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# 加权正则函数的一些性质

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**摘 要**: 正则函数是Clifford分析中的一类重要函数, 加权正则函数是正则函数的进一步发展, 也是一类重要的函数, 因此具有一定的研究意义. 在正则函数的研究基础上, 并利用加权正则函数自身的性质, 讨论了加权正则函数的平均值定理, 最大模原理, Weierstrass定理以及一些其它推论.

**关键词**: 实Clifford分析; 加权正则函数; 平均值定理; 最大模原理; Weierstrass定理
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## §1 引 言

Clifford代数是复数, 四元数, 外代数的推广. Clifford分析是关于Dirac算子零解的函数理论, Dirac算子是著名的Cauchy-Riemann算子在高维情形下的发展. Dirac算子特殊的结构使它成为联系偏微分方程, 微分几何以及物理学的重要桥梁, 这也使得对Dirac算子的研究具有很重要的意义. 1968年, D. Hestenes引入了Dirac算子, 并且研究了Dirac算子在Clifford分析中的重要作用, 证明了实Clifford分析中的Morera定理及Liouville定理[1]. 直到1982年, F. Brackx, R. Delanghe, F. Sommen等给出了正则函数的Cauchy-Riemann方程, Cauchy-Pompeiu公式, Cauchy积分公式, Sokhotski-Plemelj公式, 平均值定理, 最大模原理, Weierstrass定理等许多结果[2], 从而将单复分析中全纯函数在高维空间欧氏度量下进行了推广. 随后, 国内外许多学者开始致力于Clifford分析的研究. 在国外, K. Gürlebeck, W. Sprössig, U. Kähler, H. Begehr等都对Clifford分析做过大量的研究[3-6]. 在国内, 杜金元[7], 黄沙和乔玉英[8]等较早开始研究Clifford分析, 并取得了一系列较好的研究成果. 近几年, 乔玉英和王丽萍等又研究了Clifford分析中一些高阶奇异积分算子的性质及有关偏微分方程边值问题, 得到了很好的结果[9-12]. 然而, 为了能更好地描述物体的特征, 经典Dirac算子已经不能满足要求, 譬如各向同性介质中的热传导问题, 可以在经典Clifford代数结构上利用经典的Dirac算子进行描述. 但对于涉及到非均匀材料的物理问题, 譬如各向异性介质中的热传导问题, 经典的Dirac算子就不能很好地对这类

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问题进行描述, 就需要在依赖于参数的Clifford代数结构上利用加权Dirac算子进行描述. 因此, 2012年, A. N. Di. Teodoro, C. J. Vanegas在含参量Clifford代数下给出了一阶meta-monogenic算子的基本解及Cauchy-Pompeiu公式[13]. 2013 年, C. Balderrama, A. D. Teodoro, A. Infante在含参量Clifford代数下给出了$n$阶meta-monogenic算子的Cauchy-Pompeiu公式及应用[14]. 2016年, A. G. Eusebio, A. D. Teodoro研究了第一类multi-meta-weighted-monogenic函数的Cauchy-Pompeiu公式, 并给出了非齐次meta-$n$-weighted-monogenic方程的分布解[15]. 2017年, A. G. Eusebio, A. D. Teodoro又研究了第二类multi-meta-$\varphi$-monogenic函数的积分公式[16]. 2018年, J. Vanegas, F. Vargas研究了带有Clifford常数权$\gamma_i \in \mathcal{A}_n(\mathbf{R})(i=1,2,\cdots,n)$ 的Dirac算子$\mathcal{D}_\omega = \sum_{i=1}^{n}\gamma_i e_i \partial_i$的基本解, 其中$\mathcal{D}_\omega$是二阶椭圆微分算子$\widetilde{\triangle}_n = \sum_{i,j=1}^{n} B_{ij}\partial_i\partial_j$的因子, $B=(B_{ij})$是对称的正定矩阵. 随后, J. Vanegas, F. Vargas又给出了该类加权正则函数的Borel-Pompeiu公式与Cauchy-积分公式[17]. 当函数$f$满足方程$\mathcal{D}_\omega f = 0$时, 则称其为加权正则函数. 加权正则函数是Clifford分析中的一类重要的函数, 是正则函数的推广, 当矩阵$B$是单位矩阵的时候, 加权正则函数是正则函数.

本文研究加权正则函数的一些性质, 文章的结构安排如下: §2回顾了有关Clifford分析的一些基本知识, 加权Dirac算子的定义以及与加权Dirac算子有关的积分公式. §3研究了加权正则函数的平均值定理, 最大模原理, Weierstrass定理以及一些推论. 这些性质刻画了实Clifford分析中加权正则函数的基本特征.

## §2 预备知识

设$\{e_1,e_2,\ldots,e_n\}$是$n$维欧式空间$\mathbf{R}^n$的一组标准正交基, $\mathcal{A}_n(\mathbf{R})$是建立在$\mathbf{R}^n$上的$2^n$维实Clifford空间, 且基底表示为$\beta=\{e_N|N\in\Gamma_n\}$, 其中$\Gamma_n=\{0,1,2,\cdots,n,12,13,\cdots,123\cdots n\}$, $e_0$为其单位元, $\mathcal{A}_n(\mathbf{R})$中的基元素可表示为: $e_{N_1N_2\cdots N_r}=e_{N_1}e_{N_2}\cdots e_{N_r}$, 其中$1\le N_1<\cdots<N_r\le n$. 则任意的一个元素$a\in\mathcal{A}_n(\mathbf{R})$可表示为$a=\sum_N a_N e_N(a_N\in\mathbf{R})$, 且有

$$\begin{cases} e_i^2=-1, & i=1,2,\cdots,n,\\ e_ie_j=-e_je_i, & i,j=1,2,\cdots,n,i<j,\\ \overline{a\cdot b}=\bar{b}\cdot\bar{a}, & a,b\in\mathcal{A}_n. \end{cases}$$

定义$a$的模为

$$|a|=\sqrt{[a,a]_0}=\left(\sum_N |a_N|^2\right)^{\frac{1}{2}}.$$

设$\Omega\subset\mathbf{R}^n$是一个有界域且边界$\partial\Omega$充分光滑, 接下来考虑定义在$\Omega$取值于$\mathcal{A}_n(\mathbf{R})$中的函数$f$, 则$f$可表示为$f(x)=\sum_N f_N(x)e_N$, 其中$f_N(x)$是定义于$\Omega$上的实值函数. 用$F_\Omega^{(r)}$表示$\Omega$中$r$阶导数连续的函数的全体,

$$F_\Omega^{(r)}=\{f|f:\Omega\to\mathcal{A}_n(\mathbf{R}),f(x)=\sum_N f_N(x)e_N\}.$$

加权Dirac算子定义为

$$\mathcal{D}_\omega=\sum_{i=1}^{n}\gamma_i e_i\partial_i. \tag{2.1}$$

其中加权$\gamma_i\in\mathcal{A}_n(\mathbf{R})(i=1,2,\cdots,n)$是Clifford常数, 希望$\mathcal{D}_\omega$是下式二阶椭圆微分算子$\widetilde{\triangle}_n$的因

子, 即$\widetilde{\triangle}_n=\mathcal{D}_\omega\overline{\mathcal{D}}_\omega$,

$$\widetilde{\triangle}_n=\sum_{i=1}^{n}B_{ii}\partial_i^2+2\sum_{1\leq i<j\leq n}B_{ij}\partial_i\partial_j, \tag{2.2}$$

其中$B_{ij}$是矩阵$B$的元素, 矩阵$B$是对称的正定矩阵, 从而矩阵$B$有逆矩阵$A$和平方根$B^{\frac{1}{2}}$, 矩阵$A$和$B^{\frac{1}{2}}$也是对称的正定矩阵, 且满足$A^{\frac{1}{2}}=B^{-\frac{1}{2}}$.

便于计算, 将$\mathcal{D}_\omega$表示为

$$\mathcal{D}_\omega=\sum_{i=1}^{n}\psi_i\partial_i, \tag{2.3}$$

其中$\psi_i=\gamma_i e_i$, $i=1,2,\cdots,n$.

为了使得$\widetilde{\triangle}_n=\mathcal{D}_\omega\overline{\mathcal{D}}_\omega$, 经计算需要满足

$$\psi_i\overline{\psi_j}+\psi_j\overline{\psi_i}=2B_{ij},\quad i,j=1,2,\cdots,n \tag{2.4}$$

或

$$\overline{\psi_i}\psi_j+\overline{\psi_j}\psi_i=2B_{ij},\quad i,j=1,2,\cdots,n. \tag{2.5}$$

当$\psi_i=e_i$时, 则(2.4), (2.5)变为

$$e_i\bar{e}_j+e_j\bar{e}_i=2B_{ij},\quad i,j=1,2,\cdots,n, \tag{2.4$'$}$$

$$\bar{e}_ie_j+\bar{e}_je_i=2B_{ij},\quad i,j=1,2,\cdots,n, \tag{2.5$'$}$$

其中矩阵$B$为单位矩阵, 加权Dirac算子$D_\omega$退化为经典Dirac算子$D=\sum_{i=1}^{n}e_i\partial_i$.

