11} - \sum_{j \neq 1}^n |b_{1j}|w_{j1}(\mathbf{B}_D^+), 1 \right\}} \\ &\leq \frac{1}{\min \left\{ b_{11} - \sum_{j \neq 1}^n |b_{1j}|w_{j1}(\mathbf{B}^+), 1 \right\}} = \frac{1}{\min \{ \alpha_1(\mathbf{B}^+), 1 \}}, \end{aligned} \quad (13)$$

根据引理2.2和(9)式有

$$\begin{aligned} \frac{1}{(1 - d_i + b_{ii}d_i)(1 - u_i(\mathbf{B}_D^+)p_i(\mathbf{B}_D^+))} &= \frac{1}{(1 - d_i + b_{ii}d_i) - \sum_{j=i+1}^n |b_{ij}|d_i p_i(\mathbf{B}_D^+)} \\ &\leq \frac{1}{\min \left\{ b_{ii} - \sum_{j=i+1}^n |b_{ij}|p_i(\mathbf{B}_D^+), 1 \right\}} \\ &\leq \frac{1}{\min \left\{ b_{ii} - \sum_{j=i+1}^n |b_{ij}|p_i(\mathbf{B}^+), 1 \right\}} = \frac{1}{\min \{ \hat{\beta}_i, 1 \}}, \end{aligned} \quad (14)$$

$$\begin{aligned} \frac{u_j(\mathbf{B}_D^+)}{1 - u_j(\mathbf{B}_D^+)p_j(\mathbf{B}_D^+)} &= \frac{\sum_{k=j+1}^n |b_{jk}|d_j}{1 - d_j + b_{jj}d_j - \sum_{k=j+1}^n |b_{jk}|d_j p_j(\mathbf{B}_D^+)} \\ &\leq \frac{\sum_{k=j+1}^n |b_{jk}|}{b_{jj} - \sum_{k=j+1}^n |b_{jk}|p_j(\mathbf{B}^+)} = \frac{\sum_{k=j+1}^n |b_{jk}|}{\hat{\beta}_j}, \end{aligned} \quad (15)$$

根据引理2.3和(9)有

$$\begin{aligned} \frac{1}{1 - u_j(\mathbf{B}_D^+)p_j(\mathbf{B}_D^+)} &= \frac{1 - d_j + b_{jj}d_j}{1 - d_j + b_{jj}d_j - \sum_{k=j+1}^n |b_{jk}|d_j p_j(\mathbf{B}_D^+)} \\ &\leq \frac{b_{jj}}{b_{jj} - \sum_{k=j+1}^n |b_{jk}|p_j(\mathbf{B}^+)} = \frac{b_{jj}}{\hat{\beta}_j}. \end{aligned} \quad (16)$$

于是, 根据(9)式和(13)-(16)式有

$$\|(\mathbf{B}_D^+)^{-1}\|_\infty \leq \max \left\{ \frac{1}{\min \{ \alpha_1(\mathbf{B}^+), 1 \}} + \sum_{i=2}^n \left(\frac{1}{\min \{ \hat{\beta}_i, 1 \}} \prod_{j=1}^{i-1} \frac{\sum_{k=j+1}^n |b_{jk}|}{\hat{\beta}_j} \right) \right\},$$

$$\frac{p_1(\mathbf{B}^+)}{\min\{\alpha_1(\mathbf{B}^+), 1\}} + \sum_{i=2}^n \left(\frac{p_i(\mathbf{B}^+)}{\min\left\{\hat{\beta}_i, 1\right\}} \prod_{j=1}^{i-1} \frac{b_{jj}}{\hat{\beta}_j} \right) \Bigg\}. \quad (17)$$

综上所述, 由(6)式和(17)式成立知(5)式成立, 故定理2.1得证.

下面对(3)式, (4)式, 及(5)式进行理论上的比较.

定理2.2 设 $\mathbf{M} = (m_{ij}) \in \mathbf{R}^{n \times n}$ 为 B -矩阵, 且 $\mathbf{M} = \mathbf{B}^+ + \mathbf{C}$, 其中 $\mathbf{B}^+ = (b_{ij})$ 如(1)式所示, 那么