基于正定矩阵的不同分解, 例如Cholesky分解$B=LL^{\mathrm{T}}$($L$是下三角矩阵, 且对角线上元素$L_{ii}>0, i=1,2,\cdots,n$), 平方根分解$L^{'}=B^{\frac{1}{2}}$, 以及根据奇异值分解$B=U\Lambda U^{\mathrm{T}}$得到$L^{''}=U\Lambda^{\frac{1}{2}}$, 来构造加权$\psi_i$. 如果定义$\psi_i$是矩阵$L$的第$i$行嵌入到了$\mathbf{R}^n$的标准基中, 即

$$\psi_i=\sum_{j=1}^{n}L_{ij}e_j. \tag{2.6}$$

经计算, 满足

$$\psi_i\overline{\psi}_j+\psi_j\overline{\psi}_i=\sum_{p,q=1}^{n}L_{ip}L_{jq}(2e_p\overline{e}_q+e_q\overline{e}_p)=2\sum_{p,q=1}^{n}L_{ip}L_{jq}\delta_{pq}=2\sum_{p=1}^{n}L_{ip}L_{jp}=2B_{ij}. \tag{2.7}$$

另一个公式也可通过计算得到.

根据(2.6)所构造的加权$\psi_i$是向量值函数, 也可以构造不是向量值的加权$\psi_i$.

定义$\mathit{\Gamma}_n^o\subset\mathit{\Gamma}_n$为

$$\mathit{\Gamma}_n^o=\{N\in\mathit{\Gamma}_n:\#N\equiv1(\mathrm{mod}4)\text{或}\#N\equiv2(\mathrm{mod}4)\}.$$

设$\mathit{\Gamma}_*^-\subset\mathit{\Gamma}_n^o$为非空子集, 当$\forall A,B\in\mathit{\Gamma}_*^-$, $A\neq B$时, 满足$AB\neq BA$, $AB, A\triangle B\in\mathit{\Gamma}_n^o$.

考虑$\mathcal{A}_n(\mathbf{R})$的子空间$\mathcal{A}_n^-(\mathbf{R})$, 定义

$$\mathcal{A}_n^-(\mathbf{R})=\mathrm{gen}\{e_N:N\in\mathit{\Gamma}_n^-\},$$

其中$\mathit{\Gamma}_n^-=\{0\}\cup\mathit{\Gamma}_*^-$. 令$m=\#\mathit{\Gamma}_*^-$, $p_i$是$\mathit{\Gamma}_*^-$的第$i$个元素, $i=1,2,\cdots,m$, 且$p_0=0$.

**引理2.1**[2, p 7] $\mathcal{A}_n^-(\mathbf{R})$中的元素有如下运算性质.

1. $\overline{e_N}=-e_N$, $N\in\mathit{\Gamma}_n^o$.
2. 任意$a\in\mathcal{A}_n^-(\mathbf{R})$, 有$a+\bar{a}=2[a]_0$.

3. $e_N^2=-1$, $e_Ne_M+e_Me_N=0$, $N,M\in\Gamma_*^-$, $N\neq M$.
4. 若$a,b\in\mathcal{A}_n^-(\mathbf{R})$, 则$[a\bar{b}]_0=\sum_{N\in\Gamma_n^-}[a]_N[b]_N=[\bar{b}a]_0$.
5. 若$a\in\mathcal{A}_n^-(\mathbf{R})$, 则$a\bar{a}=\bar{a}a=|a|^2=\sum_{N\in\Gamma_n^-}[a]_N^2$.

定义加权$\psi_i$是矩阵$L$的第$i$行嵌入到$\mathcal{A}_n^-(\mathbf{R})$中的元素, 即

$$\psi_i=\sum_{N\in\Gamma_n^-}[\psi_i]_Ne_N=\sum_{k=1}^m L_{ik}e_{p_k},\quad i=1,2,\cdots,n. \tag{2.8}$$

利用引理2.1中的(2), 有

$$\psi_i\overline{\psi_j}+\psi_j\overline{\psi_i}=\psi_i\overline{\psi_j}+\overline{\psi_i\overline{\psi_j}}=2[\psi_i\overline{\psi_j}]_0=2\sum_{k=1}^m L_{ik}L_{jk}=2B_{ij},\quad i,j=1,2,\cdots,n.$$

之后只考虑(2.8)给定的加权$\psi_i$的形式.

**定义2.1** 设$f\in F_\Omega^{(1)}$, 若对任意$x\in\Omega$, 有$\mathcal{D}_\omega f=0(f\mathcal{D}_\omega=0)$, 则称$f$为$\Omega$上的加权左(右)正则函数. 也简称加权左正则函数为加权正则函数.

**定义2.2** 对于$\mathbf{R}^n$内任意两点$\xi=(\xi_1,\xi_2,\cdots,\xi_n)$和$x=(x_1,x_2,\cdots,x_n)$定义它们之间的非欧氏距离$\rho$为

$$\rho^2(x,\xi)=\sum_{i,j=1}^n A_{ij}(x_i-\xi_i)(x_j-\xi_j)=\langle(x-\xi),A(x-\xi)\rangle, \tag{2.9}$$

其中$A_{ij}$是矩阵$A$的元素.

对于$\mathbf{R}^n$内任意两点$x,\xi$, 当$x\neq\xi$时, 设它们之间的欧氏距离为$r$, 即$r=|x-\xi|$, 则有$x-\xi=ry(|y|=1)$, 把此点$y$和$(0,\cdots,0)$之间的非欧氏距离记作$\rho_0$, 则可以证得$\rho_0\geq c>0$.

这是因为$y=(y_1,\cdots,y_n)\in\mathbf{R}^n$且$|y|=1$, 从而$\sum_{i=1}^n y_i^2=1$. 则

$$\rho_0^2=y^{\mathrm{T}}Ay=\begin{pmatrix} y_1,\cdots,y_n\end{pmatrix}\begin{pmatrix} A_{11} & \cdots & A_{1n}\\ \vdots & \ddots & \vdots\\ A_{1n} & \cdots & A_{nn}\end{pmatrix}\begin{pmatrix} y_1\\ \vdots\\ y_n\end{pmatrix}>0. \tag{2.10}$$

令$\varphi(u_1,\cdots,u_n)=\dfrac{\rho_0^2}{\sum_{i=1}^n y_i^2}$, 其中$u_i=\dfrac{y_i}{\sqrt{\sum_{i=1}^n y_i^2}}$, 因为$|y|^2=\sum_{i=1}^n y_i^2=1$, 则

$$\varphi(u_1,\cdots,u_n)=\rho_0^2>0.$$

又根据$\rho_0$的定义, 显然$\varphi(u_1,\cdots,u_n)$还可以表示为下式

$$\begin{aligned}&\varphi(u_1,\cdots,u_n)\\ =&\begin{pmatrix}\dfrac{y_1}{\sqrt{\sum_{i=1}^n y_i^2}},\cdots,\dfrac{y_n}{\sqrt{\sum_{i=1}^n y_i^2}}\end{pmatrix}\begin{pmatrix} A_{11} & \cdots & A_{1n}\\ \vdots & \ddots & \vdots\\ A_{1n} & \cdots & A_{nn}\end{pmatrix}\begin{pmatrix}\dfrac{y_1}{\sqrt{\sum_{i=1}^n y_i^2}}\\ \vdots\\ \dfrac{y_n}{\sqrt{\sum_{i=1}^n y_i^2}}\end{pmatrix}\\ =&\begin{pmatrix} u_1,\cdots,u_n\end{pmatrix}\begin{pmatrix} A_{11} & \cdots & A_{1n}\\ \vdots & \ddots & \vdots\\ A_{1n} & \cdots & A_{nn}\end{pmatrix}\begin{pmatrix} u_1\\ \vdots\\ u_n\end{pmatrix}\\ =&\sum_{i,j=1}^n A_{ij}u_iu_j>0,\end{aligned}$$

其中$u_1^2+\cdots+u_n^2=1$. 则$\varphi(u_1,\cdots,u_n)$是紧集$\{(u_1,\cdots,u_n)|u_1^2+\cdots+u_n^2=1\}$上的连续函数, 从而存在最小值, 记为$c_0\geq 0$, 又由于$u_1,\cdots,u_n$不能同时为零, 则$c_0>0$, 则

$$\varphi(u_1,\cdots,u_n)\geq c_0,\ \rho_0^2=\varphi(u_1,\cdots,u_n)\geq c_0.$$

故$\rho_0\geq\sqrt{c_0}=c$.

还可证$\rho=r\rho_0$. 事实上, 由$\rho^2(x,\xi)=\langle(x-\xi),A(x-\xi)\rangle$, $\rho_0^2=y^{\mathrm{T}}Ay$, $r=|x-\xi|$, 则$r^2=\langle(x-\xi),E(x-\xi)\rangle$, $E$为单位矩阵. 由于$|y|=1$, 不妨设$y=\dfrac{x-\xi}{|x-\xi|}$, 故

$$r^2\rho_0^2=(x-\xi)^{\mathrm{T}}E(x-\xi)\frac{(x-\xi)^{\mathrm{T}}}{|x-\xi|}A\frac{x-\xi}{|x-\xi|}=\rho^2.$$

从而得到$\rho=r\rho_0$. 又$\rho_0\geq c$, 则

$$\rho\geq cr. \tag{2.11}$$

用类似于[18-19]的方法可以证得二阶椭圆微分算子$\widetilde{\triangle}_n$的基本解为

$$\widetilde{K}(x,\xi)=\frac{1}{\det(B)^{\frac12}\omega_n}\begin{cases}\ln\rho, & n=2\\ \dfrac{-1}{n-2}\dfrac{1}{\rho^{n-2}}, & n\geq 3\end{cases},\quad x\neq\xi, \tag{2.12}$$

其中$\omega_n$表示$\mathbf{R}^n$中单位球的表面积.

$$E_\omega(x,\xi)=\overline{D}_\omega\widetilde{K}(x,\xi)=\frac{1}{\det(B)^{\frac12}\omega_n}\frac{1}{\rho^n}\sum_{i,j=1}^n\overline{\psi_i}A_{ij}(x_j-\xi_j). \tag{2.13}$$

由于$\widetilde{\triangle}_n=\mathcal{D}_\omega\overline{\mathcal{D}}_\omega$, 从而$\mathcal{D}_\omega E_\omega(x,\xi)=\mathcal{D}_\omega\overline{\mathcal{D}}_\omega\widetilde{K}(x,\xi)=\widetilde{\triangle}_n\widetilde{K}(x,\xi)=0$. 则$E_\omega(x,\xi)$是加权左正则函数, 同样可以得到$E_\omega(x,\xi)$是加权右正则函数.

设$\Omega,\partial\Omega$如上所述, 对于任意$\xi\in\Omega$, 以$\xi$为心, $\varepsilon$为半径, 做$n$维非欧氏距离超球$U_\varepsilon(\xi)=\{x\in\Omega:\rho(x,\xi)<\varepsilon\}$, $\partial U_\varepsilon(\xi)$的外法向量取正方向, 则曲面$\partial U_\varepsilon(\xi)$的参数化方程可表示为

$$x(t)=\varepsilon B^{\frac12}r(t)+\xi,\ t\in\mathbf{R}^{n-1}, \tag{2.14}$$

其中$r(t)$是$\mathbf{R}^n$中欧氏距离下单位球的参数方程, 又由于$A=B^{-1}$, $B^{\frac12}$是对称正定矩阵, 显然

$$\rho^2(x,\xi)=(x-\xi)^{\mathrm{T}}A(x-\xi)=\varepsilon(B^{\frac12})^{\mathrm{T}}rA\varepsilon B^{\frac12}r=\varepsilon^2 rB^{\frac12}AB^{\frac12}r=\varepsilon^2.$$

且对上述参数化方程Jacobian矩阵有如下关系:

$$J_tx=\varepsilon B^{\frac12}J_tr\in\mathbf{R}^{n\times n-1}.$$

**引理2.2(Green积分公式)**[17, p 7] 设$\Omega,\partial\Omega$如上所述, $u,v:\Omega\mapsto\mathcal{A}_n$是$\Omega$上的连续可微函数, 则对于加权$\mathcal{D}_\omega$算子有如下公式成立

$$\int_\Omega(v\mathcal{D}_\omega\cdot u+v\cdot\mathcal{D}_\omega u)\mathrm{d}x=\int_{\partial\Omega}v\cdot\mathrm{d}\sigma\cdot u, \tag{2.15}$$

其中$\mathrm{d}\sigma=\sum_{i=1}^n\psi_i\mathcal{N}_i\mathrm{d}\mu$是$\partial\Omega$在坐标系$\{\psi_1,\cdots,\psi_n\}$下$\mathcal{A}_n(\mathbf{R})$值的面积微元, $\mathcal{N}=(\mathcal{N}_1,\cdots,\mathcal{N}_n)$是$\partial\Omega$上的外法向量, $\mathrm{d}\mu$是$\partial\Omega$的标量面积微元.

**证** 由

$$\mathrm{d}\hat{x}_i=\mathrm{d}x_1\wedge\cdots\wedge\mathrm{d}x_{i-1}\wedge\mathrm{d}x_{i+1}\cdots\wedge\mathrm{d}x_n,\ i=1,2,\cdots,n.$$

$$\begin{aligned}\mathrm{d}\sigma &= \sum_{i=1}^{n}\psi_i\mathcal{N}_i\mathrm{d}\mu=\psi_1\mathcal{N}_1\mathrm{d}\mu+\psi_2\mathcal{N}_2\mathrm{d}\mu+\cdots+\psi_n\mathcal{N}_n\mathrm{d}\mu\\ &= \psi_1\mathrm{d}x_2\wedge\mathrm{d}x_3\wedge\cdots\wedge\mathrm{d}x_n-\psi_2\mathrm{d}x_1\wedge\mathrm{d}x_3\wedge\cdots\wedge\mathrm{d}x_n+\cdots+(-1)^{n-1}\psi_n\mathrm{d}x_1\wedge\mathrm{d}x_2\\ &\wedge \cdots\wedge\mathrm{d}x_{n-1}\\ &= \sum_{i=1}^{n}(-1)^{i-1}\psi_i\mathrm{d}\hat{x}_i.\end{aligned}$$

从而

$$\begin{aligned}\int_{\partial\Omega}v\cdot\mathrm{d}\sigma\cdot u &= \int_{\partial\Omega}\sum_{N,i,M}(-1)^{i-1}e_N\psi_ie_Mv_Nu_M\mathrm{d}\hat{x}_i\\ &= \sum_{N,i,M}e_N\psi_ie_M\int_{\Omega}(-1)^{i-1}\partial_{x_i}(v_Nu_M)\mathrm{d}x\\ &= \int_{\Omega}\sum_{N,i,M}[(\partial_{x_i}v_N)e_N\psi_iu_Me_M+v_Ne_N\psi_ie_M(\partial_{x_i}u_M)]\\ &= \int_{\Omega}(v\mathcal{D}_\omega\cdot u+v\cdot\mathcal{D}_\omega u)\mathrm{d}x.\end{aligned}$$

**引理2.3(Borel-Pompeiu公式)**[17, p 11] 设$\Omega,\partial\Omega$如上所述, $u(x):\Omega\subset\mathbf{R}^n\to\mathcal{A}_n(\mathbf{R})$, 且$u(x)$在$\mathbf{R}^n$上连续可微, 则有

$$\int_{\partial\Omega}E_\omega(x,\xi)\cdot d\sigma\cdot u-\int_{\Omega}E_\omega(x,\xi)\cdot\mathcal{D}_\omega u\cdot\mathrm{d}x=\begin{cases}u(\xi), & \xi\in\Omega,\\ 0, & \xi\in\overline{\Omega}^c.\end{cases}\tag{2.16}$$

**引理2.4(Cauchy-积分公式)**[17, p 11] 设$\Omega,\partial\Omega$如上所述, 若$u$是$\Omega$内的加权正则函数, 则Borel-Pompeiu公式即为Cauchy-积分公式

$$\int_{\partial\Omega}E_\omega(x,\xi)\cdot\mathrm{d}\sigma\cdot u=\begin{cases}u(\xi), & \xi\in\Omega,\\ 0, & \xi\in\overline{\Omega}^c.\end{cases}\tag{2.17}$$

# §3 加权正则函数的有关性质

**定理3.1(平均值定理)** 设$\Omega$如上所述, 若$f$是$\Omega$上的加权正则函数, 则对于任意$\xi\in\Omega$, 有

$$f(\xi)=\frac{1}{\det(B)^{\frac{1}{2}}R^nV_n}\int_{D(\xi,R)}f(x)\mathrm{d}x,\tag{3.1}$$

其中$\overline{D}(\xi,R)=\{x\in\Omega:\rho(x,\xi)\le R\}$, $V_n=\dfrac{\omega_n}{n}$ 是$n$维单位球的体积.

**证** 根据引理2.2和引理2.4, 有

$$
\begin{aligned}
f(\xi) &= \int_{\partial D(\xi,R)} E_\omega(x,\xi)\,\mathrm{d}\sigma\, f(x) \\
&= \int_{\partial D(\xi,R)} \frac{1}{\det(B)^{\frac12}\omega_n}\frac{1}{\rho^n}\sum_{i,j=1}^n \overline{\psi_i}A_{ij}(x_j-\xi_j)\,\mathrm{d}\sigma\, f(x) \\
&= \frac{1}{\det(B)^{\frac12}\omega_n R^n}\int_{\partial D(\xi,R)}\sum_{i,j=1}^n \overline{\psi_i}A_{ij}(x_j-\xi_j)\,\mathrm{d}\sigma\, f(x) \\
&= \frac{1}{\det(B)^{\frac12}\omega_n R^n}\int_{D(\xi,R)}\sum_{i,j=1}^n \overline{\psi_i}A_{ij}(x_j-\xi_j)\mathcal{D}_\omega\cdot f(x) \\
&+ \frac{1}{\det(B)^{\frac12}\omega_n R^n}\int_{D(\xi,R)}\sum_{i,j=1}^n \overline{\psi_i}A_{ij}(x_j-\xi_j)\cdot\mathcal{D}_\omega f(x).
\end{aligned}
\tag{3.2}
$$

又因为$f$是$\Omega$上的加权正则函数, 所以对任意$x\in D\subset\Omega$, 有$\mathcal{D}_\omega f(x)=0$, 则上式变为

$$
f(\xi)=\frac{1}{\det(B)^{\frac12}\omega_n R^n}\int_{D(\xi,R)}\sum_{i,j=1}^n \overline{\psi_i}A_{ij}(x_j-\xi_j)\mathcal{D}_\omega\cdot f(x).
$$