$$\begin{aligned} & \max \left\{ \frac{(n-1)}{\min\{\alpha_1(\mathbf{B}^+), 1\}} + \sum_{i=2}^n \left(\frac{(n-1)}{\min\left\{\hat{\beta}_i, 1\right\}} \prod_{j=1}^{i-1} \frac{\sum_{k=j+1}^n |b_{jk}|}{\hat{\beta}_j} \right), \right. \\ & \quad \left. \frac{(n-1)p_1(\mathbf{B}^+)}{\min\{\alpha_1(\mathbf{B}^+), 1\}} + \sum_{i=2}^n \left(\frac{(n-1)p_i(\mathbf{B}^+)}{\min\left\{\hat{\beta}_i, 1\right\}} \prod_{j=1}^{i-1} \frac{b_{jj}}{\hat{\beta}_j} \right) \right\} \\ & \leq \frac{n-1}{\min\left\{\bar{\beta}_1, 1\right\}} + \sum_{i=2}^n \frac{n-1}{\min\left\{\bar{\beta}_i, 1\right\}} \prod_{j=1}^{i-1} \left(1 + \frac{1}{\bar{\beta}_j} \sum_{k=j+1}^n |b_{jk}| \right) \\ & \leq \sum_{i=1}^n \frac{n-1}{\min\{\tilde{\beta}_i, 1\}} \prod_{j=1}^{i-1} \frac{b_{jj}}{\tilde{\beta}_j}. \end{aligned} \quad (18)$$

其中 $\bar{\beta}_i$ 与 $\tilde{\beta}_i$ 分别如(3)式与(4)式所示, $\alpha_1(\mathbf{B}^+)$ 和 $\hat{\beta}_i$ 如(5)式所示.

证 根据 $\mathbf{B}^+$ 为具有正对角元的严格对角占优矩阵, 易知 $0 < w_{j1}(\mathbf{B}^+) \leq l_1(\mathbf{B}^+) < 1$, $0 < p_i(\mathbf{B}^+) \leq l_i(\mathbf{B}^+) < 1$, $i \in N$, 故有

$$\begin{aligned} \bar{\beta}_1 &= b_{11} - \sum_{j=2}^n |b_{1j}| l_1(\mathbf{B}^+) \leq b_{11} - \sum_{j \neq 1}^n |b_{1j}| w_{j1}(\mathbf{B}^+) = \alpha_1(\mathbf{B}^+), \\ \bar{\beta}_i &= b_{ii} - \sum_{j=i+1}^n |b_{ij}| l_i(\mathbf{B}^+) \leq b_{ii} - \sum_{j=i+1}^n |b_{ij}| p_i(\mathbf{B}^+) = \hat{\beta}_i, \\ \tilde{\beta}_i &= b_{ii} - \sum_{j=i+1}^n |b_{ij}| \leq b_{ii} - \sum_{j=i+1}^n |b_{ij}| l_i(\mathbf{B}^+) = \bar{\beta}_i, \end{aligned}$$

进而有

$$\frac{1}{\min\{\alpha_1(\mathbf{B}^+), 1\}} \leq \frac{1}{\min\left\{\bar{\beta}_1, 1\right\}}, \quad (19)$$

$$\frac{1}{\min\left\{\hat{\beta}_i, 1\right\}} \leq \frac{1}{\min\left\{\bar{\beta}_i, 1\right\}}, \quad (20)$$

$$\frac{1}{\min\left\{\bar{\beta}_i, 1\right\}} \leq \frac{1}{\min\left\{\tilde{\beta}_i, 1\right\}}, \quad (21)$$

且对任意的 $1 \leq j \leq n-1$, 有

$$\frac{\sum_{k=j+1}^n |b_{jk}|}{\hat{\beta}_j} \leq 1 + \frac{1}{\bar{\beta}_j} \sum_{k=j+1}^n |b_{jk}|, \quad (22)$$

$$\begin{aligned} \frac{b_{jj}}{\hat{\beta}_j} &\leq \frac{b_{jj} - \sum_{k=j+1}^n |b_{jk}| l_j(\mathbf{B}^+) + \sum_{k=j+1}^n |b_{jk}| l_j(\mathbf{B}^+)}{\bar{\beta}_j} \\ &= \frac{\bar{\beta}_j + \sum_{k=j+1}^n |b_{jk}| l_j(\mathbf{B}^+)}{\bar{\beta}_j} = 1 + \frac{\sum_{k=j+1}^n |b_{jk}| l_j(\mathbf{B}^+)}{\bar{\beta}_j} \leq 1 + \frac{1}{\bar{\beta}_j} \sum_{k=j+1}^n |b_{jk}|, \end{aligned} \quad (23)$$

$$1 + \frac{1}{\bar{\beta}_j} \sum_{k=j+1}^n |b_{jk}| \leq 1 + \frac{1}{\tilde{\beta}_j} \sum_{k=j+1}^n |b_{jk}| = \frac{1}{\tilde{\beta}_j} (\tilde{\beta}_j + \sum_{k=j+1}^n |b_{jk}|) = \frac{b_{jj}}{\tilde{\beta}_j}. \quad (24)$$

于是由 $0 < p_i(\mathbf{B}^+) < 1$, (19)-(20)式, 以及(22)-(23)式知(18)式的第一个不等式成立; 由(21)式和(24)式知(18)式的第二个不等式成立; 故(18)式成立, 定理2.2得证.