由于

$$
\begin{aligned}
\sum_{i,j=1}^n \overline{\psi_i}A_{ij}(x_j-\xi_j)\mathcal{D}_\omega &= \sum_{k=1}^n\left(\sum_{i,j=1}^n \overline{\psi_i}A_{ij}(x_j-\xi_j)\partial_{x_k}\psi_k\right) \\
&= \left(\sum_{i,j=1}^n \overline{\psi_i}A_{ij}(x_j-\xi_j)\right)\partial_{x_1}\psi_1+\left(\sum_{i,j=1}^n \overline{\psi_i}A_{ij}(x_j-\xi_j)\right)\partial_{x_2}\psi_2 \\
&+ \cdots+\left(\sum_{i,j=1}^n \overline{\psi_i}A_{ij}(x_j-\xi_j)\right)\partial_{x_n}\psi_n \\
&= \overline{\psi_1}A_{11}\psi_1+\overline{\psi_2}A_{21}\psi_1+\cdots+\overline{\psi_n}A_{n1}\psi_1+\overline{\psi_1}A_{12}\psi_2+\overline{\psi_2}A_{22}\psi_2 \\
&+ \cdots+\overline{\psi_n}A_{n2}\psi_2+\cdots+\overline{\psi_1}A_{1n}\psi_n+\overline{\psi_2}A_{2n}\psi_n\cdots+\overline{\psi_n}A_{nn}\psi_n \\
&= \sum_{i=1}^n B_{ii}A_{ii}+\sum_{i\neq j}(\overline{\psi_i}\psi_j+\overline{\psi_j}\psi_j)A_{ij} \\
&= \sum_{i=1}^n B_{ii}A_{ii}+2\sum_{i\neq j}B_{ij}A_{ij}.
\end{aligned}
$$

又由于矩阵$AB=E$, $E$为单位矩阵, 则矩阵$AB$对角线上的元素的和为

$$
\sum_{i=1}^n B_{ii}A_{ii}+2\sum_{i\neq j}B_{ij}A_{ij}=n.
$$

从而

$$
\sum_{i,j=1}^n \overline{\psi_i}A_{ij}(x_j-\xi_j)\mathcal{D}_\omega=n.
$$

代入(3.2), 再根据Green积分公式, 有

$$
f(\xi)=\frac{n}{\det(B)^{\frac12}R^n\omega_n}\int_{D(\xi,R)}f(x)\mathrm{d}x=\frac{1}{\det(B)^{\frac12}R^nV_n}\int_{D(\xi,R)}f(x)\mathrm{d}x.
$$

**定理3.2(最大模原理)** 设$\Omega$是$R^n$中的域, 若$f$是$\Omega$上的加权正则函数, 如果存在$a\in\Omega$, 使得

$$|f(\xi)|\leq|f(a)|.$$

对所有$\xi\in\Omega$都成立, 则$f$在$\Omega$内为常函数.

**证** 设$|f(a)|=\lambda$. 令$\Omega_\lambda=\{\xi||f(\xi)|=\lambda\}$, 则由于$a\in\Omega_\lambda$, 则$\Omega_\lambda$非空. 对于任意$\xi\in\Omega\setminus\Omega_\lambda$, 有$|f(\xi)|<\lambda$, 又由于$|f(\xi)|$在$\Omega$内连续, 则存在以$\xi$为心, $R^{'}$为半径的$n$维超球$D(\xi,R^{'})=\{x\in\Omega:\rho(x,\xi)<R^{'}\}$, 且对于任意的$u\in D(\xi,R^{'})$有$|f(u)|<\lambda$, 则$D(\xi,R^{'})\subset\Omega\setminus\Omega_\lambda$, 这说明$\Omega_\lambda$是相对闭的.

任取$\xi\in\Omega_\lambda$, 作$n$维超球$D(\xi,R^{''})=\{x\in\Omega:\rho(x,\xi)<R^{''}\}$, 且$\overline{D}(\xi,R^{''})\subset\Omega$, 根据定理3.1有

$$f(\xi)=\frac{1}{\det(B)^{\frac{1}{2}}R''^nV_n}\int_{D(\xi,R'')}f(x)\mathrm{d}x.$$

则

$$\lambda^2=|f(\xi)|^2=\frac{1}{\det(B)R''^{2n}V_n^2}\sum_N\left(\int_{D(\xi,R'')}f_N(x)\mathrm{d}x\right)^2.$$

由Hölder不等式, 有

$$\lambda^2\quad\leq\quad\frac{1}{\det(B)R''^{2n}V_n^2}\sum_N\left(\int_{D(\xi,R'')}\mathrm{d}x\right)\left(\int_{D(\xi,R'')}f_N^2(x)\mathrm{d}x\right).$$

由平均值定理, 当$f\equiv 1$时, 有

$$1=\frac{1}{\det(B)^{\frac{1}{2}}R''^nV_n}\int_{D(\xi,R'')}\mathrm{d}x.$$

从而$\int_{D(\xi,R'')}\mathrm{d}x=\det(B)^{\frac{1}{2}}R^{''n}V_n$. 则

$$\lambda^2\quad\leq\quad\frac{1}{\det(B)^{\frac{1}{2}}R''^nV_n}\int_{D(\xi,R'')}|f(x)|^2\mathrm{d}x,$$

因此

$$0\leq\frac{1}{\det(B)^{\frac{1}{2}}R''^nV_n}\int_{D(\xi,R'')}(|f(x)|^2-\lambda^2)\mathrm{d}x\leq 0.$$

所以对于任意$x\in D(\xi,R^{''})$, 有$|f(x)|=\lambda$, 即$D(\xi,R^{''})\subset\Omega_\lambda$, 因此$\Omega_\lambda$在$\Omega$中是相对开的. 又由于$\Omega$是$\mathbf{R}^n$的非空开的连通子集, 所以由球连法, 可得在$\Omega$上对于任意的$\xi\in\Omega$, 有$|f(\xi)|_0=\lambda$.

当$\lambda=0$时, 显然有$f(\xi)=0$, 即$f(\xi)$在$\Omega$内为常数.

当$\lambda>0$时, 则对于任意的$\xi\in\Omega$, 有$\sum_N f_N^2(\xi)=\lambda^2$, 对$\xi_i$求偏导, 有

$$\sum_N f_N(\xi)\partial_{\xi_i}f_N(\xi)=0, i=1,2,\cdots,n.\tag{3.3}$$

对(3.3)关于$\xi_i$求偏导, 有

$$\sum_N\partial_{\xi_i}f_N(\xi)\partial_{\xi_i}f_N(\xi)+\sum_N f_N(\xi)[\partial^2_{\xi_i\xi_i}f_N(\xi)]=0, i,=1,2,\cdots,n.\tag{3.4}$$

再对(3.3)关于$\xi_j(j\neq i)$求偏导, 有

$$2\sum_N\partial_{\xi_i}f_N(\xi)\partial_{\xi_j}f_N(\xi)+2\sum_N f_N(\xi)(\partial^2_{\xi_i\xi_j}f_N(\xi))=0, i,j=1,2,\cdots,n.\tag{3.5}$$

在(3.4)等式两边乘以$B_{ii}$, 再对(3.5)等式两边乘以$B_{ij}$关于$i,j=1,2,\cdots,n$分别进行求和, 得到

$$\sum_{N,i,j} B_{ij}\partial_{\xi_i} f_N(\xi)\partial_{\xi_j} f_N(\xi)+\sum_N f_N(\xi)\widetilde{\Delta}_n f_N(\xi)=0.$$

又因为$f(\xi)$是$\Omega$内的加权正则函数, 且$\widetilde{\Delta}_n=D_\omega\overline{D}_\omega$, 则$\widetilde{\Delta}_n f(\xi)=0$, 从而$\widetilde{\Delta}_n f_N(\xi)=0$. 故上式可简化为

$$\sum_{N,i,j} B_{ij}\partial_{\xi_i} f_N(\xi)\partial_{\xi_j} f_N(\xi)=0.$$

展开即

$$\sum_N \begin{pmatrix} \partial_{\xi_1} f_N,\partial_{\xi_2} f_N,\cdots,\partial_{\xi_n} f_N \end{pmatrix}\begin{pmatrix} B_{11} & B_{12} & \cdots & B_{1n}\\ B_{12} & B_{22} & \cdots & B_{2n}\\ \vdots & \vdots & \vdots & \vdots\\ B_{1n} & B_{2n} & \cdots & B_{nn}\end{pmatrix}\begin{pmatrix}\partial_{\xi_1} f_N\\ \partial_{\xi_2} f_N\\ \cdots\\ \partial_{\xi_n} f_N\end{pmatrix}=0. \quad (3.6)$$

根据正定矩阵$B$的Cholesky分解$B=LL^{\mathrm{T}}$($L$是下三角矩阵, 且对角线上元素$L_{ii}>0, i=1,2,\cdots,n$)则将下式

$$\begin{pmatrix} B_{11} & B_{12} & \cdots & B_{1n}\\ B_{12} & B_{22} & \cdots & B_{2n}\\ \vdots & \vdots & \vdots & \vdots\\ B_{1n} & B_{2n} & \cdots & B_{nn}\end{pmatrix}=\begin{pmatrix} L_{11} & 0 & \cdots & 0\\ L_{21} & L_{22} & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots\\ L_{n1} & L_{n2} & \cdots & L_{nn}\end{pmatrix}\begin{pmatrix} L_{11} & L_{21} & \cdots & L_{n1}\\ 0 & L_{22} & \cdots & L_{n2}\\ \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & \cdots & L_{nn}\end{pmatrix}.$$

代入(3.6)得

$$\sum_N \begin{pmatrix} \sum_{i=1}^n L_{i1}\partial_{\xi_i} f_N, \sum_{i=2}^n L_{i2}\partial_{\xi_i} f_N,\cdots, L_{nn}\partial_{\xi_n} f_N \end{pmatrix}\begin{pmatrix}\sum_{i=1}^n L_{i1}\partial_{\xi_i} f_N\\ \sum_{i=2}^n L_{i2}\partial_{\xi_i} f_N\\ \cdots\\ L_{nn}\partial_{\xi_n} f_N\end{pmatrix}=0.$$

即

$$\sum_N\left[\left(\sum_{i=1}^n L_{i1}\partial_{\xi_i} f_N\right)^2+\left(\sum_{i=2}^n L_{i2}\partial_{\xi_i} f_N\right)^2+\cdots+(L_{nn}\partial_{\xi_n} f_N)^2\right]=0.$$

从而对所有的$N$, 有

$$\left(\sum_{i=1}^n L_{i1}\partial_{\xi_i} f_N\right)^2+\left(\sum_{i=2}^n L_{i2}\partial_{\xi_i} f_N\right)^2+\cdots+(L_{nn}\partial_{\xi_n} f_N)^2=0.$$

从而$\partial_{\xi_i} f_N=0, i=1,2,\cdots,n$. 所以$f(\xi)$在$\Omega$内为常数.

**推论3.1** 设$\Omega$如上所述, 若$f(\xi)$在$\overline{\Omega}$内连续, 在$\Omega$内是加权正则函数, 则

$$\sup_{\xi\in\overline{\Omega}}|f(\xi)|=\sup_{\xi\in\partial\Omega}|f(\xi)|. \quad (3.7)$$

**证** 如果$f(\xi)$是常函数, 则结论成立.

假设$f(\xi)$是非常函数的加权正则函数, 由于$\Omega$是有界域, 且$f(\xi)$在$\overline{\Omega}$内连续, 则存在点$a\in\overline{\Omega}$, 使得

$$\sup_{\xi\in\overline{\Omega}}|f(\xi)|=|f(a)|.$$

如果$a \in \partial\Omega$, 则

$$\sup_{\xi\in\partial\Omega}|f(\xi)| = |f(a)|.$$

则结论成立.

假设$a \in \Omega$, 则将$\Omega$分解为$\Omega = \Omega_1 \cup \Omega_2 \cup \cdots$, 其中$\Omega_j$是有界的连通开集, $j = 1, 2, \cdots$, 则存在$j$使得$a \in \Omega_j$, 由于对于任意$\xi \in \Omega_j$, 有$|f(\xi)| \le |f(a)|$, $f(\xi)$是加权正则函数, 根据定理3.2, 则$f(\xi)$在$\Omega_j$ 内为常函数, 与已知矛盾, 因此

$$\sup_{\xi\in\overline{\Omega}}|f(\xi)| = |f(a)| = \sup_{\xi\in\partial\Omega_j}|f(\xi)| \le \sup_{\xi\in\partial\Omega}|f(\xi)| \le \sup_{\xi\in\overline{\Omega}}|f(\xi)|.$$

即

$$\sup_{\xi\in\overline{\Omega}}|f(\xi)| = \sup_{\xi\in\partial\Omega}|f(\xi)|.$$

**推论3.2** 设$\Omega$如上所述, 若$f(\xi)$在$\Omega$内是非常量的加权正则函数, 则对$\Omega$内任意一点$\xi$, 有

$$|f(\xi)| < \sup_{\zeta\in\partial\Omega}\overline{\lim_{u\to\zeta,u\in\Omega}}|f(u)|. \tag{3.8}$$

**证** 不妨设$\sup\limits_{\zeta\in\partial\Omega}\overline{\lim\limits_{u\to\xi,u\in\Omega}}|f(u)| < +\infty$, 否则显然成立.

令

$$\varphi(\xi) = \begin{cases} |f(\xi)|, & \xi \in \Omega, \\ \overline{\lim\limits_{u\to\xi,u\in\Omega}}|f(u)|, & \xi \in \partial\Omega. \end{cases} \tag{3.9}$$

由$\varphi(\xi)$的上半连续性, $\overline{\Omega}$为紧集, 则存在一点$a \in \overline{\Omega}$, 使得$\varphi(a) = \{\sup\varphi(u), u \in \overline{\Omega}\}$. 由于$f$在$\Omega$内是非常量的加权正则函数, 则$\varphi(\xi)$在$\Omega$内也是非常量加权正则函数, 由定理3.2, 则$a \in \partial\Omega$. 即

$$|f(\xi)| = \varphi(\xi) < \varphi(a) = \sup_{\zeta\in\partial\Omega}\overline{\lim_{u\to\zeta,u\in\Omega}}|f(u)|,\ \xi \in \Omega. \tag{3.10}$$

从而

$$|f(\xi)| < \sup_{\zeta\in\partial\Omega}\overline{\lim_{u\to\zeta,u\in\Omega}}|f(u)|.$$

**定理3.3** 设$E_\omega(x, \xi)$如上所述, 则存在$M_0$, 使得

$$\left|\frac{\partial^p}{\partial_{\xi_{\alpha_1}}\partial_{\xi_{\alpha_2}}\cdots\partial_{\xi_{\alpha_p}}}\frac{1}{\rho^n}\sum_{i,j=1}^{n}\overline{\psi_i}A_{ij}(x_j - \xi_j)\right| \le \frac{M_0}{\rho^{n+p-1}},\ x \ne \xi.$$

从而

$$\left|\frac{\partial^p E_\omega(x,\xi)}{\partial_{\xi_{\alpha_1}}\partial_{\xi_{\alpha_2}}\cdots\partial_{\xi_{\alpha_p}}}\right| \le \frac{1}{\det(B)^{\frac{1}{2}}\omega_n}\frac{M_0}{\rho^{n+p-1}},\ x \ne \xi$$

对所有$(\alpha_1, \alpha_2, \cdots, \alpha_p) \in \{1, 2, \cdots, n\}^p$成立.

**证** 利用数学归纳法证明.

$j = 1$时, $\alpha_1 \in (1, 2, \cdots, n)^1$,

$$\frac{\partial}{\partial_{\xi_{\alpha_1}}}\left(\frac{1}{\rho^n}\sum_{i,j=1}^{n}\overline{\psi_i}A_{ij}(x_j - \xi_j)\right)$$

$$= \left( \frac{\partial}{\partial_{\xi_{\alpha_1}}} \frac{1}{\rho^n} \cdot \sum_{i,j=1}^{n} \overline{\psi_i} A_{ij}(x_j - \xi_j) + \frac{1}{\rho^n} \cdot \frac{\partial}{\partial_{\xi_{\alpha_1}}} \sum_{i,j=1}^{n} \overline{\psi_i} A_{ij}(x_j - \xi_j) \right)$$

$$= \left( \frac{n}{\rho^{n+2}} \sum_{i,j=1}^{n} \overline{\psi_i} A_{ij}(x_j - \xi_j) \sum_{k=1}^{n} A_{\alpha_1,k}(x_k - \xi_k) - \frac{1}{\rho^n} \sum_{i=1}^{n} \overline{\psi_i} A_{i,\alpha_1} \right).$$

由于

$$\sum_{i,j=1}^{n} \overline{\psi_i} A_{ij}(x_j - \xi_j) \sum_{l,k=1}^{n} \psi_l A_{lk}(x_k - \xi_k)$$

$$= [\overline{\psi_1} A_{11}(x_1 - \xi_1) + \cdots + \overline{\psi_1} A_{1n}(x_n - \xi_n) + \cdots + \overline{\psi_n} A_{1n}(x_1 - \xi_1) + \cdots + \overline{\psi_n} A_{nn}(x_n - \xi_n)]$$

$$\cdot [\psi_1 A_{11}(x_1 - \xi_1) + \cdots + \psi_1 A_{1n}(x_n - \xi_n) + \cdots + \psi_n A_{1n}(x_1 - \xi_1) + \cdots + \psi_n A_{nn}(x_n - \xi_n)]$$

$$= [A_{11}(A_{11}B_{11} + \cdots + A_{1n}B_{1n}) + \cdots + A_{1n}(A_{11}B_{1n} + \cdots + A_{1n}B_{nn})](x_1 - \xi_1)^2$$

$$+ \cdots$$

$$+ [2A_{1n}(A_{11}B_{11} + \cdots + A_{1n}B_{1n}) + \cdots + 2A_{nn}(A_{11}B_{1n} + \cdots + A_{1n}B_{nn})](x_1 - \xi_1)(x_n - \xi_n)$$

$$+ \cdots$$

$$+ [A_{n1}(A_{1n}B_{11} + \cdots + A_{nn}B_{1n}) + \cdots + A_{nn}(A_{1n}B_{1n} + \cdots + A_{nn}B_{nn})](x_n - \xi_n)^2.$$

又由于$AB = E$, 则

$$\begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{12} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} & \cdots & B_{1n} \\ B_{12} & B_{22} & \cdots & B_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ B_{1n} & B_{2n} & \cdots & B_{nn} \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$