§3 数值算例

例3.1 考虑 B -矩阵

$$\mathbf{M}_k = \begin{pmatrix} 1.5 & 0.5 & 0.4 & 0.5 \\ -0.1 & 1.7 & 0.7 & 0.6 \\ 0.8 & -0.1 \frac{k}{k+1} & 1.8 & 0.7 \\ 0 & 0.7 & 0.8 & 1.8 \end{pmatrix},$$

其中 $k \geq 1$. $\mathbf{M}_k = \mathbf{B}_k^+ + \mathbf{C}_k$, 其中

$$\mathbf{B}_k^+ = \begin{pmatrix} 1 & 0 & -0.1 & 0 \\ -0.8 & 1 & 0 & -0.1 \\ 0 & -0.1 \frac{k}{k+1} - 0.8 & 1 & -0.1 \\ -0.8 & -0.1 & 0 & 1 \end{pmatrix}.$$

根据(4)式求得

$$\max_{d \in [0,1]^4} \|(\mathbf{I} - \mathbf{D} + \mathbf{D}\mathbf{M}_k)^{-1}\|_\infty \leq \sum_{i=1}^4 \left(\frac{3}{\min \left\{ \tilde{\beta}_i, 1 \right\}} \prod_{j=1}^{i-1} \frac{b_{jj}}{\tilde{\beta}_j} \right) \approx 15.2675.$$

根据(3)式求得

$$\begin{aligned} \text{当 } k=1 \text{ 时, } \max_{d \in [0,1]^4} \|(\mathbf{I} - \mathbf{D} + \mathbf{D}\mathbf{M}_k)^{-1}\|_\infty &\leq 14.8064; \\ \text{当 } k=2 \text{ 时, } \max_{d \in [0,1]^4} \|(\mathbf{I} - \mathbf{D} + \mathbf{D}\mathbf{M}_k)^{-1}\|_\infty &\leq 14.8229; \\ &\dots\dots\dots \\ \text{当 } k=100 \text{ 时, } \max_{d \in [0,1]^4} \|(\mathbf{I} - \mathbf{D} + \mathbf{D}\mathbf{M}_k)^{-1}\|_\infty &\leq 14.8550; \\ &\dots\dots\dots \\ \text{当 } k=\infty \text{ 时, } \max_{d \in [0,1]^4} \|(\mathbf{I} - \mathbf{D} + \mathbf{D}\mathbf{M}_k)^{-1}\|_\infty &\leq 14.8560. \end{aligned}$$

根据(5)式求得

$$\begin{aligned} \text{当 } k=1 \text{ 时, } \max_{d \in [0,1]^4} \|(\mathbf{I} - \mathbf{D} + \mathbf{D}\mathbf{M}_k)^{-1}\|_\infty &\leq 10.0404; \\ \text{当 } k=2 \text{ 时, } \max_{d \in [0,1]^4} \|(\mathbf{I} - \mathbf{D} + \mathbf{D}\mathbf{M}_k)^{-1}\|_\infty &\leq 10.2566; \\ &\dots\dots\dots \end{aligned}$$

当 $k = 100$ 时, $\max_{d \in [0,1]^4} \|(I - D + DM_k)^{-1}\|_\infty \leq 10.6867$;

.....

当 $k = \infty$ 时, $\max_{d \in [0,1]^4} \|(I - D + DM_k)^{-1}\|_\infty \leq 10.7001$.

综上, (5) 式优于 (3)-(4) 式.

例3.2 考虑 B -矩阵

$$M = \begin{pmatrix} 1.3 & 0.3 & 0.2 & 0.3 & 0.2 & 0.3 & 0.2 & 0.3 & 0.2 & 0.3 \\ 0 & 1.4 & 0.4 & 0.3 & 0.4 & 0.3 & 0.4 & 0.3 & 0.4 & 0.3 \\ 0.4 & -0.1 & 1.4 & 0.3 & 0.3 & 0.4 & 0.3 & 0.4 & 0.3 & 0.4 \\ 0.5 & 0.4 & 0.5 & 1.5 & 0.5 & 0.4 & 0.5 & 0.4 & 0.5 & 0.4 \\ 0.4 & 0.6 & 0.5 & 0.6 & 1.6 & 0.6 & 0.5 & 0.6 & 0.5 & 0.6 \\ 0.5 & -0.1 & 0.5 & 0.4 & 0.5 & 1.5 & 0.5 & 0.4 & 0.5 & 0.4 \\ 0.1 & 0.2 & 0.3 & 0.3 & 0.2 & 0.3 & 1.3 & 0.3 & 0.2 & 0.3 \\ -0.1 & 0.2 & 0.1 & 0.2 & 0.2 & 0.1 & 0.2 & 1.2 & 0.2 & 0.1 \\ 0.1 & -0.4 & 0.1 & 0 & 0.1 & 0.1 & 0 & 0.1 & 1.1 & 0.1 \\ 0.1 & 0.2 & 0.3 & 0.3 & 0.3 & 0.3 & 0.3 & 0.2 & 0.3 & 1.3 \end{pmatrix},$$