从而上式化简为

$$\sum_{i,j=1}^{n} \overline{\psi_i} A_{ij}(x_j - \xi_j) \sum_{l,k=1}^{n} \psi_l A_{lk}(x_k - \xi_k)$$

$$= A_{11}(x_1 - \xi_1)^2 + \cdots + 2A_{1n}(x_1 - \xi_1)(x_n - \xi_n) + \cdots + A_{nn}(x_n - \xi_n)^2 = \rho^2.$$

且由于

$$\left| \sum_{k=1}^{n} A_{\alpha_i,k}(x_k - \xi_k) \right| \leq \sum_{k=1}^{n} |A_{\alpha_i,k}||x_k - \xi_k|.$$

令$M_1=\max\{|A_{\alpha_i,1}|,|A_{\alpha_i,2}|,\cdots,|A_{\alpha_i,n}|\}$, 则

$$\left|\sum_{k=1}^{n}A_{\alpha_i,k}(x_k-\xi_k)\right|\le\sum_{k=1}^{n}|A_{\alpha_i,k}||x_k-\xi_k|\le M_1\sum_{k=1}^{n}|x_k-\xi_k|\le nM_1r\le\frac{nM_1}{c}\rho\le M_2\rho.$$

则

$$\left|\sum_{i,j=1}^{n}\overline{\psi_i}A_{ij}(x_j-\xi_j)\right|\left|\sum_{k=1}^{n}A_{\alpha_1,k}(x_k-\xi_k)\right|\le M_2\rho^2.$$

另一方面

$$\left|\sum_{i=1}^{n}\overline{\psi_i}A_{i,\alpha_1}\right|=\left|\sum_{i,k=1}^{n}L_{ik}A_{i,\alpha_1}\overline{e}_{p_k}\right|\le\sum_{i,k=1}^{n}|L_{ik}A_{i,\alpha_1}||\overline{e}_{p_k}|\le M_3.$$

从而有

$$\begin{aligned}&\left|\frac{\partial}{\partial_{\xi_{\alpha_1}}}\left(\frac{1}{\rho^n}\sum_{i,j=1}^{n}\overline{\psi_i}A_{ij}(x_j-\xi_j)\right)\right|\\ \le&\left(\frac{n}{\rho^{n+2}}\left|\sum_{i,j=1}^{n}\overline{\psi_i}A_{ij}(x_j-\xi_j)\sum_{k=1}^{n}A_{\alpha_1,k}(x_k-\xi_k)\right|+\frac{1}{\rho^n}\left|\sum_{i=1}^{n}\overline{\psi_i}A_{i,\alpha_1}\right|\right)\\ \le&\left(\frac{nM_2\rho^2}{\rho^{n+2}}+\frac{M_3}{\rho^n}\right).\end{aligned}$$

则存在$M_4>0$, 使得

$$\left|\frac{\partial}{\partial_{\xi_{\alpha_1}}}\left(\frac{1}{\rho^n}\sum_{i,j=1}^{n}\overline{\psi_i}A_{ij}(x_j-\xi_j)\right)\right|\le\frac{M_4}{\rho^n}.$$

假设$j\le p-1$时成立, 即

$$\left|\frac{\partial^j}{\partial_{\xi_{\alpha_1}}\partial_{\xi_{\alpha_2}}\cdots\partial_{\xi_{\alpha_j}}}\left(\frac{1}{\rho^n}\sum_{i,j=1}^{n}\overline{\psi_i}A_{ij}(x_j-\xi_j)\right)\right|\le\frac{M_5}{\rho^{n+j-1}},\quad(\alpha_1,\alpha_2,\cdots,\alpha_p)\in\{1,2,\cdots,n\}^j.$$

下证$j=p$时, 成立

$$\left|\frac{\partial^p}{\partial_{\xi_{\alpha_1}}\partial_{\xi_{\alpha_2}}\cdots\partial_{\xi_{\alpha_p}}}\left(\frac{1}{\rho^n}\sum_{i,j=1}^{n}\overline{\psi_i}A_{ij}(x_j-\xi_j)\right)\right|\le\frac{M_0}{\rho^{n+p-1}},\quad(\alpha_1,\alpha_2,\cdots,\alpha_p)\in\{1,2,\cdots,n\}^p.$$

事实上

$$\begin{aligned}&\frac{\partial^p}{\partial_{\xi_{\alpha_1}}\partial_{\xi_{\alpha_2}}\cdots\partial_{\xi_{\alpha_p}}}\left(\frac{1}{\rho^n}\sum_{i,j=1}^{n}\overline{\psi_i}A_{ij}(x_j-\xi_j)\right)\\ =&\frac{\partial^{p-1}}{\partial_{\xi_{\alpha_1}}\partial_{\xi_{\alpha_2}}\cdots\partial_{\xi_{\alpha_{p-1}}}}\left[\frac{\partial}{\partial_{\xi_{\alpha_p}}}\left(\frac{1}{\rho^n}\sum_{i,j=1}^{n}\overline{\psi_i}A_{ij}(x_j-\xi_j)\right)\right]\end{aligned}$$

$$= \frac{\partial^{p-1}}{\partial_{\xi_{\alpha_1}}\partial_{\xi_{\alpha_2}}\cdots\partial_{\xi_{\alpha_{p-1}}}}\left(\frac{n}{\rho^{n+2}}\sum_{i,j=1}^{n}\overline{\psi}_i A_{ij}(x_j-\xi_j)\sum_{k=1}^{n}A_{\alpha_p,k}(x_k-\xi_k)-\frac{1}{\rho^n}\sum_{i=1}^{n}\overline{\psi}_i A_{i,\alpha_p}\right)$$

$$= \frac{\partial^{p-1}}{\partial_{\xi_{\alpha_1}}\partial_{\xi_{\alpha_2}}\cdots\partial_{\xi_{\alpha_{p-1}}}}\left(\frac{n}{\rho^{n+2}}\sum_{i,j=1}^{n}\overline{\psi}_i A_{ij}(x_j-\xi_j)\sum_{k=1}^{n}A_{\alpha_p,k}(x_k-\xi_k)\right)$$

$$+\frac{\partial^{p-1}}{\partial_{\xi_{\alpha_1}}\partial_{\xi_{\alpha_2}}\cdots\partial_{\xi_{\alpha_{p-1}}}}\left(-\frac{1}{\rho^n}\sum_{i=1}^{n}\overline{\psi}_i A_{i,\alpha_p}\right)$$

$$= I_1 + I_2.$$

下面首先讨论$I_1$. 若$\alpha_1,\alpha_2,\cdots,\alpha_{p-1}$中存在一数与$\alpha_p$相同, 不妨设$\alpha_{p-1}=\alpha_p$, 则

$$\begin{aligned}
I_1 &= \frac{\partial^{p-2}}{\partial_{\xi_{\alpha_1}}\partial_{\xi_{\alpha_2}}\cdots\partial_{\xi_{\alpha_{p-2}}}}\frac{\partial}{\partial_{\xi_{\alpha_{p-1}}}}\left(\frac{n}{\rho^{n+2}}\sum_{i,j=1}^{n}\overline{\psi}_i A_{ij}(x_j-\xi_j)\sum_{k=1}^{n}A_{\alpha_p,k}(x_k-\xi_k)\right)\\
&= \frac{\partial^{p-2}}{\partial_{\xi_{\alpha_1}}\partial_{\xi_{\alpha_2}}\cdots\partial_{\xi_{\alpha_{p-2}}}}\left(\frac{\partial}{\partial_{\xi_{\alpha_{p-1}}}}\frac{1}{\rho^{n+2}}\sum_{i,j=1}^{n}\overline{\psi}_i A_{ij}(x_j-\xi_j)\cdot n\sum_{k=1}^{n}A_{\alpha_p,k}(x_k-\xi_k)\right)\\
&- \frac{\partial^{p-2}}{\partial_{\xi_{\alpha_1}}\partial_{\xi_{\alpha_2}}\cdots\partial_{\xi_{\alpha_{p-2}}}}\left(nA_{\alpha_p,\alpha_p}\frac{1}{\rho^{n+2}}\sum_{i,j=1}^{n}\overline{\psi}_i A_{ij}(x_j-\xi_j)\right)\\
&= \frac{\partial^{p-1}}{\partial_{\xi_{\alpha_1}}\partial_{\xi_{\alpha_2}}\cdots\partial_{\xi_{\alpha_{p-1}}}}\frac{1}{\rho^{n+2}}\sum_{i,j=1}^{n}\overline{\psi}_i A_{ij}(x_j-\xi_j)\cdot n\sum_{k=1}^{n}A_{\alpha_p,k}(x_k-\xi_k)\\
&- nA_{\alpha_p,\alpha_p}\frac{\partial^{p-2}}{\partial_{\xi_{\alpha_1}}\partial_{\xi_{\alpha_2}}\cdots\partial_{\xi_{\alpha_{p-2}}}}\frac{1}{\rho^{n+2}}\sum_{i,j=1}^{n}\overline{\psi}_i A_{ij}(x_j-\xi_j).
\end{aligned}$$

由假设存在$M_6>0$, $M_7>0$, 使得

$$\left|\frac{\partial^{p-1}}{\partial_{\xi_{\alpha_1}}\partial_{\xi_{\alpha_2}}\cdots\partial_{\xi_{\alpha_{p-1}}}}\frac{1}{\rho^{n+2}}\sum_{i,j=1}^{n}\overline{\psi}_i A_{ij}(x_j-\xi_j)\right|\le\frac{M_6}{\rho^{n+p}}.$$

$$\left|\frac{\partial^{p-2}}{\partial_{\xi_{\alpha_1}}\partial_{\xi_{\alpha_2}}\cdots\partial_{\xi_{\alpha_{p-2}}}}\frac{1}{\rho^{n+2}}\sum_{i,j=1}^{n}\overline{\psi}_i A_{ij}(x_j-\xi_j)\right|\le\frac{M_7}{\rho^{n+p-1}}.$$

则有

$$|I_1|\le\frac{M_6}{\rho^{n+p}}\cdot n\left|\sum_{k=1}^{n}A_{\alpha_p,k}(x_k-\xi_k)\right|+nA_{\alpha_p,\alpha_p}\frac{M_7}{\rho^{n+p-1}}.$$

即存在$M_8>0$, 使得

$$|I_1|\leq\frac{M_8}{\rho^{n+p-1}}.$$

若$\alpha_1,\alpha_2,...,\alpha_{p-1}$任何数都不等于$\alpha_p$, 则有

$$\begin{aligned}
|I_1| &= \left|\frac{\partial^{p-1}}{\partial_{\xi_{\alpha_1}}\partial_{\xi_{\alpha_2}}\cdots\partial_{\xi_{\alpha_{p-1}}}}\left(\frac{n}{\rho^{n+2}}\sum_{i,j=1}^{n}\overline{\psi_i}A_{ij}(x_j-\xi_j)\sum_{k=1}^{n}A_{\alpha_p,k}(x_k-\xi_k)\right)\right|\\
&\leq \left|\frac{\partial^{p-1}}{\partial_{\xi_{\alpha_1}}\partial_{\xi_{\alpha_2}}\cdots\partial_{\xi_{\alpha_{p-1}}}}\frac{n}{\rho^{n+2}}\sum_{i,j=1}^{n}\overline{\psi_i}A_{ij}(x_j-\xi_j)\cdot\sum_{k=1}^{n}A_{\alpha_p,k}(x_k-\xi_k)\right|\\
&+ \left|\frac{n}{\rho^{n+2}}\sum_{i,j=1}^{n}\overline{\psi_i}A_{ij}(x_j-\xi_j)\cdot\frac{\partial^{p-1}}{\partial_{\xi_{\alpha_1}}\partial_{\xi_{\alpha_2}}\cdots\partial_{\xi_{\alpha_{p-1}}}}\sum_{k=1}^{n}A_{\alpha_p,k}(x_k-\xi_k)\right|\\
&= \left|\frac{\partial^{p-1}}{\partial_{\xi_{\alpha_1}}\partial_{\xi_{\alpha_2}}\cdots\partial_{\xi_{\alpha_{p-1}}}}\frac{n}{\rho^{n+2}}\sum_{i,j=1}^{n}\overline{\psi_i}A_{ij}(x_j-\xi_j)\cdot\sum_{k=1}^{n}A_{\alpha_p,k}(x_k-\xi_k)\right|\\
&\leq \frac{M_6}{\rho^{n+p}}\cdot n\left|\sum_{k=1}^{n}A_{\alpha_p,k}(x_k-\xi_k)\right|\\
&\leq \frac{M_9}{\rho^{n+p-1}}.
\end{aligned}$$

下面讨论$I_2$, 由于$x\neq\xi$时, $\frac{\partial^{p-1}}{\partial_{\xi_{\alpha_1}}\partial_{\xi_{\alpha_2}}\cdots\partial_{\xi_{\alpha_{p-1}}}}\frac{1}{\rho^n}=\frac{\partial^{p-1}}{\partial_{\xi_{\alpha_{p-1}}}\cdots\partial_{\xi_{\alpha_2}}\partial_{\xi_{\alpha_1}}}\frac{1}{\rho^n}$.
从而有

$$\begin{aligned}
I_2 &= \frac{\partial^{p-1}}{\partial_{\xi_{\alpha_{p-1}}}\cdots\partial_{\xi_{\alpha_2}}\partial_{\xi_{\alpha_1}}}\left(-\frac{1}{\rho^n}\sum_{i=1}^{n}\overline{\psi_i}A_{i,\alpha_p}\right)\\
&= -\sum_{i=1}^{n}\overline{\psi_i}A_{i,\alpha_p}\frac{\partial^{p-1}}{\partial_{\xi_{\alpha_{p-1}}}\cdots\partial_{\xi_{\alpha_2}}\partial_{\xi_{\alpha_1}}}\frac{1}{\rho^n}\\
&= -\sum_{i=1}^{n}\overline{\psi_i}A_{i,\alpha_p}\frac{\partial^{p-1}}{\partial_{\xi_{\alpha_{p-1}}}\cdots\partial_{\xi_{\alpha_2}}\partial_{\xi_{\alpha_1}}}\frac{1}{\rho^{n+2}}\sum_{i,j=1}^{n}\overline{\psi_i}A_{ij}(x_j-\xi_j)\sum_{l,k=1}^{n}\psi_lA_{lk}(x_k-\xi_k)\\
&= -\sum_{i=1}^{n}\overline{\psi_i}A_{i,\alpha_p}\frac{\partial^{p-2}}{\partial_{\xi_{\alpha_{p-1}}}\cdots\partial_{\xi_{\alpha_3}}\partial_{\xi_{\alpha_2}}}\frac{\partial}{\partial_{\xi_{\alpha_1}}}\frac{1}{\rho^{n+2}}\sum_{i,j=1}^{n}\overline{\psi_i}A_{ij}(x_j-\xi_j)\cdot\sum_{l,k=1}^{n}\psi_lA_{lk}(x_k-\xi_k)\\
&- \sum_{i=1}^{n}\overline{\psi_i}A_{i,\alpha_p}\frac{\partial^{p-2}}{\partial_{\xi_{\alpha_{p-1}}}\cdots\partial_{\xi_{\alpha_3}}\partial_{\xi_{\alpha_2}}}\left(-\frac{1}{\rho^{n+2}}\sum_{i,j=1}^{n}\overline{\psi_i}A_{ij}(x_j-\xi_j)\sum_{l=1}^{n}\overline{\psi_l}A_{l,\alpha_1}\right)
\end{aligned}$$

$$
\begin{aligned}
&= -\sum_{i=1}^{n}\overline{\psi_i}A_{i,\alpha_p}\frac{\partial^{p-3}}{\partial_{\xi_{\alpha_{p-1}}}\cdots\partial_{\xi_{\alpha_4}}\partial_{\xi_{\alpha_3}}}\frac{\partial}{\partial_{\xi_{\alpha_2}}}\left[\frac{\partial}{\partial_{\xi_{\alpha_1}}}\frac{1}{\rho^{n+2}}\sum_{i,j=1}^{n}\overline{\psi_i}A_{ij}(x_j-\xi_j)\cdot\sum_{l,k=1}^{n}\psi_lA_{lk}(x_k\right.\\
&- \left.\xi_k)\right]-\sum_{i=1}^{n}\overline{\psi_i}A_{i,\alpha_p}\frac{\partial^{p-2}}{\partial_{\xi_{\alpha_{p-1}}}\cdots\partial_{\xi_{\alpha_3}}\partial_{\xi_{\alpha_2}}}\left(-\frac{1}{\rho^{n+2}}\sum_{i,j=1}^{n}\overline{\psi_i}A_{ij}(x_j-\xi_j)\sum_{l=1}^{n}\overline{\psi_l}A_{l,\alpha_1}\right)\\
&= -\sum_{i=1}^{n}\overline{\psi_i}A_{i,\alpha_p}\frac{\partial^{p-3}}{\partial_{\xi_{\alpha_{p-1}}}\cdots\partial_{\xi_{\alpha_4}}\partial_{\xi_{\alpha_3}}}\frac{\partial}{\partial_{\xi_{\alpha_2}}\partial_{\xi_{\alpha_1}}}\frac{1}{\rho^{n+2}}\sum_{i,j=1}^{n}\overline{\psi_i}A_{ij}(x_j-\xi_j)\cdot\sum_{l,k=1}^{n}\psi_lA_{lk}(x_k\\
&- \xi_k)-\sum_{i=1}^{n}\overline{\psi_i}A_{i,\alpha_p}\frac{\partial^{p-2}}{\partial_{\xi_{\alpha_{p-1}}}\cdots\partial_{\xi_{\alpha_3}}\partial_{\xi_{\alpha_1}}}\left(-\frac{1}{\rho^{n+2}}\sum_{i,j=1}^{n}\overline{\psi_i}A_{ij}(x_j-\xi_j)\sum_{l=1}^{n}\overline{\psi_l}A_{l,\alpha_2}\right)\\
&- \sum_{i=1}^{n}\overline{\psi_i}A_{i,\alpha_p}\frac{\partial^{p-2}}{\partial_{\xi_{\alpha_{p-1}}}\cdots\partial_{\xi_{\alpha_3}}\partial_{\xi_{\alpha_2}}}\left(-\frac{1}{\rho^{n+2}}\sum_{i,j=1}^{n}\overline{\psi_i}A_{ij}(x_j-\xi_j)\sum_{l=1}^{n}\overline{\psi_l}A_{l,\alpha_1}\right)\\
&= \cdots\cdots\\
&= -\sum_{i=1}^{n}\overline{\psi_i}A_{i,\alpha_p}\frac{\partial^{p-1}}{\partial_{\xi_{\alpha_{p-1}}}\cdots\partial_{\xi_{\alpha_2}}\partial_{\xi_{\alpha_1}}}\frac{1}{\rho^{n+2}}\sum_{i,j=1}^{n}\overline{\psi_i}A_{ij}(x_j-\xi_j)\cdot\sum_{l,k=1}^{n}\psi_lA_{lk}(x_k-\xi_k)\\
&- \sum_{i=1}^{n}\overline{\psi_i}A_{i,\alpha_p}\frac{\partial^{p-2}}{\partial_{\xi_{\alpha_{p-2}}}\cdots\partial_{\xi_{\alpha_2}}\partial_{\xi_{\alpha_1}}}\left(-\frac{1}{\rho^{n+2}}\sum_{i,j=1}^{n}\overline{\psi_i}A_{ij}(x_j-\xi_j)\sum_{l=1}^{n}\overline{\psi_l}A_{l,\alpha_{p-1}}\right)\\
&- \cdots\\
&- \sum_{i=1}^{n}\overline{\psi_i}A_{i,\alpha_p}\frac{\partial^{p-2}}{\partial_{\xi_{\alpha_{p-1}}}\cdots\partial_{\xi_{\alpha_3}}\partial_{\xi_{\alpha_2}}}\left(-\frac{1}{\rho^{n+2}}\sum_{i,j=1}^{n}\overline{\psi_i}A_{ij}(x_j-\xi_j)\sum_{l=1}^{n}\overline{\psi_l}A_{l,\alpha_1}\right).
\end{aligned}
$$