$M = B^+ + C$, 其中

$$B^+ = \begin{pmatrix} 1 & 0 & -0.1 & 0 & -0.1 & 0 & -0.1 & 0 & -0.1 & 0 \\ -0.4 & 1 & 0 & -0.1 & 0 & -0.1 & 0 & -0.1 & 0 & -0.1 \\ 0 & -0.5 & 1 & -0.1 & -0.1 & 0 & -0.1 & 0 & -0.1 & 0 \\ 0 & -0.1 & 0 & 1 & 0 & -0.1 & 0 & -0.1 & 0 & -0.1 \\ -0.2 & 0 & -0.1 & 0 & 1 & 0 & -0.1 & 0 & -0.1 & 0 \\ 0 & -0.6 & 0 & -0.1 & 0 & 1 & 0 & -0.1 & 0 & -0.1 \\ -0.2 & -0.1 & 0 & 0 & -0.1 & 0 & 1 & 0 & -0.1 & 0 \\ -0.3 & 0 & -0.1 & 0 & 0 & -0.1 & 0 & 1 & 0 & -0.1 \\ 0 & -0.5 & 0 & -0.1 & 0 & 0 & -0.1 & 0 & 1 & 0 \\ -0.2 & -0.1 & 0 & 0 & 0 & 0 & 0 & -0.1 & 0 & 1 \end{pmatrix}.$$

应用(4)式求得 $\max_{d \in [0,1]^{10}} \|(I - D + DM)^{-1}\|_\infty \leq 756.4076$.

应用(3)式求得 $\max_{d \in [0,1]^{10}} \|(I - D + DM)^{-1}\|_\infty \leq 520.7391$.

而应用(5)式求得 $\max_{d \in [0,1]^{10}} \|(I - D + DM)^{-1}\|_\infty \leq 68.7210$.

显然, (5) 式比 (3)-(4) 式更精确.

例3.3 考虑三对角矩阵 $M \in \mathbf{R}^{n \times n}$:

$$M = \begin{pmatrix} b + \alpha \sin(\frac{1}{n}) & c & & & & \\ a & b + \alpha \sin(\frac{2}{n}) & c & & & \\ & \ddots & \ddots & \ddots & & \\ & & a & b + \alpha \sin(\frac{n-1}{n}) & c & \\ & & & a & b + \alpha \sin(1) & \end{pmatrix}.$$

当 $b = 2.2$, $a = -1.5$, $c = -0.5$, $\alpha = \frac{1}{n^2}$ 时, M 为 B -矩阵, 因此, 可应用 (3)-(5) 式对 B -矩阵线性互补问题解的误差界进行估计, 具体结果见表1.

表1 $\max_{d \in [0,1]^n} \|(I - D + DM)^{-1}\|_\infty$ 的上界

n	$n = 10$	$n = 20$	$n = 50$	$n = 100$	$n = 400$
(4) 式	371.2869	1.1124e+04	6.6054e+07	5.2996e+13	8.3539e+47
(3) 式	356.0571	1.0101e+04	5.0242e+07	2.9960e+13	7.9581e+46
(5) 式	259.9763	5.8258e+03	1.3226e+07	2.1130e+12	2.0701e+42

由表1知应用(5)式所得的结果比应用(3)-(4)式所得的结果要小, 故(5)式比(3)-(4)式更精确.

上述数值算例充分说明: 本文所获得的结论改进了文献[6]中的定理4, 文献[7]中的定理2.

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Error bounds for linear complementarity problems of B -matrices

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Abstract: According to the definition and properties of the subclass of P -matrices B -matrices, a new estimate of the error bounds for the linear complementarity problem of B -matrices is obtained by using the upper bound estimation of the inverse infinite norm of a strictly diagonally dominant M -matrices and the scaling technique of inequality. Theoretical analysis and numerical examples show that the new estimate improves the relevant results in the existing literature.

Keywords: error bounds; linear complementarity problem; B -matrix

MR Subject Classification: 90C33; 60G50; 65F35