所以有

$$
\begin{aligned}
|I_2| &\le \left|\sum_{i=1}^{n}\overline{\psi_i}A_{i,\alpha_p}\right|\frac{M_4}{\rho^{n+p}}\cdot\left|\sum_{l,k=1}^{n}\psi_lA_{lk}(x_k-\xi_k)\right|\\
&+ \left|\sum_{i=1}^{n}\overline{\psi_i}A_{i,\alpha_p}\right|\left|\sum_{j=1}^{p-1}\sum_{l=1}^{n}\overline{\psi_l}A_{l,\alpha_j}\right|\frac{M_7}{\rho^{n+p-1}}.
\end{aligned}
$$

即存在$M_{10}>0$, 使得

$$
|I_2| \le \frac{M_{10}}{\rho^{n+p-1}}.
$$

综上, 由归纳假设可知, 存在$M_0$使得

$$\left|\frac{\partial^p}{\partial_{\xi_{\alpha_1}}\partial_{\xi_{\alpha_2}}\cdots\partial_{\xi_{\alpha_p}}}\left(\frac{1}{\rho^n}\sum_{i,j=1}^{n}\overline{\psi_i}A_{ij}(x_j-\xi_j)\right)\right|\leq\frac{M_0}{\rho^{n+p-1}},\ x\neq\xi.$$

从而

$$\left|\frac{\partial^p E_\omega(x,\xi)}{\partial_{\xi_{\alpha_1}}\partial_{\xi_{\alpha_2}}\cdots\partial_{\xi_{\alpha_p}}}\right|\leq\frac{1}{\det(B)^{\frac{1}{2}}\omega_n}\frac{M_0}{\rho^{n+p-1}},\ x\neq\xi.$$

对所有

$$(\alpha_1,\alpha_2,\cdots,\alpha_p)\in\{1,2,\cdots,n\}^p$$

都成立.

**定理3.4 (Weierstrass定理)** 假设$\{f_j\}$($j$是正整数)是$\Omega$中的一列加权正则函数列, 如果对$\Omega$中的每个紧集$K$, 任意$\varepsilon>0$, 存在自然数$N(\varepsilon,K)$, 使得

$$\sup_{\xi\in K}|f_i-f_j|<\varepsilon,\quad i,j\ \ \geq\ \ N(\varepsilon,K).$$

则存在$\Omega$上的函数$f$, 使得

(i)$f$是$\Omega$中的加权正则函数;

(ii)对任意多重指标$\beta$, 函数列$\{\partial^\beta f_j\}$($j$是正整数)在$\Omega$上是内闭一致收敛于$\partial^\beta f$.

**证** 设$K$是$\Omega$中任意一个紧集, 令序列$K_j(j=1,2,...)$是$\Omega$的一个正规穷竭, 满足$K_j$是紧集, $j=1,2,...$; $\bigcup\limits_{j=1}^{\infty}K_j=\Omega$; $K_j\subset\mathring{K}_{j+1}$, 则存在$i_0$, 使得$K\subset K_{i_0}$.

由加权正则函数的Cauchy积分公式, 以及$f_i-f_j$是$\Omega$中的加权正则函数, 从而对所有$\xi\in K$和多重指标$\beta$有

$$\partial^\beta f_i(\xi)-\partial^\beta f_j(\xi)=\int_{\partial K_{i_0}}\partial^\beta E_\omega(x,\xi)\mathrm{d}\sigma\cdot[f_i(x)-f_j(x)].$$

则当$i,j\geq N(\varepsilon,K)$时, 有

$$\begin{aligned}&|\partial^\beta f_i(\xi)-\partial^\beta f_j(\xi)|\\ \leq\ &\int_{\partial K_{i_0}}|\partial^\beta E_\omega(x,\xi)|\mathrm{d}\sigma\cdot|f_i(x)-f_j(x)|\\ \leq\ &\int_{\partial K_{i_0}}\mathrm{d}\sigma\cdot\sup_{x\in\partial K_{i_0}}|\partial^\beta E_\omega(x,\xi)|\cdot\sup_{x\in\partial K_{i_0}}|f_i(x)-f_j(x)|.\end{aligned}$$

由于$K_{i_0}$是紧集, 从而积分

$$\int_{\partial K_{i_0}}\mathrm{d}\sigma$$

有界, 因而存在$M_{11}>0$, 使得

$$\int_{\partial K_{i_0}}\mathrm{d}\sigma\leq M_{11}.$$

又由定理3.3可知

$$|\partial^\beta E_\omega(x,\xi)| \le \frac{1}{\det(B)^{\frac{1}{2}}\omega_n}\frac{M_0}{\rho^{n+p-1}},$$

则

$$\begin{aligned}
& |\partial^\beta f_i(\xi) - \partial^\beta f_j(\xi)| \\
\le\ & \int_{\partial K_{i_0}} \mathrm{d}\sigma_x \cdot \sup_{x\in\partial K_{i_0}} |\partial^\beta E_\omega(x,\xi)| \cdot \sup_{x\in\partial K_{i_0}} |f_i(x) - f_j(x)| \\
\le\ & M_{11}\frac{1}{\det(B)^{\frac{1}{2}}\omega_n}\frac{M_0}{\rho^{n+p-1}}\varepsilon \\
\le\ & M_{11}\frac{1}{\det(B)^{\frac{1}{2}}\omega_n}\frac{M_0}{d^{n+p-1}}\varepsilon,
\end{aligned}$$

其中$d = d(K, \partial K_{i_0})$是非欧距离下集合间的距离.

对于$f_i$, $f_j$的各个元素, $i, j = 1, 2, \cdots$, 有

$$\sup_{x\in\partial K_{i_0}} |\partial^\beta f_{i,N}(\xi) - \partial^\beta f_{j,N}(\xi)| \quad\le\quad M_{12}\varepsilon,\ i,j \ge N(\varepsilon, K).$$

这表明$\forall N \in \Gamma_n$, 序列$\{f_{j,N}(\xi)\}$ $(j = 1, 2, \cdots,)$ 是$E(\Omega, \mathbf{R})$中的Cauchy列, 其中$E(\Omega, \mathbf{R})$是定义在$\Omega$上取值于$\mathbf{R}$的函数全体构成的集合, 则由$E(\Omega, \mathbf{R})$的完备性可知, 存在函数$f_N \in E(\Omega, \mathbf{R})$, $N \in \Gamma_n$, 使得$\{\partial^\beta f_{i,N}(\xi)\}$在$\Omega$中内闭一致收敛于$\partial^\beta f_N(\xi)$, $\beta$为任意多重指标.

再令$f = \sum_N f_N e_N$, 从而$\{\partial^\beta f_i(\xi)\}$在$\Omega$中内闭一致收敛于$\partial^\beta f(\xi)$, 当$\beta = 1$, $\{\partial f_i(\xi)\}$在$\Omega$中内闭一致收敛于$\partial f(\xi)$, 从而有$\{D_\omega f_i(\xi)\}$在$\Omega$中内闭一致收敛于$D_\omega f(\xi)$. 由$D_\omega f_i(\xi) = 0 (i = 1, 2, \cdots)$, 从而$D_\omega f(\xi) = 0$, 所以$f$是$\Omega$中的加权正则函数.

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## Some properties of weighted regular functions

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**Abstract**: Regular function is one of the important functions in Cliford analysis and weighted regular function is the further development of regular function, which is also a kind of important function, so it is a certain research significance to study the weighted regular function. Based on the research of regular function and using the characteristics of the weighted regular function itself, mean value theorem, maximum modulus theorem, Weierstrass theorem and some corollaries of weighted regular functions are discussed.

**Keywords**: real Clifford analysis; weighted regular functions; mean value theorem; maximum modulus theorem; Weierstrass theorem

**MR Subject Classification**: 47H05; 30G30